PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 1, 1979

2-FACTORIZATION IN FINITE GROUPS

MAKOTO HAYASHI

Let G be a finite group, and S be a nonidentity 2-subgroup of G. Then, it is naturally conjectured that there exists a nonidentity $N_G(S)$-invariant subgroup of S, whose normalizer contains all the subgroups H of G with the following properties: $(\alpha)S$ is a Sylow 2-subgroup of H; $(\beta)H$ does not involve the symmetric group of degree four; and $(\gamma)C_H(O_2(H)) \subseteq O_2(H)$. The purpose of this paper is to give a partial answer to this problem.

1. Introduction. Suppose π is a set of primes, and X is a finite group. Let $\mathscr{D}(X: \pi)$ be the family of all groups D that are involved in X with the following properties; $(\alpha)D$ possesses a normal simple subgroup E, $(\beta)C_D(E) \subseteqq E$ (that is, D/C induces outer automorphisms of E), and $(\gamma)D/E$ involves a dihedral group of order $2p$ for some prime $p(\geqq 5)$ in π.

THEOREM. *Let π be a set of primes. Suppose G is a finite group, and S is a nonidentity 2-subgroup of G. Assume that for any nonidentity subgroup T of S which is normal in $N_G(S)$,*

(1) *S is normal in some Sylow 2-subgroup of $N_G(T)$; and*

(2) *$\mathscr{D}(N_G(T)/G_G(T): \pi) = \phi$.*

Then there exists a nonidentity subgroup $W(S)$ of S which satisfies the following conditions (a) *and* (b):

(a) *$W(S)$ is normal in $N_G(S)$; and*

(b) *$W(S)O(H)$ is normal in H for any solvable π-subgroup H of G which satisfies the following conditions (α) and (β):*

(α) *S is a Sylow 2-subgroup of H: and*

(β) *H is S^4-free, where S^4 denotes the symmetric group of degree four.*

REMARK 1.1. The condition (1) of the theorem is satisfied, whenever S is normal in some Sylow 2-subgroup of G.

In general, suppose p is a prime, G is a finite group, and S is a nonidentity p-subgroup of G. Let $Qd(G, S)$ be the family of all subgroups H of G that satisfy the following conditions: (α) S is a Sylow p-subgroup of H; and $(\beta)H$ is p-constrained, and p-stable (if $p = 2$, S^4-free). Then, what are the relations among the elements of $Qd(G, S)$? Furthermore, what are the relations between G and the elements of $Qd(G, S)$? These problems were proposed by G. Glauberman and J. G. Thompson, and for which amazing progresses have been made chiefly by them over the past ten years (c. f. [4], [5], and [6]). The theorem is for $p=2$ a weak analogue of the ZJ-Theorem [3], in which G. Glauberman treated p-groups, namely he succeeded in the Replacement Theorem for p-groups with class at most 2, p odd; And he applied it to prove that $ZJ(S) \triangleleft H$ for any element H of $Qd(G, S)$. In contrast with this, since we can not find a suitable characteristic subgroup of S, we must directly analyze the relations among the solvable elements of $Qd(G, S)$.

The §§3 and 4 are used for preliminaries. In the §5, we get expressions among the solvable elements of $Qd(G, S)$ through the intermediary S, and in the §6, we apply them to some groups involved in G. In his paper [5], G. Glauberman defined a new characteristic subgroup $\hat{J}(S)$ for a finite 2-group S which possesses good properties, in particular, in connection with $J_e(S)$ and $\Omega_1 Z(S)$. In this paper, these good properties are exploited which make possible to prove the existence of $W(S)$ which corresponds to $ZJ(S)$, but unfortunately, which is not in general characteristic in S. We shall treat nonsolvable subgroups in [9].

