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ON CERTAIN SEQUENCES OF LATTICE POINTS

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Let S be a finite subset of R^n. A sequence $\{z_i\}$ is an S-walk if and only if $z_{i+1}-z_i$ is an element of S for all i. In an effective manner it is shown that long S-walks in Z^2 must have an increasing number of collinear points. In Z^3, however, an infinite S-walk may have a bounded number of collinear points.

1. Introduction.

Let S be a finite subset of R^n.

DEFINITION. An *S-walk* is any (finite or infinite) sequence of vectors in R^n, say $\{z_i\}$, such that $z_{i+1} - z_i \in S$, for all i.

Given S, let M be the maximum of the Euclidean norms of the vectors in S. In [5] the following theorem is proved (see also [3] for the case $M = \sqrt{2}$):

THEOREM. *Let $S \subset Z^2$, and let K be any positive integer. There exists $N = N(K, M)$ such that any S-walk of length at least N must have K collinear points.*

With Theorem 1 of this paper we provide an effective bound on $N(K, M)$. With Theorem 2 we show that the situation of $S \subset Z^3$ is quite different, i.e., an infinite S-walk in Z^3 may have a bounded number of collinear points. In Theorem 3 we show that there are still some restrictions in Z^3, namely that if S has only three elements, then a sufficiently long S-walk must have three collinear points.

2. The Planar case.

THEOREM 1. *Let $S \subset Z^2$, let K be any positive integer, and let N be a positive integer such that*

$$\log_2 N \geqq 2^{13} M^4 (K-1)^4 + \log_2 (K-1) .$$

Then, for every S-walk $\{z_i\}_{i=0}^{N}$, there is some line L, and K choices for i, such that $z_i \in L$.

Proof. We suppose that the theorem is false for some K and derive a contradiction. Let $Q = 8 \cdot 2^{1/2} \cdot M(K-1)$. Let T denote the set of (positive and negative) Farey fractions of order no greater than Q. Let A be the set of all lines through the origin with slopes in T. Let B be the mirror image of A reflected through the line $y = x$. Enumerate the lines in the two sets A and B in order of increasing slope: $L_1, L_2, L_3, \cdots$. Let $\{z_i\}$ be a counterexample to the theorem for K. We may assume that z_0 is the origin.

Let z_J be an arbitrary point of the counterexample sequence. There are lines in the set $A \cup B$, L_n and L_{n+1}, such that z_J is on or between these lines; that is, the slope of the line through the origin and z_J is between or equal to the slopes of L_n and L_{n+1}, respectively a and b.

Dirichlet's theorem [2, page 1] gives us for $x = (a + b)/2$, integers p and q, $0 < q < Q$, such that

$$|qx - p| \leqq Q^{-1} .$$

We have either $p/q \geqq b \geqq a$, or $b \geqq a \geqq p/q$. Note that $ab \geqq 0$. We may therefore choose p/q to be the same sign as a and b. Let H_0 be the line through the origin with slope p/q and let U be the larger of the two angles between H_0 and L_n and between H_0 and L_{n+1}. Clearly, since a, b, and p/q have the same sign (viewing zero as positive and negative), the tangent of U is at most $2Q^{-1}q^{-1}$.

Enumerate the lines parallel to H_0 through points of Z^2 as $\cdots H_{-2}, H_{-1}, H_0, H_1, H_2, \cdots$ so that the distance from H_0 to H_i is $|id|$, where d is the minimum distance between such translates of H_0.

