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SUPERHARMONIC INTERPOLATION IN SUBSPACES OF $C_c(X)$

L. ASIMOW

Let E be a closed subset of the compact Hausdorff X and let A be a closed separating subspace of $C_C(X)$. Let ρ be a dominator (strictly positive, *l.s.c.*) defined on $X \times T$, T the unit circle in C. Conditions, formulated in terms of boundary measures, are discussed for approximate and exact solutions to the problem of finding ρ-dominated extensions in A of functions $g \in (A|_E)^-$ satisfying re tg $(x) \leqq \rho(x, t)$ on $E \times T$. Various interpolation theorems of Rudin-Carleson type for superharmonic dominators are incorporated into this framework.

We do not assume that A contains the constant functions. We denote $M(X) = C(X)^*$, the space of regular Borel measures on X.

We consider $N = M(E)$ as situated in $M(X)$ as the range of the projection $\pi_1\mu = \mu|_E$ and denote the complementary projection $\pi_2\mu = \mu|_{X\setminus E}$. Thus $(A|_E)^\perp$ is identified with the subspace $A^\perp \cap N$ in $M(X)$.

We call $\mu \in M(X)$ a *boundary measure* if $|\mu|$ is maximal with respect to the Choquet ordering as a measure of X (embedded by evaluation) in the w^* compact unit ball A_1^*. If $1 \in A$ then this is the same as $|\mu|$ being maximal on the state space S_A, as $X \subset S_A$, a w^* closed face of A_1^*.

For brevity we denote the boundary measures by $\partial_A M(X)$, or $\partial M(X)$, if A is understood, and in general, adopt the convention of writing $\partial_A S$ for $S \cap \partial_A M(X)$. Thus, $\partial_A A^\perp$ refers to the boundary measures annihilating A. The space A^* is the quotient space $M(X)/A^\perp$ and images under the quotient map are denoted $\hat{\mu}$ for $\mu \in M(X)$. A subset $S \subset M(X)$ is called *A-stable* if $\hat{S} = (\partial_A S)^\wedge$.

We call E an *interpolation set* if $A|_E$ is closed in $C(E)$. Gamelin [8] shows that E is an interpolation set if and only if there is a k; $0 \leqq k < \infty$, such that for each $m \in A^\perp$,

$$(1) \qquad ||\pi_1 m + A^\perp \cap N|| \leqq k||\pi_2 m|| .$$

The best value of k is called the *extension constant*, $e(A, E)$.

In [10] Roth introduces a general framework for interpolation problems by means of a *dominator*, ρ, defined as a strictly positive *l.s.c.* extended real-valued function on $X \times T$ (T the unit circle in C). We let

$$U = \{f \in C(X): re\ \mathrm{tf}\ (x)/\rho(x, t) \leqq 1 \text{ for all } (x, t) \in X \times T\}$$

and write

$$||f||_\rho = \sup\{re\, tf(x)/\rho(x, t)\colon (x, t) \in X \times T\}$$

for the Minkowski functional of U. Thus $||f||_\rho \leqq 1$ if and only if $re\, tf(x) \leqq \rho(x, t)$, $(x, t) \in X \times T$. Then $||\mu||_\rho$, $\mu \in M(X)$, refers to the polar functional given by

$$||\mu||_\rho = \sup\{re(f, \mu)\colon f \in U\} .$$

Since ρ is *l.s.c* and positive there is a constant c such that $||f||_\rho \leqq c||f||$ (the uniform norm corresponding to $\rho \equiv 1$) and if ρ is bounded above the two are equivalent.

We say E is an *approximate ρ-interpolation set* for A if E is an interpolation set and for each $g \in (A|_E)^-$ and $\varepsilon > 0$ there is an $f \in A$ such that $f|_E = g$ and $||f||_\rho < ||g||_\rho + \varepsilon$. We say E is an *exact ρ-interpolation set* if f can be chosen with $||f||_\rho = ||g||_\rho$. It is shown in [5] that for bounded ρ, E is an approximate ρ-interpolation set for A if and only if for each $m \in A^\perp$,

$$(2) \qquad ||\pi_1 m + A^\perp \cap N||_\rho \leqq ||-\pi_2 m||_\rho .$$

If, in addition, the image $\hat{U}$ of U^0 under the quotient map is *decomposable* by $\hat{N}$ then E is an exact ρ-interpolation set. If there is an s, $0 \leqq s < 1$, such that for each $m \in A^\perp$,

$$(3) \qquad ||\pi_1 m + A^\perp \cap N||_\rho \leqq s||-\pi_2 m||_\rho$$

then the above holds and E is ρ-exact for A. Gamelin's results [8] can be phrased as follows: Let G be a compact set in $X \backslash E$ and let

$$\rho(G, k)(x, t) = \begin{cases} 1 \text{ for } (x, t) \in E \times T \\ k \text{ for } (x, t) \in G \times T \\ 1 \vee k \text{ otherwise.} \end{cases}$$

Then E is an approximate $\rho(G, k)$-interpolation set for all such G if and only if (1) holds and if, in addition, $e(A, E) < 1$ then E is an exact ρ-interpolation set for any continuous T-invariant ρ such that $\rho > e(A, E)$ on $X \times T$. This was obtained in abstract form using polar techniques by Ando [3].