2. Notation and definition. All groups considered in this paper will be finite. For every finite set S, denote the number of elements of S by $|S|$. Let T and U be subsets of S. $T \backslash U$ denotes the set of all elements of T that do not belong to U. Let X be a finite group, and Y and Z be subsets of X. We write $Y \subseteq X (Y \subset X)$ to indicate that Y is a (proper) subgroup of X. Let $YZ = \{yz; y \in Y, z \in Z\}$, and $Y^Z = \{z^{-1}yz; y \in Y, z \in Z\}$. $\langle \cdots; \cdots \rangle$ denotes the group which is generated by all $\cdots$ such that $\cdots$. Let $[y, z] = y^{-1}z^{-1}yz$ for any pair of elements y and z of X, and $[Y, Z] = \langle [y, z]; y \in Y, z \in Z \rangle$. $Y \triangleleft X$ if Y is a normal subgroup of X. For a finite group W, $X \simeq W$ if X is isomorphic to W. W is involved in X if X contains subgroups X_1 and X_2 such that $X_1 \triangleright X_2$ and $X_1 / X_2 \simeq W$; otherwise X is W-free. For a set of primes π, π' denotes the set of all primes which do not belong to π. We say that X is a π-group if π contains the set of all prime divisors of $|X|$. X is π-closed if X possesses a unique maximal π-subgroup. X is a dihedral group if X is generated by two elements of order 2.

Suppose π is a set of primes. Denote by:

- $N_X(Y)$ the normalizer of Y in X;
- $C_X(Y)$ the centralizer of Y in X;
- $Z(X)$ the center of X;
- $\Omega_1(X)$ the subgroup of X which is generated by every element of X that has prime order;
- $\Phi(X)$ the Frattini subgroup of X, that is, the intersection of all maximal subgroups of X;

$O_\pi(X)$ the maximal normal π-subgroup of X, and let $O(X) = O_{2'}(X)$;

$O^\pi(X)$ the subgroup of X which is generated by all elements of X whose orders are coprime to any prime in π;

$F(X)$ the fitting subgroup of X, that is, the maximal normal nilpotent subgroup of X;

$F_\infty(X)$ the maximal normal solvable subgroup of X.

$E(X)$ the subgroup of X which is generated by all quasi-simple subnormal subgroups of X.

We say that X is quasi-simple if $X = [X, X]$ and $X/Z(X)$ is simple.

In addition to the more standard terminology, for a finite group X and a subgroup S of X, we say that X is *S-irreducible* if and only if $X = X_1$ or X_2, whenever both X_1 and X_2 are subgroups of X which contain S and $X = \langle X_1, X_2\rangle$.

In this paper, both the Thompson subgroup and the Glauberman subgroup will play very crucial roles. We define them according to G. Glauberman [6]. Suppose S is a finite 2-group. Let

$$d_e(S) = \max.\ \{|A|;\ A \text{ is an elementary Abelian subgroup of } S\}\ ,$$

$J_e(S) = \langle A; A$ ranges over all elementary Abelian subgroups of S with $|A| = d_e(S)\rangle$, and $\Omega_1 Z J_e(S) = \Omega_1(Z(J_e(S)))$.

We say that S^* is an E-group, if $Z(S^*)$ contains every normal elementary Abelian subgroup V of S^* which has the following property:

Whenever R is a nonidentity elementary Abelian subgroup of $S^*/C_{S^*}(V)$, then $|V/C_V(R)| > |R|^{3/2}$ and $|[V, R]| > |R|$.

DEFINITION. Let S be a finite 2-group. Then

$$\hat{J}(S) = \langle S^*; S^* \text{ ranges over all } E\text{-groups such that } J_e(S) \subseteqq S^* \subseteqq S\rangle$$

and

$$\Omega_1 Z\hat{J}(S) = \Omega_1(Z(\hat{J}(S)))\ .$$

REMARK 1.2. $\hat{J}(S) \supseteqq J_e(S)$ (c.f. [6, Chapter II, Remark 1.1]).

3. Preliminaries and konwn results. In this section, we shall present lemmas which will be frequently quoted and will be used to prove Propositions 3.13 and 17.

HYPOTHESIS A. Suppose H is a finite solvable group, and S is a subgroup of H. Assume:

(A.1) S is a Sylow 2-subgroup of H; and

(A.2) H is S^4-free, where S^4 denotes the symmetric group of degree 4.

