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LONG WALKS IN THE PLANE WITH FEW COLLINEAR POINTS

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Let S be a set of vectors in R^n. An S-walk is any (finite or infinite) sequence $\{z_i\}$ of vectors in R^n such that $z_{i+1}-z_i \in S$ for all i. We will show that if the elements of S do not all lie on the same line through the origin, then for each integer $K \geqq 2$, there exists an S-walk $W_K=\{z_i\}_{i=1}^{N(K)}$ such that no $K+1$ elements of W_K are collinear and $N(K)$ grows faster than any polynomial function of K.

Specifically, we will prove that

$$\log_2 N(K) > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) .$$

We will then show that if the elements of S lie on at least L distinct lines through the origin, then there exists an S-walk of length $N(K, L)$ with no $K+1$ elements collinear, such that $N(K, L) \geqq (1/4)L^* N(K-1)$, where $L-2 \leqq L* \leqq L+1$ and $L* \equiv 0 \bmod 4$. In [3] it was shown that if $S \subset Z^2$, and for all $s \in S$ we have $||s|| \leqq M$, then there does not exist an S-walk $W = \{z_i\}_{i=1}^{N(K,M)}$ such that no $K+1$ elements of W are collinear and

$$\log_2 N(K, M) > 2^{13} M^4 K^4 + \log_2 K .$$

Before proving these theorems we introduce some notation. If $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_m)$ are ordered sets of vectors, we let $RA = (a_n, \cdots, a_1)$ and we let $(A, B) = (a_1, \cdots, a_n, b_1, \cdots, b_m)$. We let $2A = (A, A)$ and, for every positive integer k, we let $(k+1)A = (kA, A)$. If J is a vector operator, we let $JA = (Ja_1, \cdots, Ja_n)$.

THEOREM 1. *Let S contain two vectors independent over R, and let K be an integer greater than or exual to 2. There exists an S-walk $W_K = \{z_p\}_{p=1}^{N(K)}$ such that no $K+1$ elements of W_K are collinear and such that*

$$\log_2 N(K) > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) .$$

Proof. If we let $(\log_2 K - 1)^2/9 - (\log_2 K - 1)/6 = \log_2 K$, then $\log_2 K = (25 + 3\sqrt{65})/4 > 12$ or $(25 - 3\sqrt{65})/4 < 1$. Therefore if $1 \leqq \log_2 K \leqq 12$, and $2 \leqq K \leqq 4096$, then

$$\frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) < \log_2 K .$$

Since W_K cannot have more than $N(K)$ collinear points, we need only consider $K > 4096$.

We may let $S = \{i, j\}$ without loss of generality, where i and j are orthonormal unit vectors.

For every positive integer m and nonnegative integer n, let $A_0^m = i$, and let

$$A_{n+1}^m = (mA_n^m, 2^n RJA_n^m) ,$$

where $Ji = j$ and $Jj = i$. Let $V = \{v_p\}_{p=1}^N = \mu A_\nu^\mu$, where μ is the greatest integer less than or equal to $((7/9)K)^{1/3}$, and ν is the least integer greater than or equal to $\log_2 \mu - 3/2$. Note that since $K > 4096$, we have $\mu \geqq 14$, and $\nu \geqq 3$. Let $z_p = \sum_{q=1}^p v_q$ for each p, and let $W = \{z_p\}_{p=1}^N$. We maintain that W has no more than K collinear points and that $\log_2 N > (\log_2 K - 1)^2/9 - (\log_2 K - 1)/6$.

Let $b_0 = 1$ and let $b_{n+1} = (\mu + 2^n)b_n$. Then b_n is the cardinality of A_n^μ, and $N = \mu b_\nu$. Clearly $b_n \geqq \mu^n$, so $N \geqq \mu^{\nu+1}$ and $\log_2 N \geqq (\nu + 1)\log_2 \mu \geqq (\log_2 \mu - 1/2)\log_2 \mu$. Since μ is the greatest integer less than or equal to $((7/9)K)^{1/3}$, and $((7/9)K)^{1/3} > 14$, we have $\mu > (14/15)((7/9)K)^{1/3} > ((1/2)K)^{1/3}$. It follows that $\log_2 N > 1/9[\log_2((1/2)K)]^2 - \log_2((1/2)K)/6 = (\log_2 K - 1)^2/9 - (\log_2 K - 1)/6$.