In [6] Briem shows that if E is a subset of the Choquet boundary, $\partial_A X$, then E is an interpolation set if and only if (1) holds only for $m \in \partial_A A^\perp$. Further, if X is metrizable then (1) holds for $\partial_A A^\perp$ if and only if E is an approximate $\rho(G, k)$-interpolation set for each compact $G \subset \partial_A X \backslash E$. The A-stability of the unit ball $M_1(X)$ (Hustad's theorem [9]) and of $N = M(E)$ (since $E \subset \partial_A X$) are essential here. If (1) holds for $\tilde{e}(A, E) < 1$ (again, $\tilde{e}$ is the smallest k such that (1) holds for all $m \in \partial A^{\perp}$) then E is $\rho(G, k)$ exact for any $G \subset \partial_A X \backslash E$ and $k > \tilde{e}$.

If (1) holds for all $m \in \partial_A A^{\perp}$ with $k = 0$ this can be expressed as

$$(4) \qquad m \in \partial_A A^{\perp} \text{ imples } \pi_1 m \in A\ .$$

The set E is called an M-set if $M(E)$ is A-stable and (4) holds. Roth [10] shows that if E is an M-set and ρ is a bounded A-*superharmonic* (if $1 \in A$ this means $\rho(x, t) \geqq \int \rho(\cdot, t) d\mu$ for any $\mu \in M_1^+(X)$ and $\hat{\mu} = x \in X \subset A_1^*$) dominator then E is an exact ρ-interpolation set for A. This generalizes the Alfsen-Hirsberg theorem [2] which deals with T-invariant ρ and $E \subset \partial_A X$.

In this note we consolidate these results by showing that for E an interpolation set with $M(E)$ A-stable and ρ A-superharmonic then E is an approximate ρ-interpolation set if and only if (2) holds for $m \in \partial_A A^{\perp}$. If in addition $\hat{U}$ is decomposable by $\hat{N}$ in A^* then the interpolation is exact. This is the case if ρ is bounded and (3) holds for $m \in \partial_A A^{\perp}$. (If ρ is bounded and (2) or (3) holds then E is already an interpolation set.) We give a measure theoretic condition for the decomposability of $\hat{U}$ and show by means of simple examples of $A(K)$ spaces that exactness of interpolation can be deduced in this way even though equality holds in (2) which, of course, precludes the use of (3).

1. Hustad-Roth stability theorems. Let A be a closed separating subspace of $C(X)$. Define $\Phi\colon C(X) \to C(X \times T)$ by $\Phi f(x, t) = tf(x)$. By separating we shall mean that the range of $\Phi|_A$ separates the points of $X \times T$. This assumption can be avoided, as is shown in Fuhr-Phelps [7], but at the expense of additional technicalities. If $\nu \in M(X \times T)$ then the Hustad map is given by

$$\mu = \Phi^* \nu \in M(X);\ \mu(f) = \int_{X \times T} tf(x) d\nu(x, t)\ .$$

If $\Phi = \Phi|_A$ has range $B \subset C(X \times T)$ and ν is a maximal probability measure on $X \times T \subset B_1^*$ representing $\tilde{L} \in B_1^*$ then Hustad's theorem says $\mu = \Phi^* \nu$ belongs to $\partial_A M(X)_1$ with $\hat{\mu} = L = \phi^* \tilde{L}$. We combine this with the following observations concerning T-invariant A-superharmonic dominators to obtain a general stability theorem due to Roth [11].

