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ON THE NONOSCILLATION OF PERTURBED FUNCTIONAL DIFFERENTIAL EQUATIONS

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We study the behavior of the solutions of the second order nonlinear functional differential equation

$$(a(t)x')' = f(t, x(t), x(g(t))) \tag{1}$$

where a, g: $[t_0, \infty) \to R$ and f: $[t_0, \infty) \times R^2 \to R$ are continuous, $a(t) > 0$, and $g(t) \to \infty$ as $t \to \infty$. We are primarily interested in obtaining conditions which ensure that certain types of solutions of (1) are nonoscillatory. Conditions which guarantee that oscillatory solutions of (1) converge to zero as $t \to \infty$ are also given. We apply these results to the equation

$$(a(t)x')' + q(t)r(x(g(t))) = e(t, x) \tag{2}$$

where q: $[t_0, \infty) \to R$, r: $R \to R$, e: $[t_0, \infty) \times R \to R$ are continuous and a and g are as above. We compare our results to those obtained by others. Specific examples are included.

In the case of nonlinear ordinary equations, the search for sufficient conditions for all solutions to be nonoscillatory has been successful; see, for example, the papers of Graef and Spikes [4-7], Singh [11], Staikos and Philos [14], and the references contained therein. The only such results known for functional equations to date are due to Graef [3], Kusano and Onose [9], and Singh [13]. Moreover, none of the results in [3], [9], or [13] apply to equation (2) if $e(t, x) \equiv 0$ or if r is superlinear, e.g., $r(x) = x^{\gamma}$, $\gamma > 1$. We refer the reader to the recent paper of Kartsatos [8] for a survey of known results on the oscillatory and asymptotic behavior of solutions of (1) and (2).

In view of a recent paper by Brands [1], it does not appear to be possible to obtain integral conditions on $q(t)$ which will guarantee that all solutions of (2) with $e(t, x) \equiv 0$ are nonoscillatory and which are similar to those usually encountered in the study of ordinary equations. (We will return to this point again in §2.) So too our main results in this direction when applied to equation (2) require that $e(t, x) \not\equiv 0$ (cf. conditions (27) and (28)). Although all the results presented here hold if $r(x)$ is sublinear, we are especially interested in the superlinear case.

2. Main results. The results in this paper pertain only to the continuable solutions of (1). A solution $x(t)$ of (1) will be called oscillatory if its set of zeros is unbounded, and it will be called nonoscillatory otherwise. Some of the results which follow concern solutions of (1) which satisfy growth estimates of the form

$$|x(t)| = O(m(t)) \quad \text{as} \quad t \longrightarrow \infty\ , \tag{3}$$

where $m\colon [t_0, \infty) \to R$ is continuous and positive. Other authors, for example Staikos and Sficas [15], have studied the asymptotic behavior of nonoscillatory solutions which satisfy estimates of this type with $m(t) = t^k$.

We will assume in the remainder of this paper that the function f satisfies an estimate of the form

$$|f(t, x, y)| \leqq F(t, |x|, |y|) \tag{4}$$

where $F\colon [t_0, \infty) \times R_+^2 \to R_+$ is continuous and such that

$$F(t, u, v) \leqq F(t, u', v') \quad \text{for} \quad 0 \leqq u \leqq u', 0 \leqq v \leqq v'\ .$$

THEOREM 1. *Suppose that*

$$\int^\infty [1/a(s)] \int_s^\infty F(u, cm(u), cm(g(u)))duds < \infty \tag{5}$$

