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CONTINUOUSLY VARYING PEAKING FUNCTIONS

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Let X be a compact metric space, $A \subseteqq C(X)$ a closed subalgebra. Let $\mathscr{P} \subseteqq X$ be the set of peak points for A. It is shown that there is a continuous function $\Phi: \mathscr{P} \to A$ such that $\Phi(x)$ peaks at x for all $x \in \mathscr{P}$.

0. Let X be a compact Hausdorff space, $C(X)$ the continuous functions on X under the uniform norm, and A a closed subspace of $C(X)$ containing 1. Let $\mathscr{P}$ be the set of peak points for A. Clearly if X has more than one point and $x \in \mathscr{P}$ then there are infinitely many functions in A which peak at x. Can one construct a function

$$\Phi: \mathscr{P} \longrightarrow A$$

so that $\Phi(x)$ peaks at x and Φ has some regularity properties?

In [4], using the von Neumann selection principle, it was shown that for $X = \bar{\mathscr{D}} \subset\subset C^n$ with smooth boundary, $A = A(\mathscr{D})$ (the analytic functions on $\mathscr{D}$ which extend continuously to $\bar{\mathscr{D}}$), one can choose Φ to be measurable. The same argument is valid under much more general circumstances.

In the present note we prove that, for quite general X and for A an algebra, Φ can be chosen to be continuous. This generalizes results in [1, Theorem 3.1] and [2, Proposition 4].

1. Throughout the discussion, X will be a fixed compact metric space with metric d. We let $C(X)$ denote the continuous, complex-valued functions on X with the uniform norm and $A \subseteqq C(X)$ will be a closed complex linear subspace. If $x \in X$, $r > 0$, then $B(x, r) = \{t \in X: d(x, t) < r\}$.

DEFINITION. A point $x \in X$ is said to be a *peak point* for A if there is an $f \in A$ with $f(x) = 1$ and, for all $y \in X \sim \{x\}$, $|f(y)| < 1$. The function f is said to peak at x.

We let $\mathscr{P}(A)$ denote the set of peak points for A.

THEOREM. *Let X be a compact metric space, $A \subseteqq C(X)$ a closed subalgebra (with or without 1). Then there is a continuous map*

$$\Phi: \mathscr{P}(A) \longrightarrow A$$

such that $\Phi(x)$ peaks at x for each $x \in \mathscr{P}(A)$.

The remainder of the paper is devoted to the proof of the theorem. We proceed via a sequence of lemmas. The plan of the proof is as follows.

For each $k \in \{1, 2, \cdots\}$ we will construct a continuous function

$$\Phi_k \colon \mathscr{P}(A) \longrightarrow A$$

such that for each $x \in \mathscr{P}(A)$ we have

(i) $\|\Phi_k(x)\| = 1$;

(ii) $[\Phi_k(x)](x) = 1$;

(iii) if $t \in X \sim B(x, 1/k)$ then $|[\Phi_k(x)](t)| \leqq 1 - 1/(k+2)$.

Once the $\{\Phi_k\}$ are constructed, the proof is immediate. For let $\Phi = \sum_{l=1}^{\infty} 2^{-l}\Phi_l$. Then Φ is continuous and for each $x \in \mathscr{P}(A)$ we have $\Phi(x) \in A$ and $[\Phi(x)](x) = 1$. Moreover, if $t \neq x$ and $k > 1/d(x, t)$ then

$$\begin{aligned} |[\Phi(x)](t)| &\leqq \sum_{l \neq k} 2^{-l}|[\Phi_l(x)](t)| + |2^{-k}[\Phi_k(x)](t)| \\ &\leqq 1 - 2^{-k} + 2^{-k}(1 - 1/(k+2)) < 1 . \end{aligned}$$

So $\Phi(x)$ peaks at x. Thus it remains to construct the Φ_k.