We now return to z_J. Among $z_J, z_{J+1}, \cdots, z_{J+(2P-1)(K-1)}$ at least one point is on some H_i with $|i| > P - 1$. Otherwise one of the H_i, with $|i| \leqq P - 1$, would contain K points of our S-walk, contrary to hypothesis. Let z_f be on a line H_i, with $|i| > P - 1$, and $J \leqq f \leqq J + (2P - 1)(K - 1)$. This point z_f is at least distance Pd from H_0. The component of z_f parallel to H_0 is at most fM. Thus, if V is the angle between z_f and H_0, we have

$$|\tan V| \geqq Pd/fM .$$

By taking P so that $(2P - 1)(K - 1) \geqq J$, we can write that

$$\begin{aligned} |\tan V| &\geqq Pd/M[J + (2P - 1)(K - 1)] \\ &\geqq Pd/2M(2P - 1)(K - 1) \\ &> d/4M(K - 1) . \end{aligned}$$

We now estimate d. We may assume that both L_n and L_{n+1} are in A, since otherwise they are both in B and the mirror image of the forthcoming analysis applies. With this assumption both a and b are in T. We may also assume that p/q is in T, for if not either $p/q \geqq 1$ or $p/q \leqq -1$. In the first case $1/1$ will play the role of p/q and in the second, $-1/1$. Thus

$$d \geqq (p^2 + q^2)^{-1/2} \geqq (2^{1/2}q)^{-1} .$$

Thus

$$|\tan V| > [4\cdot 2^{1/2}Mq(K-1)]^{-1} .$$

It is now clear that the choice of Q as $8\cdot 2^{1/2}M(K-1)$ gives us

$$|\tan V| > |\tan U| .$$

It is clear that the broken line path from z_J to z_f has crossed either L_n or L_{n+1}. In summary, given z_J on or between L_n and L_{n+1}, there is some integer t such that $0 < t \leqq (2P-1)(K-1)$ and

(i) P is the first integer such that $(2P-1)(K-1) \geqq J$ and

(ii) z_{J+t} is within M of either L_n or L_{n+1}.

By induction we choose a subsequence $\{z_{t_i}\}$ of $\{z_i\}$ such that

(i) each z_{t_i} is within M of some line in $A \cup B$ and

(ii) $t_i < t_{i+1} \leqq t_i + (2P-1)(K-1)$, where P is the first integer such that $(2P-1)(K-1) \geqq t_i$.

Note that we may choose $t_0 = 0$ and $t_1 = 1$. In general, if $t_i \leqq j_i(K-1)$, then the P for t_{i+1} satisfies

$$2P - 1 \leqq j_i + 1 .$$

Thus, $t_{i+1} \leqq (2j_i + 1)(K-1)$. Thus, if $j_i \leqq 2^i - 1$, we have $j_{i+1} \leqq 2^{i+1} - 1$.

We now count the number of lines in $A \cup B$. It is less than $2Q^2$. For any given line in $A \cup B$, the number of translates of it through points of Z^2 which are within distance M of it is at most $2M/d$, where d is the minimum distance between such translates. If their common slope is p/q in T, we have

$$d \geqq (p^2 + q^2)^{-1/2} \geqq (2^{1/2}Q)^{-1} .$$

If their common slope with respect to the y-axis is in T, the mirror image analysis applies. Thus, in all cases, $2M/d \leqq 2\cdot 2^{1/2}MQ$. Finally, $(2Q^2)(2\cdot 2^{1/2}MQ) = 4\cdot 2^{1/2}MQ^3$ is an upper bound on the number of lines which the subsequence $\{z_{t_i}\}$ can occupy. If the index i on t_i is at least $(K-1)(4\cdot 2^{1/2}MQ^3)$, one of these lines will have K points of $\{z_{t_i}\}$. All that is required is that $t_i \leqq N$. Since $t_i \leqq (K-1)(2^i - 1)$, it suffices to have

$$\log_2 (K-1) + 4\cdot 2^{1/2}MQ^3(K-1) \leqq \log_2 N .$$

Since $Q = 8\cdot 2^{1/2}M(K-1)$, we have $4\cdot 2^{1/2}MQ^3(K-1) = 2^{13}M^4(K-1)^4$. By our choice of N this is satisfied. This contradiction establishes the theorem.