Thus let ρ be a strictly positive *l.s.c.* extended real-valued function on X such that for each $x \in X$ and $\mu \in M_1^+(X)$ with $\hat{\mu} = x \in A^*$, we have $\rho(x) \geqq \int_X \rho d\mu$, that is, ρ is A-superharmonic. If $U = \{f \in C(X)\colon ref/\rho \leqq 1\}$ then U^0 is a w^* compact convex subset of the positive cone $M^+(X)$, and we let $\hat{U}$ be the quotient image in A^*. Take $\bar{R}^+$ to be the one-point-compactification of R^+ and

$$X_0 = \{(x, s) \in X \times \bar{R}^+ : \rho(x) \leqq s \leqq +\infty\} ,$$
$$Y_0 = \{(x, \rho(x)) \in X_0 : \rho(x) < \infty\} ,$$
$$Y_\infty = \{(x, \rho(x)) \in X_0 : \rho(x) = +\infty\} .$$

Since ρ is *l.s.c.*, $Y_0 \cup Y_\infty$ and Y_∞ are both G_δ subsets of X_0 so that Y_0 is a Borel set. Define

$$\psi : C(X) \longrightarrow C(X_0);\ \psi f(x, s) = f(x)/s ,$$

and let $\theta = \psi|_A$ with (not necessarily closed) range $B \subset C(X_0)$. Since ρ is strictly positive ψ is bounded and θ^* is one-to-one from B^* into A^*. Let

$$\phi_0 : X_0 \longrightarrow B_1^*$$

be the evaluation map and let $\hat{V} = w^* - \overline{co}\phi_0(X_0)$.

PROPOSITION 1.1. *Let ρ be a T-invariant A-superharmonic dominator on X as above.*

(1) *ϕ_0 is one-to-one on $X_0 \backslash (X \times \{\infty\})$, $X \times \{\infty\} = \phi_0^{-1}(0)$, and $\theta^* \hat{V} = \hat{U}$.*

(2) *If ν is a maximal probability measure on $\hat{V}$ then $\nu[\phi_0(Y_0) \cup \{0\}] = 1$ and ν may be identified with the measure on Y_0 given by $\nu \circ \phi_0$.*

(3) If *ν is as in* (2) *and $\mu = \psi^* \nu$ then for any bounded Borel function h on X*

$$\int_X h d\mu = \int_{Y_0} (h(x)/\rho(x) d\nu(x, \rho(x)) .$$

In particular, $\mu \in U^0$.

(4) *Let $\mu_0 \in M_1^+(X)$ with $\hat{\mu}_0 = x_0 \in X \subset A_1^*$ and define $\tilde{\mu}_0 \in M(X_0)$ by*

$$\tilde{\mu}_0(F) = (1/\rho(x_0)) \int_X F(x, \rho(x)) \rho(x) d\mu_0(x) .$$

Then for any bounded Borel function h on X

$$\int_{X_0} (h(x)/s) d\tilde{\mu}_0(x, s) = (1/\rho(x_0)) \int_X h d\mu_0 .$$

In particular $\tilde{\mu}_0 \geqq 0$, $\tilde{\mu}_0(X_0) = \tilde{\mu}_0(Y_0) \leqq 1$, and $\tilde{\mu}_0$ represents $(x_0, \rho(x_0)) \in \hat{V}$.

(5) *If ν is maximal on $\hat{V}$ then $\mu = \psi^* \nu$ is maximal on $K = \overline{co}X \subset A^*$.*

Proof. (1) The separation theorem shows $\hat{U} = w^*\overline{co}\{x/s: (x, s) \in X_0\}$. Now

$$\theta^* \circ \phi_0(x, s) = x/s \in A^*$$

so the rest of (1) follows from the fact that A separates points in X. For (2) let $p = 1 - \chi_{\{0\}}$ on $\hat{V}$ and note that the lower envelope $\breve{p}$ is the Minkowski functional of $\hat{V}$. Since ν is maximal,

$$1 = \nu[\{x: p(x) = \breve{p}(x)\}] = \nu[\{x: \breve{p}(x) = 1 \text{ or } 0\}] .$$

Now $\lambda \geqq 1$ implies $\phi_0(x, \lambda s) = (1/\lambda)\phi_0(x, s)$, so that

$$\nu[\phi_0(Y_0) \cup \{0\}] = 1 .$$

If $f \in C(X)$ then $\psi^*\nu(f) = \int_{X_0}(f(x)/s)d\nu(x, s) = \int_{Y_0}(f(x)/\rho(x))d\nu(x, \rho(x))$ and so (3) holds.

(4): If $F \in C(X_0)$ and $0 \leqq F \leqq 1$ then

$$0 \leqq \tilde{\mu}_0(F) \leqq (1/\rho(x_0))\int_X \rho d\mu_0 \leqq 1 .$$

Thus $\tilde{\mu}_0 \geqq 0$, $\tilde{\mu}_0(X_0) \leqq 1$ and $\mu_0[\{x: \rho(x) = +\infty\}] = 0$. For $F = \psi h$,

$$\begin{aligned}\tilde{\mu}_0(F) &= \int_{X_0}(h(x)/s)d\tilde{\mu}_0(x, s) \\ &= (1/\rho(x_0))\int_X h d\mu_0 .\end{aligned}$$