We now prove that W has no more than K collinear points.

Let $C_n^\alpha = \{z_p : \alpha b_n \leqq p \leqq (\alpha + 1)b_n\}$. For each n, all C_n^α are congruent; specifically one can get from any one to any other by a translation plus, possibly, a reflection about the major diagonal (i.e., a reflection about the line passing through the vector $i + j$, which interchanges i and j), followed by a rotation about the origin of 180°. This reflection plus rotation is equivalent to a reflection about the line perpendicular to the major diagonal (i.e., the line passing through the vector $i - j$). We will refer to this latter line as the minor diagonal. Let

$$U_n^\beta = \{C_n^\alpha : \beta(\mu + 2^n) \leqq \alpha < (\beta + 1)(\mu + 2^n)$$
$$\text{if } n \neq \nu \text{ and } U_\nu^0 = \{C_\nu^\alpha : 0 \leqq \alpha \leqq \mu\} .$$

Note that $C_{n+1}^\beta = \{z_p : \beta(\mu + 2^n)b_n \leqq p \leqq (\beta + 1)(\mu + 2^n)b_n\}$, so U_n^β is a partition of C_{n+1}^β and U_ν^0 is a partition of W. We now consider a line with slope m and determine for each n, the maximum number of elements of U_n^β which the line can intersect (the maximum number cannot depend on β, since all C_{n+1}^β are congruent). Let r_n be this maximum number. Then the line cannot intersect more than $r = \prod_{n=0}^\nu r_n$ points of W.

Let s_n be the slope of z_{b_n}; i.e., $s_n = y_n/x_n$ where $z_{b_n} = x_n\boldsymbol{i} + y_n\boldsymbol{j}$. The slope of $z_{(\alpha+1)b_n} - z_{\alpha b_n}$ is then either s_n or s_n^{-1}, depending on whether C_n^α is a simple translation of C_n^0, or a translation of the reflection of C_n^0 about the minor diagonal. We wish to find a lower bound on s_n/s_{n-1}.

Now $x_0 = 1$, $y_0 = 0$, $x_{n+1} = \mu x_n + 2^n y_n$, and $y_{n+1} = \mu y_n + 2^n x_n$. It follows that x_n, y_n, and s_n are strictly positive for all $n \geqq 1$. We now prove by induction that $s_n < 2^n/\mu$. Clearly $s_0 = 0 < 2^0/\mu$ and $s_1 = 1/\mu < 2^1/\mu$. Suppose $s_n < 2^n/\mu$. Let $t_n = 2^n/s_n\mu$. Then $t_n > 1$. Now

$$\begin{aligned} s_{n+1} &= (\mu y_n + 2^n x_n)/(\mu x_n + 2^n y_n) \\ &= (\mu s_n + 2^n)/(\mu + 2^n s_n) \\ &= (\mu s_n + \mu s_n t_n)/(\mu + \mu s_n^2 t_n) \\ &= (s_n + s_n t_n)/(1 + s_n^2 t_n) \ . \end{aligned}$$

Thus

$$\begin{aligned} t_{n+1} &= 2^{n+1}/s_{n+1}\mu = 2s_n t_n/s_{n+1} \\ &= 2s_n t_n(1 + s_n^2 t_n)/(s_n + s_n t_n) \\ &= 2t_n(1 + s_n^2 t_n)/(t_n + 1) \ . \end{aligned}$$

We now view t_{n+1} as a function of the real variables t_n and s_n, and compute its partial derivatives:

$$\partial t_{n+1}/\partial t_n = 2(s_n^2 t_n^2 + 2s_n^2 t_n + 1)/(t_n + 1) > 0$$

and

$$\partial t_{n+1}/\partial s_n = 4t_n^2 s_n/(t_n + 1) > 0 \ .$$

Since t_{n+1} has the value 1 when $s_n = 0$ and $t_n = 1$, it follows that $t_{n+1} > 1$ when $s_n \geqq 0$ and $t_n > 1$, as is the case here. Therefore $s_{n+1} < 2^{n+1}/\mu$.