REMARK 1. Theorem 1 remains true in n-dimensional space with the same relations between N, M and K if we use $n-1$ dimensional hyperplanes for L instead of lines. The proof consists of projecting the S-walk onto Z^2, finding a line there and taking its pre-image under the projection.

REMARK 2. Professor Carl Pomerance of the University of Georgia [4] has extended this theorem by considering walks whose average step size is bounded. His theorem is stated below. Let $d(V) = \sum_{i=0}^{m-1} ||z_{i+1} - z_i||$ for a finite sequece $V = \{z_i\}_{i=0}^{m} \subset Z^2$.

THEOREM. *For every positive integer K and every positive real number M, there exists $m_0 = m_0(M, K)$ such that if $m > m_0$ and $d(V)/m \leqq M$, then there are K points of V which are collinear.*

An effective bound on m_0 is not known for Pomerance's theorem.

III. Three dimensional case.

THEOREM 2. *If S is a set of vectors which do not all lie in the same plane, then there exists an infinite S-walk in which no $5^{11} + 1$ vectors are collinear.*

NOTATION. If $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ and $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$ are ordered sets of vectors, and β is a vector operator, we let $RA = (\boldsymbol{a}_n, \cdots, \boldsymbol{a}_1)$, $(A, B) = (a_1, \cdots, a_n, b_1, \cdots, b_m)$, and $\beta A = (\beta a_1, \cdots, \beta a_n)$. Let $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ be the three orthonormal unit vectors. For a vector $z = z_1\boldsymbol{i} + z_2\boldsymbol{j} + z_3\boldsymbol{k}$, let $||z||^{||} = z_1 + z_2 + z_3$ and $||z||^{\perp} = (z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1)^{1/2}$. Note that $||z||^{||}$ and $||z||^{\perp}$ are proportional to the components of z parallel and perpendicular respectively to the vector $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. Let γ be the length of the component of $\boldsymbol{i}, \boldsymbol{J}$, or $\boldsymbol{k}$ perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. Then $\gamma = (2/3)^{1/2}$ and in general the perpendicular component of z has length $\gamma ||z||^{\perp}$.

Proof. It suffices to prove Theorem 2 for the case where $S = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Let α and β be vector operators such that $\alpha\boldsymbol{i} = \boldsymbol{j}$, $\alpha\boldsymbol{j} = \boldsymbol{i}$, $\alpha\boldsymbol{k} = \boldsymbol{k}$, $\beta\boldsymbol{i} = \boldsymbol{i}$, $\beta\boldsymbol{j} = \boldsymbol{k}$, and $\beta\boldsymbol{k} = \boldsymbol{j}$. We define inductively ordered sets of vectors A_n. Let $A_0 = (\boldsymbol{i})$, and let $A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$. Note that A_n has 7^n elements and that the sequence A_{n+1} begins with A_n. It follows that there exists a unique infinite sequence of vectors $\{\boldsymbol{v}_p\}$ such that $(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{7^n}) = A_n$ for all n. Let $z_p = \sum_{q=1}^{p} \boldsymbol{v}_q$ for all positive integers p. Then $W = \{z_p\}$ is an S-walk. We claim that no $5^{11} + 1$ elements of W are collinear.

For convenience of notation we let z_0 be the zero vector. Let $C_n^0 = \{z_0, z_1, \cdots, z_{7^n}\}$. We prove by induction that the projection of

C_n^0 onto the plane perpendicular to $i + j + k$ lies within a trapezoid with base $4^n\gamma$, base angles $60°$, and adjacent sides $4^n\gamma/3$, with z_0 and z_{7^n} lying at extreme ends of the base. We will refer to such a trapezoid as a trapezoid of order n. The case $n = 0$ is trivial. Assume it is true for n. Note that A_n, αA_n, $R\beta A_n$, and $R\beta\alpha A_n$ are all mirror images of each other, either in space or in time (i.e., one can get from one to the others by permuting the unit vectors, by reversing the order of the sequence, or both). It follows that the set $C_n^\nu = \{z_{7^n\nu}, \cdots, z_{7^n(\nu+1)}\}$ is congruent to C_n^0, or its mirror image, for $0 \leqq \nu \leqq 6$. Therefore the projection of C_n^ν lies within a trapezoid of order n, with $z_{7^n\nu}$ and $z_{7^n(\nu+1)}$ lying at extreme ends of the base. From the definition of A_{n+1}, it follows that the seven trapezoids of order n fit together within a trapezoid of order $n + 1$, as illustrated in Figure 1.