(5): Let f be a continuous convex function of K and denote the upper envelope of f by $\hat{f}(K)$, where [1, I. 3.6]

$$\hat{f}(K)(x_0) = \sup\{\mu(f): \mu \in M_1^+(X) \text{ and } \hat{\mu} = x_0 \in A^*\} .$$

If $g = \psi(f|_X)$ then $g \in C(X_0)$ with $g \equiv 0$ on $X \times \{\infty\}$. If $\tilde{\mu}_0 = x_0$ and $\tilde{\mu}_0$ is as in (4) then $\tilde{\mu}_0$ represents $(x_0, \rho(x_0)) \in \hat{V}$ and the upper envelope, $\hat{g}(\hat{V})$, satisfies

$$\hat{g}(\hat{V})(x_0, \rho(x_0)) \geqq \sup\{\tilde{\mu}_0(g): \hat{\mu}_0 = x_0\} = (1/\rho(x_0))\hat{f}(K)(x_0)$$

by part (4). Thus, using part (3), and [1, I. 4.5],

$$\int_X [\hat{f}(K) - f]d\mu = \int_{Y_0}[\hat{f}(K) - f]/\rho d\nu \leqq \int_{Y_0}[\hat{g}(\hat{V}) - g]d\nu = 0$$

since ν is maximal. Hence, μ is maximal on K.

We now consider the case where ρ is defined on $X \times T$. We say such a ρ is *A-superharmonic* if for each $(x, t) \in X \times T$ and $\mu \in M(X \times T)_1^+$ with

$$\int_{X\times T} sf(y)d\mu(y, s) = tf(x) \text{ for all } f \in A$$

we have $\rho(x, t) \geqq \int_{X\times T} \rho d\mu$.

THEOREM 1.2 (Hustad-Roth). *If ρ is an A-superharmonic dominator then U^0 is A-stable.*

Proof. Let $\Phi: C(X) \to C(X \times T)$; $\Phi f(x, t) = tf(x)$ and let

$$U^1 = \{F \in C(X \times T): reF(x, t)/\rho(x, t) \leqq 1\}$$

and $\phi = \Phi|_A$ with range B.

Let $\Psi: C(X \times T) \to C(X_0)$; $\Psi F(x, t, s) = F(x, t)/s$, where X_0 is the closed epigraph of ρ in $(X\times T)\times \bar{R}^+$. Now $\Phi U \subset U^1$ and $\phi(A \cap U) = B \cap U^1$. Given $L \in \hat{U}$, let $\tilde{L} \in (U^1)^\wedge \subset B^*$ and $L' \in \hat{V}$ (as in Proposition 1.1) with $\theta^* L' = \tilde{L}$ and $\phi^* \tilde{L} = L$. We have $B_1^* = w^*\overline{co}(X \times T)$ and the hypothesis says ρ on $X \times T$ is B-superharmonic. Hence the results of Proposition 1.1 apply. Thus if ν' is maximal on $\hat{V}$ representing L' then 1.1 (3) and (5) show $\nu = \Psi^*\nu'$ is maximal on B_1^* representing $\tilde{L} \in (U^1)^\wedge$. Then $\mu = \phi^*\nu \in U^0$ and $\hat{\mu} = L \in \hat{U}$. Furthermore, Hustad's theorem shows μ is a boundary measure.

If $1 \in A$ then the condition for A-superharmonicity is somewhat simpler.

PROPOSITION 1.3. *If $1 \in A$ then ρ is A-superharmonic if and only if for each $\mu \in M_1^+(X)$ with $\hat{\mu} = x$,*

$$\rho(x, t) \geqq \int_X \rho(\cdot, t) d\mu .$$

Proof. If ρ is A-superharmonic and $\mu \in M_1^+(X)$ with $\hat{\mu} = x$ we can embed X as $X \times \{t\} \subset X \times T$ so that the measure μ satisfies

$$\int_{X\times T} sf(y)d\mu = tf(x)$$

and hence

$$\rho(x, t) \geqq \int_{X\times\{t\}} \rho(x, t)d\mu = \int_X \rho(\cdot, t)d\mu .$$

Conversely, if $\mu \in M_1^+(X \times T)$ and represents tx then, since $1 \in A$, we have $t\overline{co}X = tS_A (S_A$ the state space of $A)$ is a face of A_1^*. Hence $\text{supp}\mu \subset X \times \{t\}$ and the measure $\mu_1(f) = \int_{X\times T} f(x)d\mu$ represents x so that

$$\rho(x, t) \geqq \int_X \rho(\cdot, t) d\mu_1 = \int_{X\times T} \rho d\mu .$$

2. **Dominated interpolation.** If E is a compact subset of X we let

$$M = \{f \in C(X): f|_E = 0\}$$

and denote $M \cap A$ by $E^\perp$. It is well known that E is an interpolation set for A if and only if $A + M$ is closed in $C(X)$ and this in turn is equivalent to $\hat{N}$ being w^* (or norm) closed in A^*. The following characterization of approximate ρ-interpolation sets follows from results in [5; 4.2]. We denote $N = M(E) \subset M(X)$.

