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ON COMPACT SUBMANIFOLDS WITH NONDEGENERATE PARALLEL NORMAL VECTOR FIELDS

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In this paper we obtain characterizations of spherical submanifolds in Euclidean space of codimension $\geqq 1$. Such characterizations are given here in terms of certain relationships involving the elementary symmetric functions of principal radii of curvature and the support function of a submanifold.

1. **Introduction.** For hypersurfaces similar characterizations are well known. For example, let M be a closed convex hypersurface in Euclidean space, h the support function of M, and S_l the elementary symmetric function of order l of principal curvatures. It has been proved by several authors (see Simon [8], and further references given there) that if for some integer $l(1 \leqq l \leqq \dim M)$ everywhere on M $h^l S_l = \text{const}$, then M is a hypersphere. Other results of this type are also known [8], [9].

Our proofs are based on a differential analogue of the Minkowski-Hsiung formulas, relating the support function and elementary symmetric functions of various orders of the principal radii of curvature. Those formulas are obtained for submanifolds which possess a nondegenerate normal vector field parallel in the normal bundle.

Finally, we note that characterizations of spherical submanifolds in terms of the elementary symmetric functions of principal curvatures are obtained by Chen [2] and Chen and Yano [4] (see also Chen [3], Chapter 6).

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2. **Preliminaries.** In this section we shall present local formulas relating the second fundamental form and the support function of a submanifold in Euclidean space. We shall use the following convention on the ranges of indices:

$$1 \leqq i, j, k, l, r \leqq m\ , \qquad 1 \leqq \alpha \leqq n\ ,$$

and as usual, it is agreed that repeated lower and upper indices are summed over the respective ranges. We denote by E the Euclidean space of dimension $m + n$, and we fix the origin at some point O. Consider a smooth, orientable submanifold M of dimension $m(\geqq 2)$ immersed in E, and represented by the position vector field

$$X = X(u^1, \cdots, u^m) ,$$

where $\{u^i\}$ are the local coordinates on M. Let x be a point of M. We denote by $T_x(M)$ and $N_x(M)$ the restrictions of the tangent bundle $T(M)$ and normal bundle $N(M)$ at x.

Put

$$X_i = \partial_i X , \qquad \partial_i = \partial/\partial u^i .$$

The metric I, or the first fundamental form induced on M from E via X, is $G_{ij} = \langle X_i, X_j \rangle$, where $\langle , \rangle$ denotes the inner product in E. Let ξ be an arbitrary unit normal vector field defined in a neighborhood U of $x \in M$. The second fundamental form at x with respect to ξ is $II(\xi) = b_{ij}(\xi) du^i du^j$, where $b_{ij}(\xi) = -\langle X_i, \xi_j \rangle$. Let η be a unit normal vector field in U not necessarily different from ξ. The mixed third fundamental form is $III(\xi, \eta) = g_{ij}(\xi, \eta) du^i du^j$, where $g_{ij}(\xi, \eta) = \langle \xi_i, \eta_j \rangle$. We write $III(\xi) \equiv III(\xi, \xi)$, and $g_{ij}(\xi) \equiv g_{ij}(\xi, \xi)$. Evidently, $g_{ij}(\xi, \eta) = g_{ji}(\eta, \xi)$, but, in general, no other symmetries exist. For a unit normal vector field $\xi \in N(M)$, $h(\xi)$ denotes the support function of M with respect to ξ, that is, $h(\xi) = -\langle X, \xi \rangle$.

Recall that a nondegenerate normal vector field on M is a unit normal vector field ξ such that $\det(b_{ij}(\xi)) \neq 0$ everywhere on M (see [2], and also [3], p. 59).

Vectors $\{X_i\}$ form a basis in $T_x(M)$, $x \in M$, and we denote by $\{N(\alpha)\}$ a field of orthonormal frames in $N(M)$. Put

$$X_{ij} = \partial_{ij} X , \quad \partial_{ij} = \partial^2/\partial u^i \partial u^j .$$

At first we note that $b_{ij}(\xi) = \langle X_{ij}, \xi \rangle$, and $b_{ij}(\xi) = b_{ji}(\xi)$ for a unit normal vector field ξ. Also, $g_{ij}(\xi) = -\langle \xi_{ij}, \xi \rangle = g_{ji}(\xi)$. Suppose that ξ is parallel in $N(M)$, that is, $\xi_i \in T(M)$, $i = 1, \cdots, m$, everywhere on M, and let η be an arbitrary unit normal vector field on M. Then $g_{ij}(\xi, \eta) = -\langle \xi_{ij}, \eta \rangle = g_{ji}(\xi, \eta)$.