It is straightforward to prove, by induction on n, that for any positive integer ν, the projections of C_{n+1}^ν and $C_{n+1}^{\nu+1}$ can fit together in one of only three possible configurations (ignoring rotations, reflections, and reversals of the sequence), namely those illustrated in Figure 2.

It follows that the distance between two points lying in non-adjacent trapezoids of order n must be at least $3^{-1/2}\cdot 4^n\gamma$, and that the distance between two points lying in adjacent trapezoids, or

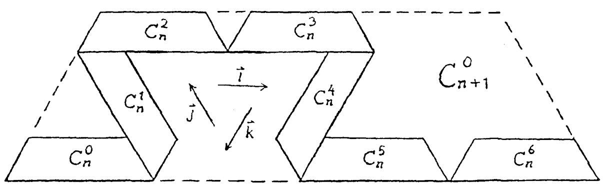

FIGURE 1

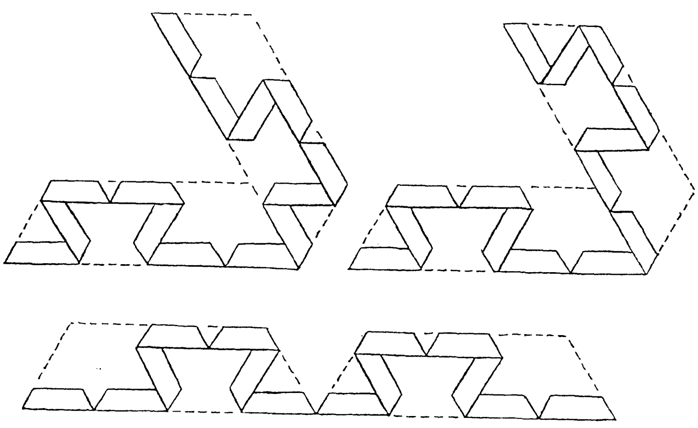

FIGURE 2

the same trapezoid, of order n can be at most $2 \cdot 4^n \gamma$.

Now let p and q be positive integers such that $7^n \leqq |p - q| < 7^{n+1}$. Then, if $n \geqq 1$, z_p and z_q cannot lie in adjacent trapezoids of order $n - 1$, so $\|z_p - z_q\|^{\perp} \geqq 3^{-1/2} \cdot 4^{n-1}$; if $n = 0$, this inequality is trivially satisfied. Likewise, z_p and z_q must lie in adjacent trapezoids, or the same trapezoid, of order $n + 1$, so $\|z_p - z_q\|^{\perp} \leqq 2 \cdot 4^{n+1}$. Since $\|z_p - z_q\|^{\|} = |p - q|$, we have

$$3^{-1/2} \cdot 4^{n-1} \cdot 7^{-(n+1)} < \|z_p - z_q\|^{\perp} / \|z_p - z_q\|^{\|} \leqq 2 \cdot 4^{n+1} \cdot 7^{-n} .$$

Now let r and s be positive integers such that $7^m \leqq |r - s| < 7^{m+1}$, with $m \geqq n$, so that

$$3^{-1/2} \cdot 4^{m-1} \cdot 7^{-(m+1)} < \|z_r - z_s\|^{\perp} / \|z_r - z_s\|^{\|} \leqq 2 \cdot 4^{m+1} \cdot 7^{-m} .$$