THEOREM 2.1. *Let ρ be a (strictly positive l.s.c) dominator on X such that either ρ is bounded or E is an interpolation set. The following are equivalent*:

(i) *E is an approximate ρ-interpolation set for A,*

(ii) *$A + M$ is closed in $C(X)$ and*

$$(A + M) \cap (U + M) = (A \cap U + M)^- ,$$

(iii) $\hat{U} \cap \hat{N} = (U^0 \cap N)^\wedge$,

(iv) $\|\mu + A^\perp \cap N\|_\rho = \|\mu + A^\perp\|_\rho$ *for all* $\mu \in N$,

(v) $\|\pi_1 m + A^\perp \cap N\|_\rho \leqq \|-\pi_2 m\|_\rho$ *for all* $m \in A^\perp$.

For $x \in A^*$ we write $\|x\|_\rho$ for the Minkowski functional of $\hat{U}$ so that if $\hat{\mu} = x$

$$\|x\|_\rho = \|\mu + A^\perp\|_\rho .$$

The set U^0 is *split*, that is, $\|\mu\|_\rho = \|\pi_1\mu\|_\rho + \|\pi_2\mu\|_\rho$ [10, 5].

PROPOSITION 2.2. *Let N and U^0 be A-stable sets in $M(X)$. Then for $\mu \in \partial_A M(X)$,*

(1) $\|\mu + A^\perp\|_\rho = \|\mu + \partial A^\perp\|_\rho = \|\hat{\mu}\|_\rho$,

(2) $\|\mu + N + A^\perp\|_\rho = \|\pi_2\mu + \pi_2\partial A^\perp\|_\rho$ $(\pi_2\mu = \mu|_{X\setminus E})$,

(3) If $\|\mu\|_\rho = \|\hat{\mu}\|_\rho$ *then*

$$\|\pi_i\mu\|_\rho = \|(\pi_i\mu)^\wedge\|_\rho \quad (i = 1, 2) .$$

Proof. If $\mu \in \partial M(X)$ and $\|\hat{\mu}\|_\rho \leqq r$ then $\mu = r\nu + m$ with $\nu \in U^0$ and $m \in A^\perp$. The stability of U^0 shows we can assume $\nu \in \partial U^0$, so that $m \in \partial A^\perp$. Then (1) follows. If $\mu = r\nu + \eta + \zeta$ with $\nu \in \partial U^0$, $\eta \in \partial N$, $\zeta \in A^\perp$, then $\zeta \in \partial A^\perp$ and $\pi_2\mu = r\pi_2\nu + \pi_2\zeta \in r\pi_2 U^0 + \pi_2\partial A^\perp$. Conversely, if $\pi_2\mu = r\nu + \pi_2\zeta$, $\nu \in \partial U^0$, $\zeta \in \partial A^\perp$ then

$$\mu = r\nu + (\pi_1\mu - \pi_1\zeta) + \zeta \in rU^0 + \partial N + \partial A^\perp .$$

For (3), we have

$$\begin{aligned}||\pi_1\mu||_\rho &\geqq ||(\pi_1\mu)^\wedge||_\rho = ||\pi_1\mu + A^\perp||_\rho = ||\mu - \pi_2\mu + A^\perp||_\rho \\ &\geqq ||\mu||_\rho - ||\pi_2\mu + A^\perp||_\rho \geqq ||\mu||_\rho - ||\pi_2\mu||_\rho = ||\pi_1\mu||_\rho .\end{aligned}$$

Since we do not assume $1 \in A$, we take the *Choquet boundary*, $\partial_A X$, to be $X \cap extA_1^*$. There are two main instances where the A-stability of N can be deduced.

PROPOSITION 2.3. *Let E be a closed subset of X such that either*

(a) $E \subset \partial_A X$ *or*

(b) $E = F \cap X$, *F a w^* closed face of A_1^*.*

Then N is A-stable.

Proof. In the case (a) each probability measure on E is maximal and so the result follows since $\overline{co}E$ spans $\hat{N}$. In case (b) each maximal probability measure μ with $\hat{\mu} \in \overline{co}E$ has its support on $(ext\ F)^- \subset E$.