In the frame $X_1, \cdots, X_m, N(1), \cdots, N(n)$ we have for an arbitrary unit vector field $\xi \in N(M)$:

$$\xi_i = -b_i^j(\xi) X_j + \sum_\alpha \langle \xi_i, N(\alpha) \rangle N(\alpha) , \tag{1}$$

where $b_i^j(\xi) = b_{il}(\xi) G^{lj}$, and G^{lj} being the inverse of G_{lj}. From here, for two unit normal vector fields ξ and η, we find

$$g_{ij}(\xi, \eta) = b_i^r(\xi) b_{rj}(\eta) + \sum_\alpha \langle \xi_i, N(\alpha) \rangle \langle \eta_j, N(\alpha) \rangle . \tag{2}$$

If ξ or η is parallel, then

$$g_{ij}(\xi, \eta) = b_i^r(\xi) b_{rj}(\eta) . \tag{3}$$

Note that when M, ξ and η are such that $II(\xi)$ and $II(\eta)$ are positive definite then so is $III(\xi, \eta)$. However, the form $III(\xi)$ is nonnegative definite for an arbitrary $II(\xi)$. If ξ is nondegenerate everywhere on M, then $III(\xi)$ induces a Riemannian metric on M. We denote by $dO(\xi)$ the corresponding volume-element. From formula (1) it follows that if ξ is nondegenerate and parallel, then vectors $\{\xi_i\}$ form a basis in $T_x(M)$, $x \in M$, and according to the Gauss equation we have:

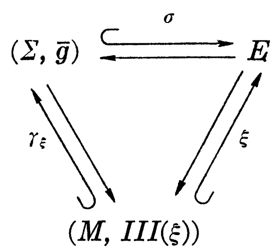
$$\xi_{ij} = \Gamma^k_{ij}(\xi)\xi_k - \sum_\alpha g_{ij}(\xi, N(\alpha))N(\alpha) , \tag{4}$$

where $\Gamma^k_{ij}(\xi)$ denote the Cristoffel symbols of the second kind with respect to $III(\xi)$.

When ξ is nondegenerate, then translating it parallel to itself in E to the origin O we can define an immersion $\gamma_\xi \colon M \to \Sigma$, where Σ is a unit hypersphere in E centered at O. In codimension one γ_ξ is the standard Gauss map.

Proposition 2.1. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field on M. Then γ_ξ is an isometric immersion of M with the metric $III(\xi)$ into Σ.*

Proof. Let the symbol $\hookrightarrow$ denote an immersion, and $\rightarrow$ a pullback of the mertic from the ambient space. Then the following diagram is commutative in $\hookrightarrow$ and $\rightarrow$.



where σ is the standard imbedding of Σ in E, and $\bar{g}$ is the metric induced on Σ from E. The Proposition is proved.

For convenience we write $h(\alpha) \equiv h(N(\alpha))$. The position vector field X of a submanifold M can be decomposed into two parts:

$$X = X_T + X_N , \tag{5}$$

where $X_T \in T(M)$, $X_N \in N(M)$. In the frame $X_1, X_2, \cdots, X_m, N(1), \cdots, N(n)$ we have

$$\begin{aligned} X_T &= G^{ij}\langle X, X_i\rangle X_j , \\ X_N &= -\sum_\alpha h(\alpha)N(\alpha) . \end{aligned} \tag{6}$$

If ξ is nondegenerate and parallel, then from (1) we see that

$$X_T = -g^{ij}(\xi)h_i(\xi)\xi_j \tag{7}$$

where $g^{ij}(\xi)$ are the elements of $(g_{ij}(\xi))^{-1}$, and $h_i(\xi) = \partial_i h(\xi)$.