for all $c > 0$. *If* $x(t)$ *is an oscillatory solution of* (1) *satisfying* (3), *then* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let $x(t)$ be an oscillatory solution of (1) satisfying (3); then $|x(t)| \leqq cm(t)$, $|x(g(t))| \leqq cm(g(t))$ for all $t \geqq t_1 \geqq t_0$ and some $c > 0$. Suppose that $\limsup_{t\to\infty} |x(t)| > 2M$ for some $M > 0$. Then there exist sequences $\{a_n\}$ and $\{b_n\}$ of zeros of $x(t)$ such that $a_n < b_n$, $a_n, b_n \to \infty$ as $n \to \infty$, $|x(t)| > 0$ on (a_n, b_n), and $M_n = \max\{|x(t)|\colon a_n \leqq t \leqq b_n\} > M$ for $n = 1, 2, \cdots$. Now choose t_n in (a_n, b_n) so that $|x(t_n)| = M_n$ for $n = 1, 2, \cdots$. Integrating equation (1) from t in $[a_n, t_n]$ to t_n, we have

$$a(t)x'(t) = -\int_t^{t_n} f(s, x(s), x(g(s)))ds\ .$$

A second integration yields

$$x(t_n) = -\int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} f(u, x(u), x(g(u)))duds\ .$$

Thus

$$M_n = |x(t_n)| \leqq \int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} F(u, cm(u), cm(g(u)))duds\ .$$

Condition (5) implies that the ri ghthand side of the above inequality converges to zero as $n \to \infty$. This contradicts $|x(t_n)| = M_n > M$ for $n = 1, 2, \cdots$ and completes the proof of the theorem.

The following corollary is an immediate consequence.

COROLLARY 2. *If condition* (5) *holds with* $m(t) \equiv K$ *for every constant* $K > 0$, *then all bounded oscillatory solutions of* (1) *converge to zero as* $t \to \infty$.

In our next theorem the following sublinearity type condition will be used. There exists a continuous function $H: [t_0, \infty) \to R$ such that

$$\limsup_{v \to \infty} F(t, v, v)/v \leqq H(t) . \tag{6}$$

THEOREM 3. *In addition to* (6) *assume that condition* (5) *holds with* $m(t) \equiv K$ *for any constant* $K > 0$,

$$g(t) \leqq t \tag{7}$$

and

$$\int^{\infty} [1/a(s)] \int_s^{\infty} H(u) du ds < \infty . \tag{8}$$

Then every oscillatory solution of (1) *converges to zero as* $t \to \infty$.

Proof. We will first show that all oscillatory solutions are bounded. Suppose that $x(t)$ is an oscillatory solution of (1) and $\limsup_{t \to \infty} |x(t)| = \infty$. Then there exists a sequence of intervals $\{(a_n, b_n)\}$ such that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \infty$, $x(a_n) = x(b_n) = 0$, $|x(t)| > 0$ on (a_n, b_n), and $M_n = \max\{|x(t)|: t \leqq b_n\} = \max\{|x(t)|: a_n \leqq t \leqq b_n\}$ and M_n increases to infinity as $n \to \infty$ with $M_1 \geqq K$. As in the proof of Theorem 1 we obtain

$$M_n = |x(t_n)| \leqq \int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} F(u, M_n, M_n) du ds$$

where $t_n \in (a_n, b_n)$. Hence

$$1 \leqq \int_{a_n}^{t_n} [1/a(s)] \int_s^{t_n} H(u) du ds$$

which contradicts (8) as $n \to \infty$.

Since $x(t)$ is bounded the conclusion of the theorem then follows from Corollary 2.

THEOREM 4. *Suppose that there exist continuous functions* $G: [t_0, \infty) \times R_+^2 \to R_+$ *and* $h: [t_0, \infty) \to R$ *such that*

$$G(t, u, v) \leqq G(t, u', v') \quad \textit{for } 0 \leqq u \leqq u', \quad 0 \leqq v \leqq v', \tag{9}$$

$$|f(t, x, y) - h(t)| \leqq G(t, |x|, |y|) \quad \textit{for} \quad x, y \in R, \tag{10}$$

$$\int^{\infty} [1/a(s)] \int_s^{\infty} |h(u)| duds < \infty, \tag{11}$$

and

$$\int^{\infty} [1/a(s)] \int_s^{\infty} G(u, cm(u), cm(g(u)))duds < \infty \tag{12}$$

for all $c > 0$. *If there exists* $c_0 > 0$ *such that either*

$$\lim_{t\to\infty} \int_T^t [1/a(s)] \int_T^s \{h(u) + G(u, c_0, c_0)\} duds = -\infty \tag{13}$$

or

$$\lim_{t\to\infty} \int_T^t [1/a(s)] \int_T^s \{h(u) - G(u, c_0, c_0)\} duds = +\infty \tag{14}$$

for all large T, *then any solwtion* $x(t)$ *of* (1) *satisfying* (3) *is nonoscillatory.*

