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ANNIHILATION OF IDEALS IN COMMUTATIVE RINGS

JAMES A. HUCKABA AND JAMES M. KELLER

Four theorem are proved concerning the annihilation of finitely generated ideals contained in the set of zero divisors of a commutative ring.

1. Introduction. An important theorem in commutative ring theory is that if I is an ideal in a Noetherian ring and if I consists entirely of zero divisors, then the annihilator of I is nonzero. This result fails for some non-Noetherian rings, even if the ideal I is finitely generated. We say that a commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero divisors has nonzero annihilator. Property (A) was originally studied by Y. Quentel in [7]. (Our Property (A) is Quentel's Condition (C).) Theorem 1 shows that all nontrivial graded rings have Property (A). (For our purposes a *nontrivial graded ring* is a ring R graded over the integers such that R contains an element x, not a zero divisor, of positive homogenous degree.) Theorem 2 completely characterizes those reduced rings with Property (A).

Property (A) is closely connected with two other conditions on a reduced ring. One is the *annihilator condition* (a.c.): If (a, b) is an ideal of R, then there exists $c \in R$ such that $\mathrm{Ann}(a, b) = \mathrm{Ann}(c)$. The other condition is that $\mathrm{MIN}(R)$, the space of minimal prime ideals of R, is compact. Our Theorem 3 shows that for a reduced coherent ring R Property (A), (a.c.), and the total quotient ring of R being a von Neumann regular ring are equivalent conditions; and that each (and hence all) of these conditions imply that $\mathrm{MIN}(R)$ is compact. Finally, in Theorem 4, we prove that every reduced nontrivial graded ring satisfies (a.c.).

We assume that all rings are commutative with identity. If R is such a ring, let $T(R)$ be the total quotient ring of R, let $Z(R)$ be the set of zero divisors of R, and let $Q(R)$ denote the complete ring of quotients of R as defined in [5]. Elements of R that are not zero divisors are called *regular elements*.

2. Graded rings.. Y. Quentel, [7, p. 269], proved that if R is a reduced ring, then the polynomial ring $R[X]$ satisfies Property (A). We generalize this to arbitrary nontrivial graded rings, and hence to polynomial rings that are not necessarily reduced.

THEOREM 1. *If R is nontrivial graded ring, then R satisfies Property* (A).

Proof. Let $I = (a_1, \cdots, a_p)$ be an ideal of R contained in $Z(R)$. For $i = 1, \cdots, p$, let $a_i = \sum_{k=m_i}^{n_i} b_k^{(i)}$ be the homogeneous decomposition of a_i, where $\deg b_k^{(i)} = k$. Let x be a regular homogeneous element in R of degree $t > 0$. Construct an element a as follows:

$$a = a_1 + a_2x^{s_2} + \cdots + a_px^{s_p} ,$$

where the s_i are integers such that $ts_2 + m_2 > n_1$, and $ts_i + m_i > n_{i-1} + ts_{i-1}$; $i = 3, \cdots, p$. There exists a nonzero homogeneous element c such that $ca = 0$. (The proof of this is identical to the proof of McCoy's Theorem: If f is a zero divisor in $R[X]$, then there is a nonzero $b \in R$ such that $bf = 0$.)

Since $deg[b_k^{(i)}x^{s_i}] \neq \deg[b_h^{(j)}x^{s_j}]$ unless $i = j$ and $k = h$, the homogeneous compontets of a are $\{b_k^{(i)}x^{s_i}\}_{i=1,\cdots,p}^{k=m_i,\cdots,n_i}$. Thus, by the unique representation in terms of the homogeneous components $cb_k^{(i)}x^{s_i} = 0$ for all i, k. Since $x \notin Z(R)$, $cb_k^{(i)} = 0$ for all i, k. Therefore, $c \in \operatorname{Ann}(I)$.

COROLLARY 1. *If R is any ring, then the polynomial ring $R[X]$ satisfies Property* (A).

3. **Reduced rings.** In this section all rings are assumed to be reduced.

THEOREM 2. *For a reduced ring R, the following statements are equivalent*:

(1) *R has Property* (A);

(2) *$T(R)$ has property* (A);

(3) *If I is a finitely generated ideal of R contained in $Z(R)$, then I is contained in a minimal prime ideal of R;*

(4) *Every finitely generated ideal of R contained in $Z(R)$, extends to a proper ideal in $Q(R)$.*

Proof. (1) $\leftrightarrow$ (2) is clear.