LEMMA 1. *Let $x_0 \in \mathscr{P}(A)$. Let p be a strictly positive continuous function on X with $p(x_0) = 1$. Then there is an $f \in A$ with $f(x_0) = 1$ and $|f(x)| \leqq p(x)$ for all $x \in X$.*

Proof. This is a special case of Theorem 12.5 of Gamelin [3], p. 58.

COROLLARY 2. *With hypotheses as in Lemma 1, there is a $g \in A$ such that $g(x_0) = 1$, $|g(x)| < p(x)$ for all $x \in X \sim \{x_0\}$.*

Proof. Immediate.

LEMMA 3. *Let $x_0 \in \mathscr{P}(A)$. Let $\psi \in A$ peak at x_0. There is a map*

$$\Psi \colon \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\} \longrightarrow A$$

so that

(i) *$\Psi(x)$ peaks at x for each $x \in \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\}$,*

(ii) *$\Psi(x_0) = \psi$,*

(iii) *Ψ is continuous at x_0.*

Proof. For each $x \in \mathscr{P}(A) \sim \{x_0\}$ choose, by Corollary 2, a function $\varphi_x \in A$ such that $\varphi_x(x) = 1$ and

$$(*) \qquad |\varphi_x(t)| < \min\{(2 - |\psi(x)| - |\psi(t)|)/2(1 - |\psi(x)|), 1\} \quad \text{for all} \quad t \in X \sim \{x\}\ .$$

Now for each $x \in \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\}$ we define

$$\Psi(x) = \begin{cases} [2(1-|\psi(x)|)\varphi_x + \overline{\operatorname{sgn} \psi(x)}\psi]/[2-|\psi(x)|] & \text{if} \quad x \neq x_0,\ |\psi(x)| > 1/2\ , \\ \psi & \text{if} \quad x = x_0\ . \end{cases}$$

Here $\operatorname{sgn} z \equiv z/|z|$, any $z \in \boldsymbol{C} \sim \{0\}$.

Clearly if $x \neq x_0$ and x is sufficiently close to x_0 then $|\psi(x)| > 1/2$ and we have

$$\begin{aligned} ||\Psi(x) - \psi|| &\leqq ||\Psi(x) - \overline{\operatorname{sgn} \psi(x)} \cdot \psi|| + ||\overline{\operatorname{sgn} \psi(x)} \cdot \psi - \psi|| \\ &\leqq ||[2(1 - |\psi(x)|)\varphi_x + \overline{\operatorname{sgn} \psi(x)} \cdot \psi]/[2 - |\psi(x)|] - \overline{\operatorname{sgn} \psi(x)} \cdot \psi|| \\ &\quad + ||\psi(1 - \overline{\operatorname{sgn} \psi(x)})|| \\ &\leqq \{[2(1 - |\psi(x)|)\, ||\varphi_x - \overline{\operatorname{sgn} \psi(x)} \cdot \psi|| \\ &\quad + (1 - |\psi(x)|)\, ||\overline{\operatorname{sgn} \psi(x)} \cdot \psi||]\}/[2 - |\psi(x)|] + |1 - \overline{\operatorname{sgn} \psi(x)}| \\ &\leqq 5(1 - |\psi(x)|) + |1 - \overline{\operatorname{sgn} \psi(x)}| \\ &\longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0\ . \end{aligned}$$

It remains to verify that $\Psi(x)$ peaks at x when $|\psi(x)| > 1/2$. For such x, we have $[\Psi(x)](x) = 1$. Further, if $t \neq x$ then by (*) we have

$$2(1 - |\psi(x)|)\,|\varphi_x(t)| < 2 - |\psi(x)| - |\psi(t)|$$

or

$$|2(1 - |\psi(x)|)\varphi_x(t)| + |\psi(t)| < 2 - |\psi(x)|$$

whence

$$|2(1 - |\psi(x)|)\varphi_x(t) + \overline{\operatorname{sgn} \psi(x)}\psi(t)| < 2 - |\psi(x)|$$

or

$$|[\Psi(x)](t)| < 1\ .$$

LEMMA 4. *Fix a positive integer k. There is a sequence $\{\Phi_k^j\}_{j=1}^{\infty}$ of functions,*

$$\Phi_k^j\colon \mathscr{P}(A) \longrightarrow A$$

satisfying, for each $z \in \mathscr{P}(A)$ and every j,

(i) $||\Phi_k^j(x)|| = 1$;