Next, recall that $\nu - 1 < \log_2 \mu - 3/2$, so if $n \leqq \nu - 1$, then $2^n \leqq 2^{\nu-1} < 2^{-3/2}\mu$. Since $2^n > s_n\mu$, it follows firstly that $s_n < 2^{-3/2}$, and secondly that

$$\begin{aligned} s_{n+1}/s_n &= (\mu s_n + 2^n)/(\mu s_n + 2^n s_n^2) \\ &> 2\mu s_n/(\mu s_n + 2^{-3/2}\mu s_n^2) \\ &= 2/(1 + 2^{-3/2}s_n) > 2\Big/\left(1 + \frac{1}{8}\right) = \frac{16}{9} \ . \end{aligned}$$

It follows that, given m, there is at most one n such that $(3/4)s_n \leqq m \leqq (4/3)s_n$. Suppose there exists λ such that $(3/4)s_\lambda \leqq m \leqq (4/3)s_\lambda$. Then $m < (3/4)s_{\lambda+1}$ and $m > (4/3)s_{\lambda-1}$. Moreover, for all $n > \lambda + 1$, we have $m < (27/64)s_n < (1/2)s_n$, and for all $n < \lambda - 1$, we have $m > (64/27)s_n > 2s_n$. All of the above also holds if we replace s_n by s_n^{-1}, except that some of the inequalities are reversed and constants replaced by their reciprocals in the obvious way.

We now calculate for each of the five cases, $n = \lambda$, $n = \lambda + 1$, $n = \lambda - 1$, $n > \lambda + 1$, and $n < \lambda - 1$, the maximum number r_n of elements of U_n^β which a line of slope m can intersect. We can assume without loss of generality that C_{n+1}^β is a simple translation of C_{n+1}^0; if C_{n+1}^β is a translation of the reflection of C_{n+1}^0 about the minor diagonal, then we can apply the same argument, replacing s_n by s_n^{-1}. Then C_n^α is a simple translation of C_n^0 for $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$, and a translation of the reflection of C_n^0 for $\beta(\mu + 2^n) + \mu \leqq \alpha < (\beta + 1)(\mu + 2^n)$. For each α, the first point of $C_n^{\alpha+1}$ coincides with the last point of C_n^α. It is easy to prove by induction on n that C_n^0 (and therefore C_n^α for all α) lies entirely within a right triangle, with sides x_n and y_n adjacent to the right angle, and with the first and last points of C_n^0 at opposite ends of the hypotenuse. Therefore the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ lie within congruent right triangles, whose hypotenuses are adjacent segments of a line with slope s_n (see Fig. 1). It follows

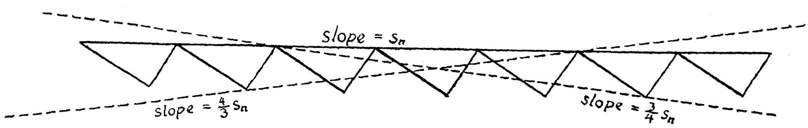

FIGURE 1

that a line with slope $m > s_n q/(q-1)$ or $m < s_n(q-1)/q$ can intersect at most q of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ at distinct points (i.e., assign the last point of each set C_n^α to the set $C_n^{\alpha+1}$, and do not count the line as intersecting C_n^α if it only intersects this last point). Suppose $m \leqq 1$. Then $m < (1/2)s_\lambda^{-1}$, and a line of slope m can intersect no more than two of the sets C_n^α: $\beta(\mu + 2^n) + \mu \leqq \alpha < (\beta + 1)(\mu + 2^n)$. If $n = \lambda$, then a line of slope m can intersect all μ of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ for a total of $\mu + 2$. If $n = \lambda + 1$ or $\lambda - 1$, the line can intersect at most 4 of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$, for a total of 6, while if $n > \lambda + 1$ or $n < \lambda - 1$, the line can intersect at most two of the sets C_n^α: $\beta(\mu + 2^n) \leqq \alpha < \beta(\mu + 2^n) + \mu$ for a total of 4. If $m > 1$, then we obtain essentially the same results by redefining λ so that $(3/4)s_\lambda^{-1} \leqq m \leqq (4/3)s_\lambda^{-1}$, the only difference being that μ is replaced by 2^n, which in any case is less than μ. Therefore we have $r_n \leqq \mu + 2$ if $n = \lambda$, $r_n \leqq 6$ if $n = \lambda - 1$ or $\lambda + 1$, and $r_n \leqq 4$ for all other n. Finally, we have