Put

$$h_{ij}(\xi) = \partial_{ij}h(\xi),\ \nabla_{ij}h(\xi) = h_{ij}(\xi) - \Gamma^k_{ij}(\xi)h_k(\xi)\ .$$

Under the above assumptions on ξ we obtain with the use of (4)

$$b_{ij}(\xi) = \nabla_{ij}h(\xi) + \sum_\alpha g_{ij}(\xi, N(\alpha))h(\alpha)\ . \tag{8}$$

3. The elementary symmetric functions of principal radii of curvature and the associated differential equations. Let ξ be a unit normal vector at a point $x \in M$. The principal radii of curvature associated with ξ are denoted by $R_{\xi 1}, \cdots, R_{\xi m}$ and defined as the roots of the determinantal equation

$$\det(b_{ij}(\xi) - Rg_{ij}(\xi)) = 0\ .$$

If ξ is a restriction to x of a nondegenerate vector field, then $III(\xi)$ is positive definite, and in this case the $R_{\xi i}$ are well defined. Moreover, in this case they do not vanish. Let $g(\xi) \equiv \det(g_{ij}(\xi))$. The elementary symmetric function of order k in $R_{\xi i}$ (nonnormed)

$$S_{\xi k}(R) = \sum_{i_l \neq i_r} R_{\xi i_1} \cdots R_{\xi i_k}\ ,$$

and it is the coefficient at $(-R)^{m-k}$ of the polynomial

$$\frac{\det(b_{ij}(\xi) - Rg_{ij}(\xi))}{g(\xi)} = (-R)^m + S_{\xi 1}(R)(-R)^{m-1} + \cdots + S_{\xi m}(R)\ . \tag{9}$$

Set $a_{ij}(\xi) = b_{ij}(\xi) - \lambda g_{ij}(\xi)$, where λ is real. Consider a polynomial in λ defined by the equation

$$a^{ij}(\xi) = \sum_{k=1}^{m} (-\lambda)^{m-k} S^{ij}_{\xi k}\ , \tag{10}$$

where $a^{ij}(\xi)$ is the cofactor of the element $a_{ij}(\xi)$.

PROPOSITION 3.1. *Let M be a submanifold of E and ξ is a parallel unit normal vector field defined in a neighborhood of $x \in M$ and such that $II(\xi) > 0$ at x. Then the quadratic forms $S^{ij}_{\xi k}\nu_i\nu_j$, $k=2, \cdots, m$, are positive definite at x. Here $\nu_1, \cdots, \nu_m$ are arbitrary real parameters, $\nu^2 = \nu_1^2 + \nu_2^2 + \cdots + \nu_m^2 \neq 0$. If M is compact, ξ is defined on M, parallel, and $II(\xi) \neq 0$ everywhere on M, then those quadratic forms are definite everywhere and by selecting a proper orientation of M and E, they can be made positive definite. When $k = 1$ this* *assertion is true under the only assumption that ξ is parallel and nondegenerate.*

The proof of this Proposition is standard and we omit it here.

Suppose now that ξ is a nondegenerate parallel vector field. Then in view of (8), (9), and (10) we put

$$P_{\xi k}(h) \equiv \frac{1}{g(\xi)} S^{ij}_{\xi k} \nabla_{ij} h(\xi) ,$$

$$Q_{\xi k} = \frac{1}{g(\xi)} \sum_{\alpha} S^{ij}_{\xi k} g_{ij}(\xi, N(\alpha)) h(\alpha) ,$$

$$M_{\xi k}(h) \equiv P_{\xi k}(h) + Q_{\xi k} . \tag{11}$$

It is not difficult to see that

$$M_{\xi k}(h) = k S_{\xi k}(R) . \tag{12}$$

PROPOSITION 3.2. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field defined in a neighborhood of $x \in M$. Then*

$$Q_{\xi k} = (m - k + 1) S_{\xi k-1}(R) h(\xi) + \langle H_{\xi k}, X \rangle , \quad (S_{\xi 0} \equiv 1) , \tag{13}$$

where $H_{\xi k}$ is a uniquely defined vector in $N'_x(M) = N_x(M) \ominus \xi$ independent on the choice of basis in $N'_x(M)$. If $k = 1$, then $-H_{\xi 1}$ is the m times mean curvature vector of the submanifold $\gamma_\xi(M) \subset \Sigma$.