THEOREM 3.1. *(G. Glauberman) Suppose H is a finite solvable group, and S is a subgroup of H. Assume the pair (H, S) satisfies Hypothesis A. Then*

(a) $H = C_H(Z(S))N_H(J_e(S))O(H)$;

(b) $H = C_H(\Omega_1 Z\hat{J}(S))N_H(J_e(S))O(H)$; *and*

(c) $H = C_H(Z(S))N_H(\hat{J}(S))O(H)$.

Proof. See [6, Chapter II., Theorem B].

LEMMA 3.2. *Suppose S is a finite 2-group. Then*

(a) $J_e(T) = J_e(S)$, *whenever* $J_e(S) \subseteq T \subseteq S$;

(b) $\hat{J}(T) = \hat{J}(S)$, *whenever* $\hat{J}(S) \subseteq T \subseteq S$; *and*

(c) $\Omega_1 Z\hat{J}(T) \supseteq \Omega_1 Z\hat{J}(S)$, *whenever* $J_e(S) \subseteq T \subseteq S$.

Proof. (a) and (b) follow from [6, Chapter II. Lemma 2.1(b) and (d)]. If T^* is an E-subgroup of T, then T^* is an E-subgroup of S by definition. So, (c) follows from (a).

LEMMA 3.3. *Suppose X is a finite group, and V is a normal subgroup of X. Let S be a Sylow p-subgroup of X. Then*

(a) $S \cap V$ *is a Sylow p-subgroup of V*;

(b) SV/V *is a Sylow p-subgroup of X/V*;

(c) *(Frattini argument)* $X = VN_X(S \cap V)$; *and*

(d) $X = O^p(O^{p'}(X))N_X(S)$.

Proof. (a), (b) and (c) follow from [8, Theorem 1.3.7 and 8, page 12]. (d) is a special case of (c).

LEMMA 3.4. *(W. Burnside) Let p be a prime. Suppose P is a finite p-group, and A is a finite group which acts on P. Assume that A acts trivially on $P/\Phi(P)$. Then,* $[P, O^p(A)] = 1$.

Proof. See [8, Theorem 5.1.4, page 174].

LEMMA 3.5. *(P. Hall) Suppose p is a prime, and π is a set of primes which contains p. Let X be a finite solvable group, D_1 and D_2 be Hall π-subgroups of X, and S be a Sylow p-subgroup of D_1. Then*

(a) *D_1 and D_2 are conjugate in X; and*

(b) *if S is also a Sylow p-subgroup of D_2, then there exists an element x of $N_X(S)$ such that $D_1^x = D_2$.*

Proof. (a) follows from [8, Theorem 6.4.1 (ii), page 231]. (b) follows from (a).

LEMMA 3.6. *Suppose X is a finite group, and W is a normal subgroup of X. Then, $E(W)$ is a central product of uniquely determined quasi-simple groups which are permuted by conjugation of X.*

Proof. See [7, Lemma (2.1)(a), page 73-74].

LEMMA 3.7. *Let r be a prime. Suppose V is a finite group, and H is a finite group which acts on V. Assume:*

(1) *H stabilizes a normal series of V: $V \supseteqq V_1 \supseteqq 1$; and*

(2) V_1 is an r-group.

Then, $[V, O^r(H)] = 1$.

Proof. Let Q be a Sylow q-subgroup of H, where q is a prime which is distinct from r. By (1), R normalizes some Sylow p-subgroup of V for any prime divisor p of $|V|$, and centralizes it by [8, Theorem 5.3.2, page 178]. Since q is an arbitrary prime with $q \neq r$, by (2), we get $R \subseteqq C_H(V) \lhd H$, which implies this lemma.

We use the following famous result without notice:

LEMMA 3.8. (*W. Burnside*) *Let X be a finite group. Assume that the number of prime divisors of $|X|$ is at most 2. Then X is solvable.*

Proof. See [8, Theorem 4.3.3, page 131].