(ii) $[\Phi_k^j(x)](x) = 1$;

(iii) $\limsup_{\mathscr{P}(A) \ni y \to x} ||\Phi_k^j(x) - \Phi_k^j(y)|| \leqq 4^{-j} \cdot (1/k)$;

(iv) *for every* $t \in X \sim B(x, (1 - 2^{-j}) \cdot (1/k))$,

$$|[\Phi_k^j(x)](t)| \leqq (1 - 2/(k + 2)) + \sum_{i=1}^{j} 2^{-i} \cdot (1/(k + 2)) ;$$

(v) $||\Phi_k^j(x) - \Phi_k^{j-1}(x)|| \leqq 2^{-j} \cdot (1/k)$, $j \geqq 2$.

Proof. This lemma is the heart of the matter. We construct the Φ_k^j inductively on j. First consider $j = 1$. For each $x \in \mathscr{P}(A)$ construct, by Lemma 1, a function $\varphi_x \in A$ which satisfies $\varphi_x(x) = 1$ and

$$|\varphi_x(t)| \leqq \min\{1 - 8kd(x, t)/(k + 2), 1 - 2/(k + 2)\} .$$

Using $\psi = \varphi_x$, construct a function

$$(*) \qquad \Psi_x^1 : \mathscr{P}(A) \cap \{|\psi(x)| > 1/2\} \longrightarrow A$$

satisfying the conclusions of Lemma 3. Choose r_x^1, $0 < r_x^1 < 1/4k$ so that $t \in B(x, r_x^1)$ implies that $|\varphi_x(t)| > 1/2$ and

$$||\Psi_x^1(x) - \Psi_x^1(t)|| < 4^{-2} \cdot (1/(k + 2)) .$$

Now observe that if $y \in B(x, r_x^1)$ and $t \notin B(y, 1/2k)$ then

$$d(x, t) \geqq d(y, t) - d(y, x) \geqq 1/4k .$$

Therefore for such y, t we have

$$(**) \qquad \begin{aligned} |[\Psi_x^1(y)](t)| &\leqq |[\Psi_x^1(x)(t)]| + |[\Psi_x^1(x)](t) - [\Psi_x^1(y)](t)| \\ &\leqq |\varphi_x(t)| + 4^{-2} \cdot (1/(k + 2)) \\ &\leqq (1 - 2/(k + 2)) + 2^{-1} \cdot (1/(k + 2)) . \end{aligned}$$

Now since $\mathscr{P}(A)$ is a metric space, it is paracompact ([5], p. 160, Cor. 35). Hence there is a locally finite refinement $\mathscr{U}^1 = \{U_\omega^1\}_{\omega \in \Omega_1}$ of the covering $\{B(x, r_x^1)\}_{x \in \mathscr{P}(A)}$ of $\mathscr{P}(A)$. Let x_ω, $\omega \in \Omega_1$, be chosen so that $U_\omega^1 \subseteqq B(x_\omega, r_\omega^1)$. Let B_ω^1 denote $B(x_\omega, r_{x_\omega}^1)$. We may assume that $\bar{U}_\omega^1 \subseteqq B_\omega^1$. Let $\{\chi_\omega^1\}$ be a continuous partition of unity subordinary to $\mathscr{U}^1$ and define

$$\Phi_k^1 = \sum_{\omega \in \Omega} \chi_\omega^1 \Psi_{x_\omega}^1 .$$

Then conclusions (i) and (ii) are immediate. Conclusion (iv) follows from (**). Conclusion (v) is vacuous for $j = 1$. It remains to verify (iii).