Proof. Let $x(t)$ be an oscillatory solution of (1) satisfying (3). In view of (11) and (12) all the hypotheses of Theorem 1 are satisfied with $F(t, u, v) = |h(t)| + G(t, u, v)$ and so $x(t) \to 0$ as $t \to \infty$. Thus there exists $T \geqq t_0$ such that $x'(T) = 0$, $|x(t)| \leqq c_0$, and $|x(g(t))| \leqq c_0$ for $t \geqq T$. Hence

$$h(t) - G(t, c_0, c_0) \leqq f(t, x(t), x(g(t))) \leqq h(t) + G(t, c_0, c_0) \tag{15}$$

for $t \geqq T$. Integrating twice we have

$$\begin{aligned} \int_T^t [1/a(s)] \int_T^s \{h(u) - G(u, c_0, c_0)\} duds &\leqq x(t) - x(T) \\ &\leqq \int_T^t [1/a(s)] \int_T^s \{h(u) + G(u, c_0, c_0)\} duds . \end{aligned}$$

If either (13) or (14) holds, then $x(t)$ cannot have arbitrarily large zeros.

REMARK. An alternate form of Theorem 4 can be obtained by replacing conditions (13) and (14) by

$$\limsup_{t\to\infty} \int_T^t \{h(u) + G(u, c_0, c_0)\} du < 0 \tag{16}$$

and

$$\liminf_{t\to\infty} \int_T^t \{h(u) - G(u, c_0, c_0)\}du > 0 \tag{17}$$

respectively. The proof in this case would follow from inequality (15) by noting that (16) or (17) implies that $x'(t)$ would have fixed sign. Condition (16) or (17) may be satisfied when (13) and (14) are not, for example, when $\int_{t_0}^\infty [1/a(s)]ds < \infty$. Similarly (13) or (14) may hold with neither (16) nor (17) being satisfied when $\int_{t_0}^\infty [1/a(s)]ds = \infty$.

THEOREM 5. *Assume that* (7) *and* (9)-(11) *hold, G is sublinear in the sense of condition* (6), i.e., *there exists* $H_G: [t_0, \infty) \to R$ *such that* $\limsup_{v\to\infty} G(t, v, v)/v \leqq H_G(t)$,

$$\int^\infty [1/a(s)] \int_s^\infty H_G(u)duds < \infty\ , \tag{18}$$

and condition (12) *holds with* $m(t) \equiv K$ *for any constant* $K > 0$. *If either* (13) *or* (14) *holds, then all solutions of* (1) *are nonoscillatory.*

Proof. Let $x(t)$ be an oscillatory solution of (1). If we let $F(t, u, v) = G(t, u, v) + |h(t)|$, then clearly (6) holds and moreover (11) and (18) imply that (8) holds with $H(t) = H_G(t) + |h(t)|$. Hence $x(t) \to 0$ as $t \to \infty$ by Theorem 3. Proceeding exactly as in the proof of Theorem 4 we again obtain a contradiction.

REMARK. Once again an alternate version of Theorem 5 can be obtained by replacing conditions (13)-(14) by (16)-(17).