$$r = \prod_{n=0}^{\nu} r_n \leqq (\mu + 2)\cdot 6^2 \cdot 4^{\nu-2} < 36(\mu + 2)\cdot 4^{\log_2 \mu - 5/2}$$
$$= \frac{36}{32}\mu^2(\mu + 2) \leqq \frac{9}{7}\mu^3 \leqq K .$$

If λ does not exist, then there are at most two values of n for which $(27/64)s_n \leqq m \leqq (64/27)s_n$, and these two values can take the place of $\lambda - 1$ and $\lambda + 1$ in our argument.

REMARK. We can use this method to get slightly better results as follows: The method works by partitioning W into a heiarchy of sets, each set of order $n + 1$ being partitioned into $\mu + 2^n$ sets of order n, and showing that for almost all n, a given line can intersect at most four sets of order n within a given set of order $n + 1$. Suppose that instead of using the partition based on the sets C_n^α, we modify this partition slightly by splitting each C_n^α into two sets of order n, namely $\{z_p : \alpha b_n \leqq p \leqq \alpha b_n + \mu b_{n-1}\}$ and $\{z_p : \alpha b_n + \mu b_{n-1} \leqq p \leqq (\alpha + 1)b_n\}$. Then each set of order $n + 1$ would have either 2μ or 2^{n+1} sets of order n, and it should not be hard to show that for almost all n, a given line can intersect at most three sets of order n within a given set of order $n + 1$. We would then have $r = c\mu \cdot 3^\nu = c\mu^{1+\log_2 3}$, where c is a constant which does not depend on K, and finally

$$\log_2 N = (1 + \log_2 3)^{-2}(\log_2 K)^2 + O(\log_2 K) .$$

However, it seems impossible to push this method any further.

THEOREM 2. *Suppose that S contains L elements which are pairwise independent over R. Then there exists an S-walk $\Omega = \{u_i\}_{i=1}^N$ containing no set of $K + 1$ collinear points, such that*

$$\log_2 N > \frac{1}{9}[\log_2 (K - 1) - 1]^2 - \frac{1}{6}[\log_2 (K - 1) - 1] + \log_2 L^* - 2 ,$$

where $L - 2 \leqq L^* \leqq L + 1$ and $L^* \equiv 0 \bmod 4$.

Proof. The L elements of S with distinct arguments must include $L/2$ elements (if L is even) or $(L + 1)/2$ elements (if L is odd) in the same half-plane. Label these elements $s_1, s_2, s_3, \cdots$ in order of their arguments. For $1 \leqq n \leqq (1/4)L^*$, let $W_n = \varphi_n W$ where W is defined as in the proof of Theorem 1, and φ_n is the linear vector operator which maps i to s_{2n-1} and j to s_{2n}. Let N_0 be the cardinality of W and let $w_n = xs_{2n-1} + ys_{2n}$ be the final element of W_n. For $1 \leqq i \leqq N_0$, let z_i be defined as in the proof of Theorem 1, and let $u_i = \varphi_1 z_1$. Let $u_{N_0 n+i} = \sum_{j=1}^{n} w_j + \varphi_{n+1} z_i$ for

$1 \leqq n \leqq (1/4)L^* - 1$. Finally, let $N = (1/4)L^* N_0$ and let $\Omega = \{\boldsymbol{u}_i\}_{i=1}^{N}$. Note that Ω is constructed by placing the W_n end to end in sequence.