Fix $x \in \mathscr{P}(A)$. Then there is a neighborhood W of x and $\{\omega_1, \cdots, \omega_m\} \subseteqq \Omega_1$ so that $W \cap \operatorname{supp} \chi_\omega \neq 0$ only if $\omega \in \{\omega_1, \cdots, \omega_m\}$. Of course m may depend on x. Letting x_i denote x_{ω_i}, $i = 1, \cdots, m$, we have that

$$\begin{aligned}\limsup_{\mathscr{P}(A)\ni y\to x}\|\Phi_k^1(x)-\Phi_k^1(y)\| &\leqq \sum_{i=1}^{m}\limsup_{\mathscr{P}(A)\cap W\ni y\to x}|\chi_{\omega_i}^1(x)-\chi_{\omega_i}^1(y)|\,\|\Psi_{x_i}(y)\| \\ &\quad+\sum_{i=1}^{m}\chi_{\omega_i}^1(x)\limsup_{\mathscr{P}(A)\cap W\ni y\to x}\|\Psi_{x_i}^1(x)-\Psi_{x_i}^1(y)\| \\ &\leqq 0+\sum_{i=1}^{m}\chi_{\omega_i}^1(x)\limsup_{\mathscr{P}(A)\cap B_{\omega_i}\ni y\to x}\|\Psi_{x_i}^1(x)-\Psi_{x_i}^1(x_i)\| \\ &\quad+\sum_{i=1}^{m}\chi_{\omega_i}^1(x)\limsup_{\mathscr{P}(A)\cap B_{\omega_i}\ni y\to x}\|\Psi_{x_i}^1(x_i)-\Psi_{x_i}^1(y)\| \\ &\leqq 2\cdot 4^{-2}/(k+2)\leqq 4^{-1}\cdot(1/k)\,.\end{aligned}$$

Now suppose that $\Phi_k^1, \cdots, \Phi_k^j$ have been constructed so that (i)-(v) are satisfied. Let $x \in \mathscr{P}(A)$. Using $\psi = \Phi_k^j(x)$, we construct a function

$$\Psi_x^{j+1}\colon \mathscr{P}(A)\cap\{|\psi(x)|>1/2\}\longrightarrow A$$

satisfying the conclusions of Lemma 3. Choose r_x^{j+1}, $0<r_x^{j+1}<2^{-j-1}\cdot(1/k)$ so that $t\in B(x, r_x^{j+1})$ implies that $|[\Phi_k^j(x)](t)|>1/2$ and both

$$\|\Psi_x^{j+1}(x)-\Psi_x^{j+1}(t)\|\leqq 4^{-j-2}\cdot(1/(k+2))$$

(***) and

$$\|\Phi_k^j(x)-\Phi_k^j(t)\|\leqq(4/3)\cdot 4^{-j}\cdot(1/k)\,.$$

If now $y\in B(x, r_x^{j+1})$, $t\notin B(y, (1-2^{-j-1})\cdot(1/k))$ then

$$d(x, t)\geqq d(y, t)-d(y, x)\geqq(1-2^{-j})(1/k)\,.$$

Hence for such y, t we have

$$\begin{aligned}|[\Psi_x^{j+1}(y)](t)| &\leqq |[\Psi_x^{j+1}(x)](t)|+|[\Psi_x^{j+1}(x)](t)-[\Psi_x^{j+1}(y)](t)| \\ &\leqq |[\Phi_k^j(x)](t)|+4^{-j-1}\cdot(1/(k+2)) \\ &\leqq (1-2/(k+2))+\sum_{i=1}^{j}2^{-i}\cdot(1/(k+2))+2^{-j-1}\cdot(1/(k+2)) \\ &= (1-2/(k+2))+\sum_{i=1}^{j+1}2^{-i}\cdot(1/(k+2))\,.\end{aligned}$$