3. Applications and discussion. We will now apply the results in the previous section to equation (4):

$$(a(t)x')' + q(t)r(x(g(t))) = e(t, x)\ .$$

Assume that

$$|e(t, u)| \leqq |e(t, v)| \quad \text{if} \quad |u| \leqq |v|\ , \tag{19}$$

and there are nonnegative constants A, B and p such that

$$|r(x)| \leqq A|x|^p + B\ . \tag{20}$$

If for some $k \geqq 0$

$$\int_{t_0}^\infty [1/a(s)] \int_s^\infty [g(u)]^{kp}|q(u)|duds < \infty \tag{21}$$

and

$$(22)\qquad \int_{t_0}^{\infty}[1/a(s)]\int_{s}^{\infty}|e(u, cu^k)|duds < \infty \quad \text{for all} \quad c \geqq 0\,,$$

then the hypotheses of Theorem 1 are satisfied with $m(t) = t^k$. Hence any oscillatory solution $x(t)$ of (2) satisfying

$$(23)\qquad |x(t)| = 0(t^k) \quad \text{as} \quad t \longrightarrow \infty\,,$$

will converge to zero as $t \to \infty$. If $k = 0$ in conditions (21) and (22) then we obtain the conclusion of Corollary 2 for equation (2). In this case we obtain Theorem 4 of Kusano and Onose [10] as a special case. They required that $r(x)$ be nondecreasing, $xr(x) > 0$ if $x \neq 0$, and $e(t, x) \equiv e(t)$; moreover if $k = 0$, conditions (13) and (14) of [10] imply conditions (21) and (22) above.

Now assume that there exist $w > 0$ and continuous functions $h_1, h_2\colon [t_0, \infty) \to R$ such that

$$(24)\qquad |e(t, x) - h_1(t)| \leqq h_2(t)|x|^w\,,$$

$$(25)\qquad \int_{t_0}^{\infty}[1/a(s)]\int_{s}^{\infty}|h_1(u)|duds < \infty\,,$$

and

$$(26)\qquad \int_{t_0}^{\infty}[1/a(s)]\int_{s}^{\infty}u^{kw}h_2(u)duds < \infty\,.$$

If (7), (19)-(21) and (24)-(26) hold with $p \leqq 1$, $w \leqq 1$, and $k = 0$, then all oscillatory solutions of (2) converge to zero by Theorem 3. Theorem 5 of [10] is a special case of this result. There the authors show that when $r(x)$ is sublinear, i.e., $\limsup_{|x|\to\infty} r(x)/x < \infty$, then the hypotheses of their Theorem 4 insure that all oscillatory solutions are bounded and hence converge to zero. In so doing they generalized Theorems 1, 2, and 3 of Singh [12] who, among other assumptions, required a bounded delay. Under a more restrictive condition on $r(x)$, namely, $0 < r(x)/x \leqq m$ for all x, Singh [13] gives sufficient conditions for all oscillatory solutions of a special case of (2) to bounded above. Under a different set of hypotheses, Kusano and Onose [9] obtained exactly the opposite result. The point to be made here is that while we are primarily interested in the case where $r(x)$ is superlinear, (cf. Theorems 1 and 4 and Corollary 2) i.e., $\limsup_{|x|\to\infty} r(x)/x = +\infty$, our condition (20) includes the sublinear forms of Kusano and Onose [9, 10] and Singh [12, 13] as special cases and, moreover, our integral conditions are similar in form and at times reduce exactly to those used in [9, 10, 12, and 13].

Relative to Theorem 4, if in addition to conditions (19)-(21) and (24)-(26), we ask that $r(0) = 0$ and there exists $N > 0$ such that either

(27) $$\lim_{t\to\infty}\int_T^t [1/a(s)]\int_T^s \{h_1(u) + N[h_2(u) + |q(u)|]\}duds = -\infty$$

or,

(28) $$\lim_{t\to\infty}\int_T^t [1/a(s)]\int_T^s \{h_1(u) - N[h_2(u) + |q(u)|]\}duds = +\infty$$

for all large T, then any solution $x(t)$ of (1) satisfying (23) is nonoscillatory. The alternate forms of (27) and (28) corresponding to (16) and (17) are respectively

(29) $$\limsup_{t\to\infty}\int_T^t \{h_1(u) + N[h_2(u) + |q(u)|]\}du < 0$$

and

(30) $$\liminf_{t\to\infty}\int_T^t \{h_1(u) - N[h_2(u) + |q(u)|]\}du > 0 .$$

We will now give some examples to illustrate our results.