Proof. Since $\xi \in N_x(M)$, we can select an orthonormal basis in $N_x(M)$ so that ξ is one of the vectors in this basis. Let us preserve the old notation for the new basis, and let $\xi = N(1)$. Then

$$\begin{aligned} Q_{\xi k} &= \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi) h(\xi) + \sum_{2 \leq \alpha \leq n} \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, N(\alpha)) h(\alpha) \\ &= (m - k + 1) S_{\xi k-1}(R) h(\xi) - \left\langle \sum_{2 \leq \alpha \leq n} \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, N(\alpha)) N(\alpha), X \right\rangle . \end{aligned}$$

The form $-(S^{ij}_{\xi k}/g(\xi)) g_{ij}(\xi, \eta)$, where $\eta \in N'_x(M)$, is linear in η. Therefore, there exists a unique element $H_{\xi k}$ in $N'_x(M)$ such that

$$-\frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, \eta) = \langle H_{\xi k}, \eta \rangle$$

for any $\eta \in N'_x(M)$. (Strictly speaking, the inner product in the last formula should be taken in $N'_x(M)$. But it is induced in $N'_x(M)$ from E, and, therefore, it is the same in either sense.) Thus, we conclude that

$$-\left\langle \sum_{2\le\alpha\le n} \frac{S^{ij}_{\xi k}}{g(\xi)} g_{ij}(\xi, N(\alpha))N(\alpha), X \right\rangle = \langle H_{\xi k}, X\rangle .$$

The rest of the Proposition follows from Proposition 2.1 and the fact that

$$\frac{S^{ij}_{\xi 1}}{g(\xi)} = g^{ij}(\xi) .$$

This completes the proof.

COROLLARY 3.1. *Let M be a submanifold of E and ξ a nondegenerate parallel vector field on M. Then* (13) *holds everywhere on M and*

$$M_{\xi k} = P_{\xi k}(h) + (m - k + 1)S_{\xi k-1}(R)h(\xi) + \langle H_{\xi k}, X\rangle \quad \text{for all } x \in M . \tag{14}$$

REMARK 1. The functions $\langle H_{\xi k}, X\rangle$ are similar to the functions $F_k(\xi)$ constructed in [4]. However, the latter are related to principal curvatures and depend on the first and the second fundamental forms, while $\langle H_{\xi k}, X\rangle$ depend on the second and the third fundamental forms in the direction ξ. It is not difficult to point out situations where $H_{\xi k}$ or $\langle H_{\xi k}, X\rangle$ vanish. For example, if $\dim E - \dim M = 1$, then $H_{\xi k} \equiv 0$ for all k. Another example is when the normal component of X has the direction ξ. Then $h(\alpha) = -\langle X, N(\alpha)\rangle \equiv 0$ for $\alpha = 2, \cdots, m$. In these examples the functions $F_k(\xi)$ introduced in [4] also vanish. One more example is given by the case where $III(\xi, N(\alpha)) \equiv 0$ for $\alpha = 2, \cdots, m$, $(N(1) = \xi)$.

REMARK 2. Let M be a submanifold of E and ξ a nondegenerate parallel normal field on M. Let f and f' be two smooth functions defined on M. Put

$$b^f_{ij}(\xi) = \nabla_{ij} f - \sum_\alpha g_{ij}(\xi, N(\alpha))\langle X, N(\alpha)\rangle .$$

Similarly to (9), (10), construct $S^{ij}_{\xi k}(f)$ and consider

$$M_{\xi k}(f, f') \equiv \frac{1}{g(\xi)} S^{ij}_{\xi k}(f)\nabla_{ij} f' + \frac{(m-k+1)}{k-1} M_{\xi k-1}(f, f)f' + \langle H_{\xi k}(f), X\rangle , \quad \text{for } k > 1 ,$$

$$M_{\xi 1}(f, f') \equiv g^{ij}(\xi)\nabla_{ij} f' + mf' + \langle H_{\xi 1}(f), X\rangle , \quad \text{for } k = 1 .$$

These differential operators proved to be useful in the study of uniqueness Theorems for convex hypersurfaces in Euclidean space [7]. (In this case they are elliptic, and the last term in the right-hand side vanishes.) It is plausible that they have applications in establishing uniqueness Theorems for submanifolds of E in codimension >1. We hope that we come back to it again elsewhere.

4. **Applications.** We begin with a slight generalization of the formula (14), which leads to an integral formula relating the elementary symmetric functions of arbitrary order. This formula is of Minkowski-Hsiung type, and in the form involving two consecutive elementary symmetric functions of principal curvatures it was derived and studied by many authors (see Chen and Yano [4], and also [3], Chapter 6; in both sources further references can be found). However, the methods of those authors do not seem to generalize so as to obtain the following formulas (16) and (17).