If z_p, z_q, z_r, and z_s are collinear, then

$$\|z_p - z_q\|^{\perp} / \|z_p - z_q\|^{\|} = \|z_r - z_s\|^{\perp} / \|z_r - z_s\|^{\|}$$

so $3^{-1/2} \cdot 4^{n-1} \cdot 7^{-(n+1)} < 2 \cdot 4^{m+1} \cdot 7^{-m}$. It follows that $(7/4)^{m-n} < 224\sqrt{3}$, and $m - n < (\log 224\sqrt{3})/(\log 7/4) < 11$, i.e., $m - n \leqq 10$. Therefore $|r - s|/|p - q| < 7^{11}$, and there are at most 7^{11} collinear points in W.

Furthermore, if X is a set of collinear points in W which all lie within the same trapezoid of order n, but not within the same trapezoid of order $n - 1$, then no two points of X can lie within the same trapezoid of order $n - 11$. However, no line can intersect more than five trapezoids of order $n - 1$ within a trapezoid of order n. For suppose a line intersected six of the trapezoids $C_n^0, C_n^1, \cdots, C_n^6$ in Figure 1. If C_n^0 where excluded, then the line would have to intersect C_n^3 and C_n^5, in which case C_n^1 would be missed. If C_n^2 were excluded, then the line would intersect C_n^0 and C_n^6, missing C_n^3. But a line intersecting C_n^0 and C_n^2 would miss C_n^6. Therefore, there are at most 5^{11} collinear points in W, and the theorem is proved.

It is obvious that this result can be sharpened considerably without changing the method of proof. For example it is not hard to convince oneself, by studying Figure 2, that in fact $4^{n-1} \leqq \|z_p - z_q\|^{\perp} \leqq 4^{n+1}$ if $7^n \leqq |p - q| < 7^{n+1}$. Also, there is no need to lump together all values of $|p - q|$ between 7^n and 7^{n+1}. By using a finer partition it ought to be possible to show that for a given value of $|p - q|$, the possible values of $\|z_p - z_q\|^{\perp} / \|z_p - z_q\|^{\|}$ range over a factor no greater than 4. Since $4 < (7/4)^3$, this would imply that W can have no more than 7^3 collinear points, all lying in the same trapezoid of order n, and no two lying in the same trapezoid of order $n - 4$. Finally, one could examine the 7^4 trapezoids of order $n - 4$ within a trapezoid of order n, preferably with the aid of a computer, and find an upper bound on the number which can be collinear, not only in the plane, but in 3-space. To clinch the argument, it might be necessary to descend to order $n-5$.

One would hope that by this method a sufficiently clever and persistent mathematician could determine the true maximum number of collinear points in W, which undoubtedly is three. However, there is no hope of sharpening Theorem 2 further than this, for we have the following theorem:

THEOREM 3. *If S has exactly three elements, then every S-walk of length nine has three collinear vectors; in fact three equally spaced collinear vectors.*

Proof. This result follows from the theorem of T. C. Brown [1] that any sequence of length nine on three symbols contains two adjacent segments which are permutations of each other. Brown's theorem can be verified in about one hour by direct computation.

An S-walk of length eight with no three collinear points is obtained by summing the sequence $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{i}, \boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{i}$.

REMARK 3. Theorem 2 also holds in the case where $S \subset \boldsymbol{R}^2$, provided that there are three elements $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$ of S, such that $\boldsymbol{e}_1 \times \boldsymbol{e}_2$, $\boldsymbol{e}_2 \times \boldsymbol{e}_3$, and $\boldsymbol{e}_3 \times \boldsymbol{e}_1$ are linearly independent over the rationals. In other words, the condition that the elements of S be lattice points is necessary for Theorem 1.

The above theorems leave unanswered the question of whether it is possible to have an infinite S-walk with no three collinear points for some $S \subset Z^n$ (in particular, can $n = 3$?).

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