EXAMPLE 1. The equation

$$x'' + x/t^2 = [\sin(\ln t)]/t^2,\ t \geqq 1$$

fails to satisfy condition (21) for $k = 0$ or condition (22). Here $x(t) = \cos(\ln t)$ is a bounded oscillatory solution which does not converge to zero.

EXAMPLE 2. The equation

$$x'' + x^3(t^{1/2})/t^3 = h_1(t),\ t \geqq 1$$

where $h_1(t) = [\sin(\ln t) - 3\cos(\ln t)]/t^3 + [\sin^3(\ln t^{1/2})]/t^{9/2}$ satisfies condition (20) with $p = 3$, condition (21) with $k = 0$, and (25). Here neither (27) nor (28) holds and we see that $x(t) = t^{-1}\sin(\ln t)$ is a bounded oscillatory solution.

EXAMPLE 3. Consider the equation

$$(t^\sigma x')' + t^{-\alpha}x^p(t^\beta) = h_1(t),\ t \geqq 1$$

where $h_1(t) = [4 + 2\cos(6\ln t) + 6\sin(6\ln t)]/t^3 + 1/t^\alpha$, $\alpha > 3$ and $\sigma > -1$. Conditions (20), (21) and (25) are satisfied provided that $\beta kp - \alpha < -1$ and $\beta kp - \alpha - \sigma < -2$. If $\sigma \leqq 1$, then (28) is satisfied while if $\sigma > 1$, then (30) is satisfied. Thus, in either case, if $x(t)$ is a solution such that

$$|x(t)| = O(t^k) \quad \text{as} \quad t \longrightarrow \infty$$

with $k < (\alpha + \sigma - 2)/\beta p$, then $x(t)$ is nonoscillatory. Notice that here the forcing term $h_1(t)$ changes signs.

The best nonoscillation theorem known to date for sublinear delay equations is the theorem of Kusano and Onose in [9]; it includes as a special case the nonoscillation criteria of Singh [13; Theorem 4.1]. There are several similarities between the conditions imposed in [9] and those used here. For example, when $k = 0$ conditions (6)-(7) of [9] imply condition (21) above. In addition, conditions (2)-(3) and (4)-(5) of [9] imply conditions (29)-(30) and (27)-(28) above respectively. On the other hand, even when $p \leqq 1$ our condition (20) on $r(x)$ is less restrictive than those used in [9] or [13]. Nor do we require $q(t) > 0$ as was needed in [9] and [13]. In both [9] and [13] the authors required that their forcing term $e(t, x) \equiv e(t)$ be either nonnegative or nonpositive; this was not done here. Other related results for sublinear equations have been obtained by Staikos and Philos [14] who studied nth order equations. They proved that for unforced advanced equations all bounded solutions are nonoscillatory and for forced delay equations all unbounded solutions are nonoscillatory. When $n = 2$, their integral conditions on $a(t)$, $q(t)$ and $e(t)$ are similar to those used in [9-13] and this paper.

Brands [1] constructed an example of an equations of the type (2) with $a(t) \equiv 1$, $g(t) = t - 1$, and $e(t, x) \equiv 0$ such that $q(t)$ satisfied

$$\int_{t_0}^{\infty} e^{\alpha t^2} q(t)dt < \infty, \ \alpha < 2 \tag{31}$$

and yet the equation possessed an oscillatory solution. This is somewhat of a surprise since many sufficient conditions for oscillation of ordinary equations have analogous counterparts (or may even be special cases of those) for functional equations (see Kartsatos [8]). Condition (31) is a far cry from the well known nonoscillation criteria of Hille

$$\int_{t_0}^{\infty} tq(t)dt < \infty$$

for linear ordinary equations.

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