In what follows, unless stated otherwise, it is assumed that M is a compact submanifold without boundary.

The following Lemma is a version of E. Hopf's Lemma on Laplace-Beltrami operator.

LEMMA. *Suppose that M is a submanifold of E, ξ is a nondegenerate parallel normal vector field on M, and h' is a smooth function on M. Put*

$$P_{\xi k}(h') \equiv \frac{1}{g(\xi)} S^{ij}_{\xi k} \nabla_{ij} h' ,$$

where the coefficients $S^{ij}_{\xi k}$ are the same as in (11). *Then*

$$P_{\xi k}(h') = \frac{1}{\sqrt{g(\xi)}} \partial_i \left(\frac{S^{ij}_{\xi k}}{\sqrt{g(\xi)}} \partial_j h \right) . \tag{15}$$

If $k = 1$ and, in addition, we assume that $P_{\xi k}(h')$ does not change its sign on M, then h' is a constant function on M. The same is true when $k > 1$ provided there exists at least one point on M where $II(\xi) \neq 0$.

Proof. It is easy to see, with the use of formula (4), that $b_{ij}(\xi)$ is a Codazzi tensor with respect to $\Gamma^k_{ij}(\xi)$. Therefore, $P_{\xi k}(h')$ can be written in the divergence form (15) (see [5, 7]). When $k > 1$ and $II(\xi) \neq 0$ at some point of M then $II(\xi) \neq 0$ everywhere on M because ξ is nondegenerate. By Proposition 3.1 the operator $P_{\xi k}(h')$ is uniformly elliptic. Now the rest of the proof runs similarly to the standard proof of E. Hopf's Lemma on the Laplace-Beltrami operator on a compact Riemannian submanifold ([6], p. 338). The Lemma is proved.

THEOREM 4.1. *Let M be a submanifold of E and ξ a nonde-*

generate parallel normal vector field on M. Then for arbitrary k and s, k = 1, ⋯, m, s = 1, ⋯, k,

$$kS_{\xi k}(R) = \sum_{l=0}^{s-1} \frac{(m-k+l)!\,(k-l-1)!}{(m-k)!\,(k-1)!}[P_{\xi k-l}(h) + \langle H_{\xi k-l}, X\rangle]h^l + \frac{(m-k+s)!\,(k-s)!}{(m-k)!\,(k-1)!} S_{\xi k-s}(R)h^s\,, \tag{16}$$

and

$$\begin{aligned} k\int_M S_{\xi k}(R)dO(\xi) &= -\sum_{l=1}^{s-1} l\cdot \frac{(m-k+l)!\,(k-l-1)!}{(m-k)!\,(k-1)!}\int_M h^{l-1}\frac{S^{ij}_{\xi k-l}}{g(\xi)} h_i h_j dO(\xi) \\ &\quad + \sum_{l=0}^{s-1} \frac{(m-k+l)!\,(k-l-1)!}{(m-k)!\,(k-1)!}\int_M \langle H_{\xi k-l}, X\rangle h^l dO(\xi) \\ &\quad + \frac{(m-k+s)!\,(k-s)!}{(m-k)!\,(k-1)!}\int_M S_{\xi k-s}(R)h^s dO(\xi)\,, \end{aligned} \tag{17}$$

where $h \equiv h(\xi)$ is the support function of M with respect to ξ.

Proof. Formula (16) follows from the formulas (12) and (14); and (17) is obtained from (16) by integrating, applying Green's formula, and the preceding Lemma.

COROLLARY 4.1. *If in Theorem 4.1 $s = k$, then*

$$k\int_M S_{\xi k}(R)dO(\xi) = (m-k+1)\int_M S_{\xi k-1}(R)h(\xi)dO(\xi) + \int_M \langle H_{\xi k}, X\rangle dO(\xi)\,. \tag{18}$$

This formula is an analogue of an integral formula due to Chen and Yano [4].

We recall that if a submanifold M (not necessarily compact) of E is contained in a hypersphere of E centered at the origin, then it is called a spherical submanifold (see [2]).

In the following we often make use of a Theorem due to Chen [2].

THEOREM A. *Let M be a submanifold (not necessarily compact) of E. If there exists a nondegenerate parallel normal vector field ξ such that $h(\xi) =$ const everywhere on M, then M is a spherical submanifold of E.*

From now on always when $k > 1$ it is assumed that there exists a point on M where $II(\xi) \neq 0$, and the orientation is such that $II(\xi) > 0$.