THEOREM 2.4. *Let E be a closed subset of X such that either*

(a) $E \subset \partial_A X$, *or*

(b) $E = F \cap X, F$ *a closed face of* A_1^*.

Let ρ be an A-superharmonic dominator such that either ρ is bounded or E is an interpolation set. Then the following are equivalent:

(i) *E is an approximate ρ-interpolation set,*

(ii) $||\mu + A^\perp \cap N||_\rho = ||\mu + \partial A^\perp||_\rho$ *for all* $\mu \in \partial N$,

(iii) $||\pi_1 m + A^\perp \cap N||_\rho \leqq ||-\pi_2 m||_\rho$ *for all* $m \in \partial A^\perp$.

Proof. The hypotheses imply that U^0 and N are A-stable and so 2.2. (1) shows for $\mu \in \partial M$,

$$||\mu + A^\perp||_\rho = ||\mu + \partial A^\perp||_\rho .$$

Thus (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) follows from 2.1. If (ii) holds and $x \in \hat{U} \cap \hat{N}$ then choose $\mu \in \partial N$ with $\hat{\mu} = x$ and $\mu \in U^0 + A^\perp$. Then

$$||\mu + A^\perp \cap N||_\rho = ||\mu + \partial A^\perp||_\rho = ||\mu + A^\perp||_\rho \leqq 1$$

so that $\mu = \nu + m$; $\nu \in U^0$, $m \in A^\perp \cap N$. Hence $\nu \in N$ and $\hat{\mu} = x = \hat{\nu} \in (U^0 \cap N)^\wedge$. Thus 2.1 (iii) holds and hence (i) is shown.

The exactness of ρ-interpolation is characterized by the sum

$A \cap U + E^{\perp}(E^{\perp}$ the ideal of functions in $C(X)$ vanishing on E) being closed in A, a condition which is implied by the decomposability of $\hat{U}$ by $\hat{N}$ in A^* [5; Theorem 3.2]. If E is an interpolation set (so that $\hat{N}$ if w^* closed in A^*) then $\hat{U}$ is said to be *decomposable* by $\hat{N}$ if there is an $\alpha \geqq 1$ such that each $x \in \hat{U}$ is a convex combination of elements y, z with $y \in \hat{U} \cap \hat{N}, z \in \hat{U}$ and $||z|| \leqq \alpha||z + \hat{N}||$ (dual uniform norm).

The condition for decomposability, and hence exact interpolation, can be formulated in terms of representing measures in $M(X)$. We illustrate this for boundary measures in the case where ρ is superharmonic.

THEOREM 2.5. *Let E be a closed subset of X and A a closed separating subspace such that either*

(a) $E \subset \partial_A X$, *or*

(b) $E = F \cap X, F$ *a closed face of* A_1^*,

and let ρ be an A-superharmonic dominator such that either ρ is bounded or E is an interpolation set.

If for each $x \in \hat{U}$ there is a $\mu \in \partial_A U^0$ with $\hat{\mu} = x$ and

$$||\pi_2\mu + \partial A^{\perp}|| \leqq \alpha||\pi_2\mu + \pi_2\partial A^{\perp}||$$

(α a constant independent of μ) then E is an exact ρ-interpolation set.)

Proof. Given $x \in \hat{U}$ choose a boundary measure μ satisfying $\hat{\mu} = x$, $||\hat{\mu}||_\rho = ||\mu||_\rho$ and $||\pi_2\mu + \partial A^{\perp}|| \leqq \alpha||\pi_2\mu + \pi_2\partial A^{\perp}||$. Now $||\mu||_\rho = ||\pi_1\mu||_\rho + ||\pi_2\mu||_\rho$ shows that μ is a convex combination of $\mu_1 \in U^0 \cap N$ and $\mu_2 \in U^0$, scalar multiples of $\pi_1\mu, \pi_2\mu$ respectively. Thus, $||\mu_2 + \partial A^{\perp}|| \leqq \alpha||\mu_2 + \pi_2\partial A^{\perp}||$ and x is a convex combination of $y \in (U^0 \cap N)^\wedge$ and $z \in \hat{U}$ with (using 2.2 (1) and (2))

$$\begin{aligned} ||z|| = ||\mu_2 + \partial A^{\perp}|| \leqq \alpha||\mu_2 + \pi_2\partial A^{\perp}|| = \alpha||\mu + N + A^{\perp}|| \\ = \alpha||z + \hat{N}|| \,. \end{aligned}$$

This shows that $(U^0 \cap N)^\wedge = \hat{U} \cap \hat{N}$ and that $\hat{U}$ is decomposable by $\hat{N}$. Therefore E is an exact ρ-interpolation set.