(1) $\rightarrow$ (3): Assume that I is a finitely generated ideal contained in $Z(R)$, but not contained in a minimal prime ideal of R. Then $cI = 0$ implies that c is in every minimal prime ideal of R; i.e., $c=0$.

(3) $\rightarrow$ (1): Let $I = (x_1, \cdots, x_n) \subset P$, P a minimal prime ideal of R. By [2, p. 111], choose $z_i \in \operatorname{Ann}(x_i)$, $z_i \notin P$. Then $z = z_1z_2 \cdots z_n \neq 0$ and $z \in \bigcap_{i=1}^n \operatorname{Ann}(x_i) = \operatorname{Ann}(I)$.

(1) $\rightarrow$ (4): If I is a finitely generated ideal contained in $Z(R)$, then $IQ(R)$ has nonzero annihilator in $Q(R)$. Hence, $IQ(R) \subsetneqq Q(R)$.
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(4) $\rightarrow$ (1): Assume that I is a finitely generated dense ideal of R such that $I \subset Z(R)$. (A subgroup H of a ring R is *dense*, if

$\operatorname{Ann} H = 0$.) Then I is dense in $Q(R)$, [5, p. 41], and whence $IQ(R)$ is dense in $Q(R)$. But $Q(R)$ is a von Neumann regular ring, [5, p. 42]; and von Neumann regular rings have Property (A), [3, p. 30]. By the equivalence of (1) and (3) of this theorem, $IQ(R)$ is not contained in any minimal prime ideal of $Q(R)$. But in $Q(R)$, minimal prime ideals are maximal. Therefore, $IQ(R) = Q(R)$, a contradiction.

The results about the compactness of $\mathrm{MIN}(R)$ that we need are summarized in Theorems A and B.

THEOREM A. *The following conditions on a reduced ring R are equivalent*:

(1) *$Q(R)$ is a flat R-module*;

(2) *$\mathrm{MIN}(R)$ is compact*;

(3) $\{M \cap R: M \in \operatorname{Spec} Q(R)\} = \mathrm{MIN}(R)$;

(4) *If $a \in R$ and if $U = \{M \in \operatorname{Spec} Q(R): a \notin M \cap R\}$, then there exists a finitely generated ideal I such that*

$$\operatorname{Spec} Q(R) \backslash U = \{M \in \operatorname{Spec} Q(R): I \not\subset M \cap R\}\ ;$$

(5) *If X is an indeterminate, then $T(R[X])$ is a von Neumann regular ring.*

Proof. A. C. Mewburn, in [6], proved the equivalence of (1) through (4). Quentel proved that (2) and (5) are equivalent, [7].

THEOREM B. *The following conditions on a reduced ring R are equivalent*:

(1) *$T(R)$ is a von Neumann regular ring*;

(2) *R satisfies Property* (A) *and* $\mathrm{MIN}(R)$ *is compact*;

(3) *R satisfies* (a.c.) and $\mathrm{MIN}(R)$ *is compact.*

Proof. In [7], Quentel proved the equivalence of (1) and (2); while M. Henriksen and M. Jerison, [2], showed that (1) and (3) are the same.

A ring R is *coherent* in case I is a finitely generated ideal of R implies there is an exact sequence $R^m \to R^n \to I \to 0$.

THEOREM 3. *For a reduced coherent ring R, the following conditions are equivalent*:

(1) *R has Property* (A);

(2) *R has* (a.c.);

(3) *$T(R)$ is a von Neumann regular ring.*

Proof. (1) $\to$ (3): In view of Theorem B(2) we must show that

MIN(R) is compact. Let $x \in R$. Since R is a coherent ring, $\text{Ann}(x) = I$ is a finitely generated ideal of R, [1, p. 462]. Let $U = \{M \in \text{Spec}\, Q(R)$: $x \notin M \cap R\}$. Assume that $I \subset M \cap R$ for some $M \in \text{Spec}\, Q(R) \backslash U$, then the ideal $(I, x) \subset M \cap R$. It is clear that $M \cap T(R)$ is a proper ideal of $T(R)$ and that $M \cap R = M \cap T(R) \cap R$. Hence, $(I, x) \subset M \cap R \subset Z(R)$. By Property (A), $\text{Ann}(I, x) \neq 0$. But this contradicts the fact that the ideal $(I, x) = xR + \text{Ann}(x)$ is dense, [5, p. 42]. By Theorem A(4), MIN(R) is compact.