Examples of submanifolds with this property can be constructed as follows. Let M_1 and M_2 be two strictly convex hypersurfaces. Then the natural imbedding of $M_1 \times M_2$ in Euclidean space of dimension $= \dim M_1 + \dim M_2 + 2$ gives such example.

The next Theorem is an immediate consequence of formulas (12), (14), the Lemma, and Theorem A.

THEOREM 4.2. *Let M be a submanifold in E and ξ is a nondegenerate parallel normal vector field on M. Assume further that for some k, $k = 1, 2, \cdots, m$, at every point of M*

$$cS_{\xi k}(R) = S_{\xi k-1}(R)h(\xi) \qquad (S_{\xi 0} \equiv 1) , \tag{19}$$

where c is a constant such that the expression

$$[k - c(m - k + 1)]S_{\xi k}(R) - \langle H_{\xi k}, X\rangle$$

is either nonnegative or nonpositive. Then M is a spherical submanifold.

Proof. In the formula (16) set $s = 1$. Then by (19)

$$[k - c(m - k + 1)]S_{\xi k}(R) - \langle H_{\xi k}, X\rangle = P_{\xi k}(h) ,$$

and the Theorem follows from the Lemma and Theorem A.

In case $k = 1$ a result similar to this Theorem has been given by Wegner [9], Satz 2. His result can be also obtained by our method, and furthermore, it can be generalized for $k > 1$.

Let $M = S^m$, where S^m is a standard m-sphere lying in $m + 1$-dimensional Euclidean space $E^{m+1} \subset E$. Then, evidently, $H_{\xi k} \equiv 0$ for all k, and ξ is the unit normal vector field on S^m in E^{m+1}. With this fact in mind we state the following

COROLLARY 4.2. *Let M be a submanifold of E and ξ a nondegenerate parallel normal vector field on M. If for some k, $k = 2, \cdots, m$, at every point of M*

$$\langle H_{\xi k}, X\rangle = 0 ,$$

and

$$cS_{\xi k}(R) = S_{\xi k-1}(R)h(\xi) ,$$

where c is a constant $\neq 0$, then M is a spherical submanifold. Furthermore, in this case it is necessary that $c = k/(m - k + 1)$. The *assertion is also true when* $k = 1$, *provided* $II(\xi) \neq 0$ *at some point of* M.

Proof. We show at first that the function $S_{\xi k}(R)$ does not change its sign on M. Let A be a point on M where $II(\xi)$ is definite. Then the principal radii of curvature $R_{\xi i}$, $i = 1, \cdots, n$, must all be of the same sign at A. Since ξ is nondegenerate $R_{\xi i}$ will all have the same sign everywhere on M. Hence, the function $S_{\xi k}(R)$ can not change its sign on M, and moreover it does not vanish on M.

Now it is clear that the expression

$$[k - c(m - k + 1)]S_{\xi k}(R)$$

is either nonnegative or nonpositive and therefore by Theorem 4.2. M is a spherical submanifold. On the other hand,

$$\int_M S_{\xi k}(R) dO(\xi) \neq 0 \ ;$$

hence, the formula (18) implies that $c = k/(m - k + 1)$. The Corollary is proved.

A Theorem similar to Theorem 4.2 can be stated with the use of Theorem 4.1.

We point out only a particular case of it.

THEOREM 4.3. *Let* M *be a submanifold in* E *and* ξ *a nondegenerate parallel normal vector field on* M. *Suppose that for some* k *and* s, $k = 1, \cdots, m$, $s = 1, \cdots, k$, *the following conditions are satisfied*:

(a) $$kS_{\xi k}(R) \geqq \frac{(m - k + s)!\ (k - s)!}{(m - k)!\ (k - 1)!} S_{\xi k-s}(R) h^s(\xi) \ ;$$

(b) $$\int_M \langle H_{\xi k-l}, X \rangle h^l(\xi) dO(\xi) \leqq 0 \quad \textit{for} \quad l = 0, \cdots, s - 1 \ ;$$

(c) $$h(\xi) > 0 \ .$$

Then M *is a spherical submanifold.*

Proof. The conditions (a), (b), (c) and Proposition 3.1 imply that all integrals in formula (17) must vanish. Hence $h(\xi) = \text{const}$, that is, M is a spherical submanifold.