If E is an M-set then $\pi_2\partial A^{\perp} \subset \partial A^{\perp}$ so that

$$||\pi_2\mu + \pi_2\partial A^{\perp}|| \geqq ||\pi_2\mu + \partial A^{\perp}||$$

and the condition of 2.5 is automatically satisfied (for A-stable U^0). More generally, if U^0 and N are A- stable and, for some $s < 1$

$$||\pi_1 m + A^{\perp} \cap N||_\rho \leqq s||-\pi_2 m||_\rho \text{ for all } m \in \partial A^{\perp}$$

then a computation based on [5; 4.8] shows the condition of Theorem 2.5 holds, so that E is an exact ρ-interpolation set.

COROLLARY 2.6. *If E is an M-set for the closed separating subspace $A \subset C(X)$ then E is an exact ρ-interpolation set for A for any A-superharmonic dominator ρ.*

Proof. If E is an M-set then $\hat{N}$ is the range of a projection in A^* so that E is an interpolation set for A. The conclusion then follows from 2.5.

3. **Examples.** We illustrate the results of § 2 with various choices of ρ. First, let X be a compact *metric* space with E a closed subset and $M(E)$ A-stable for the closed separating subspace $A \subset C(X)$. Let G be the collection of compact subsets $G \subset \partial_A X \backslash E$ and let $\rho = \rho(G, k)$ be the dominator mentioned in the introduction. Then (for $k < \infty$)

$$(1) \qquad \|\pi_1 m + A^\perp \cap N\| \leqq k\|\pi_2 m\| \text{ for all } m \in \partial A^\perp$$

if and only if E is an approximate $\rho(G, k)$-interpolation set for all $G \in \mathscr{G}$. To see this we note that since $G \subset \partial_A X$, U^0 is A-stable so that the second property holds if and only if

$$(2) \qquad \|\pi_1 m + A^\perp \cap N\|_\rho \leqq \|-\pi_2 m\|_\rho \text{ for all } m \in \partial A^\perp, G \subset \mathscr{G} .$$

It follows easily from [5; 4.1] that if $Y = X \backslash (E \cap G)$ then

$$\|\mu\|_\rho = \|\mu|_E\| + k\|\mu|_G\| + (1 \vee k)\|\mu|_Y\|$$

so that

$$\|\pi_1 m + A^\perp \cap N\| = \|\pi_1 m + A^\perp \cap N\|_\rho$$

and, since for boundary measures μ, the metrizability of X gives

$$|\mu|(X \backslash E) = |\mu|(\partial_A X \backslash E) = \sup\{|\mu|(G) \colon G \in \mathscr{G}\} ,$$

we have

$$k\|\pi_2 m\| = \sup\{\|\pi_2 m\|_\rho \colon \rho = \rho(G, k),\ G \in \mathscr{G}\} .$$

The equivalence of (1) and (2) is now immediate. If (1) holds for $k_0 < 1$ and $k_0 < k \leqq 1$ then for $\rho = \rho(G, k)$

$$\begin{aligned} \|\pi_1 m + A^\perp \cup N\|_\rho &= \|\pi_1 m + A^\perp \cap N\| \leqq k_0(\|m|_G\| + \|m|_Y\|) \\ &\leqq (k_0/k)(k\|m|_G\| + \|m|_Y\|) = (k_0/k)\|\pi_2 m\|_\rho \end{aligned}$$

so that E is an exact $\rho(G, k)$-interpolation set for A.

The study of sufficient conditions for the A-convex hull of E to be a generalized peak set (we now assume $1 \in A$) has been shown [4] to be related to an ordering on $C_c(X)$ and $M(X)$ induced by choosing P to be a closed proper convex cone with nonempty interior in $\boldsymbol{C}$. Let α, β be the generators (of modulus one) of the dual cone $P^* = \{z: re az \geqq 0 \text{ for all } a \in P\}$. We denote by e the element of P such that $ree\gamma = 1$ $(\gamma = \alpha, \beta)$. If $f \in C_c(X)$ we say $f \geqq 0(P)$ if $f(X) \subset P$ and $\mu \geqq 0(P^*)$ means $\mu(B) \in P^*$ for all Borel sets $B \subset X$. Then the function $\boldsymbol{e} \equiv e$ becomes an order unit for $C(X)$ in which the order unit norm $\|\cdot\|_e$ (equivalent to the uniform norm) is given by

$$\rho(x, t) = \begin{cases} 1 \text{ for } t = \pm\gamma \\ 1/c \text{ for } t \neq \pm\gamma\,, \end{cases} \quad \gamma = \alpha, \beta$$

where c is a constant such that

$$cz| \leqq |re\alpha z| \vee |re\beta z|\ .$$

This provides an example of a ρ which is not T-invariant.