(2) $\rightarrow$ (3): Let $x \in R$, then $\text{Ann}(x) = (z_1, \cdots, z_n)$ and $\text{Ann}\{\text{Ann}(x)\} = \text{Ann}(z_1, \cdots, z_n) = \text{Ann}(z)$. This last condition, given in [2], implies that MIN(R) is compact (even if R does not have a unit).

(3) $\rightarrow$ (1) and (3) $\rightarrow$ (2) are clear.

COROLLARY 2. *Let R be a reduced coherent ring.*

(1) *If R satisfies any (and hence all) of the conditions of Theorem* 3, *the* MIN(R) *is compact.*

(2) *If R is a nontrivial graded ring, then $T(R)$ is a von Neumann regular ring.*

THEOREM 4. *If R is a reduced nontrivial graded ring, then R satisfies* (a.c.).

Proof. Let (a, b) be an ideal in R. If $(a, b) \not\subset Z(R)$, then $\text{Ann}(a, b) = \text{Ann}(1)$. Assume that $(a, b) \subset Z(R)$, and write a and b in terms of their homogeneous components; say, $a = a_m + \cdots + a_n$ and $b = b_h + \cdots + b_k$. Let x be a homogeneous element of R, $x \notin Z(R)$, of degree $t > 0$. Choose an integer s satisfying $h + st > n$ and let $c = a_m + \cdots + a_n + b_h x^s + \cdots + b_k x^s$.

Since R in a reduced, $\text{Ann}(c) = \cap P$, where P varies over the minimal prime ideals of R not containing c. By Lemma 3 of [8, p. 153], each P is a homogeneous ideal. Hence, $\cap P = \text{Ann}(c)$ is also homogeneous.

Let d be a homogeneous element in $\text{Ann}(c)$. Then $da_i = 0$ and $db_j x^s = 0$ for all i, j. Then, $da = 0 = db$ and we have $\text{Ann}(c) \subset \text{Ann}(a, b)$. The other inclusion is obvious.

Let R be a graded ring which contains a regular homogeneous element. Define $T_q = \{a/b$: a and b are homogeneous, b is regular, and $q = \text{degree}\ a - \text{degree}\ b\}$. Just as in the integral domain case, [8, p. 157], ΣT_q is a graded ring containing R as a graded subring.

COROLLARY 3. *Let R be a reduced nontrivial graded ring. The following statements are equivalent*:

(1) MIN(R) *is compact*;

(2) $\mathrm{MIN}(T_0)$ *is compact*;

(3) $T(R)$ *is a von Neumann regular ring.*

Proof. (1) $\leftrightarrow$ (3) by Theorem B.

(1) $\leftrightarrow$ (2): If S is the set of regular homogeneous elements of R, then $R_S = \Sigma T_q$ and $\mathrm{MIN}(R)$ is homeomorphic to $\mathrm{MIN}(R_S)$. By [4, Lemma 1], there is a one-to-one order preserving correspondence between the graded prime ideals of R_S and the graded prime ideals of T_0. It follows from [8, p. 153] that the minimal prime ideals of a graded ring are graded. Thus, $\mathrm{MIN}(R_S)$ is homeomorphic to $\mathrm{MIN}(T_0)$.

REMARKS. (1) $\mathrm{MIN}(R)$ compact $\nrightarrow$ Property A or (a.c.). This follows from an example in [6]. (2) Property (A) $\nrightarrow$ $\mathrm{MIN}(R)$ compact. By [6. p. 427], there is a ring R for which $\mathrm{MIN}(R)$ is not compact. Applying Theorem B(5), $T(R[X])$ is not von Neumann regular. But $R[X]$ has Property (A), [7, p. 269]. Thus, $\mathrm{MIN}(R[X])$ cannot be compact.

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UNIVERSITY OF MISSOURI
COLUMBIA, MO 65211