Choose a locally finite refinement $\mathscr{U}^{j+1}=\{U_\omega^{j+1}\}_{\omega\in\Omega_{j+1}}$ of the covering $\{B(x, r_x^{j+1})\}_{x\in\mathscr{P}(A)}$ of $\mathscr{P}(A)$. Let $\{x_\omega\}_{\omega\in\Omega_{j+1}}$ be chosen so that $U_\omega^{j+1}\subseteqq B(x_\omega, r_{x_\omega}^{j+1})\equiv B_\omega^{j+1}$, each $\omega\in\Omega_{j+1}$. We may assume that $\bar{U}_\omega^{j+1}\subseteqq B_\omega^{j+1}$. Let $\{\chi_\omega^{j+1}\}$ be a continuous partition of unity subordinate to $\mathscr{U}^{j+1}$. Define

$$\Phi_k^{j+1}=\sum_{\omega\in\Omega_{j+1}}\chi_\omega^{j+1}\Psi_{x_\omega}^{j+1}\,.$$

It follows as in the case $j=1$ that (i), (ii), (iii), and (iv) hold. To verify (v) fix $x\in\mathscr{P}(A)$. Let $\omega_1, \cdots, \omega_m$ satisfy the property that

$\chi_\omega(x) \neq 0$ iff $\omega \in \{\omega_1, \cdots, \omega_m\}$. Let x_i denote x_{ω_i}, $i = 1, \cdots, m$. Then

$$\begin{aligned}||\Phi_k^{j+1}(x) - \Phi_k^j(x)|| &\leqq \left|\left|\sum_{l=1}^m \chi_{\omega_l}^{j+1}(x)[\Psi_{x_l}^{j+1}(x) - \Psi_{x_l}^{j+1}(x_l)]\right|\right| \\ &+ \left|\left|\sum_{l=1}^m \chi_{\omega_l}^{j+1}(x)[\Psi_{x_l}^{j+1}(x_l) - \Phi_k^j(x_l)]\right|\right| \\ &+ \left|\left|\sum_{l=1}^m \chi_{\omega_l}^{j+1}(x)[\Phi_k^j(x_l) - \Phi_k^j(x)]\right|\right| \\ &\leqq 4^{-j-2}(1/(k+2) + 0 + (4/3)4^{-j}(1/k)) \leqq 2^{-j}(1/k)\,.\end{aligned}$$

The induction is complete.

LEMMA 5. *For* $k \in \{1, 2, \cdots\}$ *there exist functions*

$$\Phi_k\colon \mathscr{P}(A) \longrightarrow A$$

such that

(i) $||\Phi_k(x)|| = 1$ *for all* $x \in \mathscr{P}(A)$,

(ii) $[\Phi_k(x)](x) = 1$;

(iii) Φ_k *is continuous*;

(iv) $|[\Phi_k(x)](t)| \leqq 1 - 1/(k+2)$ *for all* $x \in \mathscr{P}(A)$, $t \in X \sim B(x, 1/k)$.

Proof. Let Φ_k^j be as in Lemma 4 and define $\Phi_k = \lim_{j\to\infty} \Phi_k^j$. That the limit exists follows from (v) of Lemma 4. The conclusions (i)-(iv) of the present lemma now follow from the corresponding parts of Lemma 4.

By the discussion preceding Lemma 1, the proof of the theorem is complete.

REMARK. Our proof yields something more general. Indeed, instead of assuming X to be metric, one need only assume that the relative topology on $\mathscr{P}$ has a σ-locally finite base. By [5], p. 128, this is equivalent to assuming that $\mathscr{P}$ is metric, hence paracompact, and the proof goes through as before.

The referee has kindly observed that given our Lemma 3, one can use Theorem 3.1″ of [6] to prove that if X is compact Hausdorff and A is separable then the theorem holds. This is a weaker result than the one outlined in the preceding paragraph. Moreover, the proof using [6] is not essentially shorter than the elementary one presented here, and the construction of Φ as the unifiorm limit of discontinuous functions has intrinsic interest.

REMARK. It would be interesting to know whether, in the presence of differentiable structure in X and A, the peaking functions may be chosen to vary differentiably.

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