THEOREM 4.4. *Let* M *be a submanifold in* E *and* ξ *a nondegenerate parallel normal vector field on* M. *Suppose that for some* k *and* s, $k = 1, \cdots, m$, $s = 1, \cdots, k$, *the following conditions are* *satisfied*:

(a) $cS_{\xi k}(R) = S_{\xi k-s}(R)h^s(\xi)$ *everywhere on* M, *where* c *is a constant* $\neq 0$;

(b) $\langle H_{\xi k-l}, X\rangle = 0$ *for* $l = 0, \cdots, s-1$;

(c) $h(\xi) > 0$.

In case $k = 1$ *assume also that* $II(\xi) \neq 0$ *at some point of* M.

Then M *is a spherical submanifold and* $c=(m-k)!k!/(m-k+s)!(k-s)!$.

Proof. At first we show that c can have only the value indicated in the assertion. In showing that we follow Blaschke [1], p. 233. Let A be a point on M where $h(\xi)$ $(=h)$ attains its maximum. Then at A,

$$\nabla_{ij}h \leqq 0 .$$

By Proposition 3.1 the forms $S^{ij}_{\xi k}\nu_i\nu_j$, $k = 1, \cdots, m$, are definite, and therefore at the point A the expressions

$$P_{\xi k-l}(h) = \frac{1}{g(\xi)} S^{ij}_{\xi k-l}\nabla_{ij}h \quad l = 0, 1, \cdots, k-1 ,$$

are all of the same sign, and namely nonpositive. On the other hand, by Theorem 4.1 (formula (16)) in view of the conditions (a) and (b), we obtain

$$\left[k - c\frac{(m-k+s)!\ (k-s)!}{(m-k)!\ (k-1)!}\right]S_{\xi k}(R) = \sum_{l=0}^{s-1}\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!} h^l P_{\xi k-l}(h) .$$

The right-hand side is nonpositive at A, and similar to the proof of Corollary 4.2 one shows that $S_{\xi k}(R) > 0$ everywhere on M. Therefore,

$$k - c\frac{(m-k+s)!\ (k-s)!}{(m-k)!\ (k-1)!} \leqq 0 .$$

Considering the point where h attains its minimum we arrive at the opposite inequality. Thus, $c = (m-k)!\ k!/(m-k+s)!\ (k-s)!$.

Now, making use of the second part of Theorem 4.1 (formula 17)) and the conditions (a), (b), (c) with constant c taken as above, we obtain

$$-\sum_{l=1}^{s-1}\frac{(m-k+l)!\ (k-l-1)!}{(m-k)!\ (k-1)!}\int_M h^{l-1}\frac{S^{ij}_{\xi k-l}}{g(\xi)} h_i h_j dO(\xi) = 0 .$$

From here, it follows that $h = \text{const}$. Hence, M is a spherical submanifold. The Theorem is proved.

COROLLARY 4.3. *Let M be a closed strictly convex hypersurface in Euclidean space E and ξ is the unit normal vector field on M. Suppose that for some k and s, $k = 1, \cdots, m$, $s = 1, \cdots, k$,*

$$cS_k(R) = S_{k-s}(R)h^s(\xi)$$

everywhere on M, where c is a constant$\neq 0$. Then M is a hypersphere, and c is as in Theorem 4.4. (In the last equality the subscript ξ is omitted for the obvious reason.)

Proof. For a hypersurface in E, ξ is always parallel, and since M is strictly convex, ξ is nondegenerate. Also $H_{\xi l} \equiv 0$ for $l = 1, \cdots, k$. The support function $h(\xi)$ can always be made strictly positive by placing the origin of the coordinate system in E inside M. Now the Corollary follows from Theorem 4.4.

REMARK 1. As was mentioned in the introduction, this Corollary is known. In particular, the condition quoted earlier can be expressed in terms of the elementary symmetric functions of principal radii of curvature as follows:

$$cS_m(R) = S_{m-s}(R)h^s(\xi) .$$

If in Corollary 4.3 we take $k = m$, then we obtain the above result. It is due to Süss; see [8], Korollar 6.3, and other references there.

REMARK 2. Theorem 4.4 does not contain Corollary 4.2, since in the latter it is not required that $h(\xi) > 0$.

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