By Theorem 1,

$$\log_2 N > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) + \log_2 L^* - 2 .$$

We will now prove that no $K + 2$ points of Ω are collinear. Substituting $K - 1$ for the bound variable K then gives us Theorem 2 for the case $K \geqq 3$. For the case $K = 2$, we simply let $\boldsymbol{u}_i = \sum_{j=1}^{i} \boldsymbol{s}_j$. The resulting set $\{\boldsymbol{u}_i\}$, which contains at least $(1/2)L^*$ elements, is the set of vertices of a convex polygon; hence no three elements are collinear.

Let $T_n = \{u_i\}_{i=N_0(n-1)+1}^{N_0 n}$ and let $\boldsymbol{t}_n = \sum_{j=1}^{n} \boldsymbol{w}_j$, so that $\boldsymbol{t}_n$ is the final element of T_n. Let $\boldsymbol{t}_0 = 0$ and let $\boldsymbol{r}_n = \boldsymbol{t}_{n-1} + x\boldsymbol{s}_{2n-1}$ for $n \geqq 1$. Note that $\boldsymbol{t}_n = \boldsymbol{r}_n + y\boldsymbol{s}_{2n}$. Note also that from results proved previously, the set T_n must lie entirely on or in the interior of the triangle Δ_n with vertices $\boldsymbol{t}_{n-1}$, $\boldsymbol{r}_n$, and $\boldsymbol{t}_n$. Consequently any line which intersects T_n must intersect Δ_n. Now consider the polygon P with vertices $\boldsymbol{t}_0, \boldsymbol{r}_1, \boldsymbol{t}_1, \boldsymbol{r}_2, \boldsymbol{t}_2, \cdots, \boldsymbol{r}_{L^*/4}, \boldsymbol{t}_{L^*/4}$ in that order. The (directed) edges of this polygon are the vectors $x\boldsymbol{s}_1, y\boldsymbol{s}_2, x\boldsymbol{s}_3, \cdots, y\boldsymbol{s}_{L^*/2}$, and $-x\sum_{n=1}^{L^*/4} \boldsymbol{s}_{2n-1} - y\sum_{n=1}^{L^*/4} \boldsymbol{s}_{2n}$. Since the vectors $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \cdots$ are listed in order of increasing argument, and the range of all their arguments is less than 180°, it follows that the interior angles of P are all less than 180°, so P is convex. Now any line intersecting Δ_n, and in particular any line intersecting T_n, must intersect at least two sides of Δ_n (including each vertex in its two adjacent sides), and therefore must intersect P. Since P is convex, a line can only intersect P at one or two points, or along an edge. Therefore no line can intersect more than two of the T_n. Unless the slope of a line is between that of $\boldsymbol{s}_{2n-1}$ and $\boldsymbol{s}_{2n}$ inclusive, it can only intersect one point of T_n. By Theorem 1, no line can intersect more than K points of T_n. Therefore, no line can contain more than $K + 1$ points of Ω.

REMARK. In order to compare these results with the upper bound in [3], we can consider the case where $S = \{\boldsymbol{s} \in Z^2 \colon \|\boldsymbol{s}\| \leqq M\}$. Since the number of lattice points in a disc of radius R is $\pi R^2 + O(R)$ [2], we know that the number of lattice points with both coordinates divisible by q, in a disc of radius M, is $\pi M^2/q^2 + O(M/q)$. Therefore the number L of lattice points with relatively prime coordinates is

$$\pi M^2 \sum_{n=0}^{\infty} (-1)^n \sum_{q \in Q_n} q^{-2} + O(M \sum_{q \in Q} q^{-1}) ,$$

where Q is the set of square free positive integers less than or equal to M, and Q_n is the set of integers in Q with n distinct prime factors. It follows [1] that

$$L = 6M^2/\pi + O(M \log M) .$$

Finally, if we let $N(K, M)$ be the length of the longest S-walk with no more than K collinear points, and we choose any constants $c_1 < (9 \log 2)^{-1}$ and $c_2 > 2^{13} \log 2$, then we have

$$M^2 \exp [c_1(\log K)^2] < N(K, M) < \exp [c_2 M^4 K^4]$$

for all M and all but a finite number of K.

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