Let ρ^+ and ρ^- be strictly positive $l.s.c.$ functions on X and take

$$\rho(x, t) = \begin{cases} \rho^+(x) \text{ on } X \times \{1\} \\ \rho^-(x) \text{ on } X \times \{-1\} \\ +\infty \text{ otherwise.} \end{cases}$$

Then $U = \{f \in C(X): -\rho^- \leqq ref \leqq \rho^+\}$. If $\mu \in U^0$ and f is real then λ if $\in U$ for all real λ so that

$$1 \geqq re\mu(\lambda \text{ if}) = -\lambda im\mu(f)$$

and hence $im\mu(f) = 0$. Thus μ is a real measure and $U^0 \subset reM(X)$.

If A_0 is a real subspace of $C(X)$ then we can apply the results of § 2 to the self-adjoint space $A_0 + iA_0 = A$. Then $\|f\|_\rho = \|ref\|_\rho$ and $m \in A^\perp$ if and only if $m = m_1 + im_2$ with m_1, m_2 real measures in $A^\perp$. Also m is a boundary measure if and only if m_1, m_2 are boundary. Hence E is an approximate (exact) ρ-interpolation set for A if and only if it is for $A_0 = reA$, and the measure conditions of § 2 need only involve real measures in $M(X)$. If X is a compact convex subset of a locally convex space and $A_0 = A(X)$ (real affine continuous functions) then ρ is A-superharmonic if and only if $\rho^+ = (\rho^+)\hat{}$ and $\rho^- = (\rho^-)\hat{}$, that is, if and only if ρ^+ and ρ^- are concave on X.

Let X be a square in R^2 with vertices denoted $\{1, 2, 3, 4\}$ with

$E = \{1, 2\}$ diagonally opposite and $A_0 = A(X)$, ρ^+, $\rho^- \equiv 1$. Then $\partial A^\perp$ is a one-dimensional subspace of the four-dimensional space $\partial M(X)$ spanned by the point-masses $\{\delta_i\}_{i=1}^4$. A generator for $\partial A^\perp$ is $m = \delta_1 + \delta_2 - \delta_3 - \delta_4$. Clearly $A^\perp \cap N = \{0\}$ since coE is a simplex and so

$$||\pi_1 m + A^\perp \cap N|| = ||\pi_1 m|| = ||\pi_2 m|| .$$

This shows E is an approximate ρ-interpolation set for $A(X)$. Obviously E is in fact an exact interpolation set, but this cannot be concluded from a condition such as (3) in the introduction. Nevertheless, the condition of 2.5 holds, since if

$$\mu = \Sigma \lambda_i \delta_i$$

then

$$||\mu|| = \Sigma |\lambda_i|$$

and

$$||\pi_2\mu + \pi_2 \partial A^\perp|| = inf\{|\lambda_3 - \lambda| + |\lambda_4 - \lambda| : \lambda \in R\} = |\lambda_4 - \lambda_3| .$$

If λ_3 and λ_4 are opposite in sign then

$$||\pi_2\mu + \partial A^\perp|| \leqq ||\pi_2\mu|| = |\lambda_3| + |\lambda_4| = |\lambda_4 - \lambda_3| = ||\pi_2\mu + \pi_2\partial A^\perp|| .$$

If, say $0 \leqq \lambda_3 \leqq \lambda_4$, consider $\nu = \mu + \lambda_3 m$. Then $\hat{\nu} = \hat{\mu}$ and

$$||\nu|| = \Sigma|\lambda_i - \lambda_3| \leqq (|\lambda_1| + |\lambda_2| + 2|\lambda_3|) + |\lambda_4| - |\lambda_3| = ||\mu||$$

and

$$||\pi_2\nu + \partial A^\perp|| \leqq ||\pi_2\nu|| = \lambda_4 - \lambda_3 = ||\pi_2\mu + \pi_2\partial A^\perp|| .$$

We conclude with an example of an approximate interpolation set which is not exact. Let X be the unit ball of the sequence space l^1(w^* topology) and let $\rho \equiv 1$. Then take $A = c_0$, the pre-dual of l^1, so that $||a||_\rho = ||a||_\infty = sup\{|a_n|\}$. Let E be the singleton $\{x^0\}$, $x_n^0 = 1/2^n$, $n = 1, 2, \cdots$. If $(a, x^0) = 1$ then $\sum_{n=1}^\infty a_n/2^n = 1$ so that some a_n must be greater than one. Clearly we can find such an a with $||a|| \leqq 1 + \varepsilon$ for any $\varepsilon > 0$. Thus E is an approximate, but not exact, ρ-interpolation set.

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SYRACUSE UNIVERSITY
SYRACUSE, NY 13210