Lemma 3.9. *Suppose H is a finite* $\{2, 3\}$*-group. Then, the following* (a) *and* (b) *are equivalent*:

(a) $H = 0_{3,2,3}(H)$;

(b) *H is S^4-free, where S^4 denotes the symmetric group of degree four.*

Proof. Obviously, (b) follows from (a). Assume (b). Let X be an involved group in H minimal subject to satisfying $H \neq O_{3,2,3}(H)$. Then, X possesses the following properties: (α) $X = O_{2,3,2}(X)$, (β)$O_2(X)$ is a nonidentity elementary Abelian group, (γ) the order of a Sylow 3-subgroup Q of X is 3, and (δ)$C_X(Q) \cap O_2(X) = 1$. So, $N_X(Q)$ is isomorphic to the symmetric group of degree three, and $O_2(X)$ is a four group. This implies that $X \simeq S^4$, as required.

Hence, this lemma is proved.

The next lemma immediately follows from the definitions.

LEMMA 3.10. *Suppose H is a finite group, and S is a Sylow 2-subgroup of H. Then, H possesses S-irreducible subgroups $\{H_i; 1 \leqq i \leqq t\}$ with a Sylow 2-subgroup S such that $H=\langle H_i; 1\leqq i\leqq t\rangle$.*

LEMMA 3.11. *Suppose H is a finite solvable S-irreducible group with a Sylow 2-subgroup S. Let V be a normal subgroup of H. Then*

(a) *H/V is SV/V-irreducible;*

(b) *H is a $\{2, r\}$-group for some prime r;*

(c) *$H = O_{2,r,2}(H)$.*

Let R be a Sylow r-subgroup of H.

(d) *$S\Phi(R)/O_2(H)\Phi(R)$ acts irreducibly on $RO_2(H)/O_2(H)\Phi(R)$;*

(e) *if $O^2(H) \nsubseteq V$, then $S \cap V \lhd H$.*

In the following statements, we assume that H is not 2-closed.

(f) *if $S \subseteq V$, then $V = H$, that is, $H = O^{2'}(H)$;*

(g) *if $O^2(H) \nsubseteq V$, then $R \cap V \subseteq \Phi(R)$, and H/V involves a dihedral group of order $2r$.*

(h) *if $r \neq 3$, one of the following holds:*

(h.1) *H is r-closed;*

(h.2) *$H = C_H(\Omega_1 Z(S)) = SC_H(\Omega_1 Z\hat{J}(S))$;*

(h.3) *$H = N_H(J_e(S)) = N_H(\hat{J}(S))$;*

(h.4) *$H = N_H(J_e(S)) = C_H(\Omega_1 Z(S))$;*

(i) *if $r \neq 3$, there exists a (possibly trivial) characteristic subgroup T of S which is normal in H and H/T is r-closed.*

Proof. (a) follows from the definitions.

(b) follows from a theorem of P. Hall [8, Theorem 6.4.1, page 231].

To prove (c), suppose $H \supset O_{2,r,2}(H)$. Let $T = S \cap O_{2,r,2}(H)$, $H_1 = N_H(T)$, and $H_2 = O_{2,r}(H)S$. Then, $H \supset H_i \supseteqq S$, $1 \leqq i \leqq 2$, and $H = H_1H_2$ by the Frattini argument, which contradicts the fact that H is S-irreducible.

(d) follows from a theorem of H. Maschke [8, Theorem 3.3.1, page 66].

(e) By the Frattini argument, $H = N_H(S \cap V)(VS)$. As $O^2(H) \nsubseteq VS$, $S \subseteqq VS \subset H$. Since H is S-irreducible, $H = N_H(S \cap V)$, as required.

(f) Suppose $O^2(H) \nsubseteq V$. By (e), $H = N_H(S)$, which contradicts the fact that H is not 2-closed. Hence, $V \supseteqq O^2(H)S = H$, as required.

(g) By (d), $R \cap V \subseteqq \Phi(R)$. To prove the latter part of (g), we may assume $V \supseteqq O_2(H)$. Let $\bar{H} = H/V$. Then, a theorem of R. Baer [8, Theorem 3.8.2. page 105] implies that $\bar{H}$ possesses involutions $\bar{x}$ and $\bar{y}$ such that $\langle \bar{x}, \bar{y} \rangle$ is not a 2-group, which implies (g).

To prove (h), assume $H \neq O_r(H)S$; then, by Theorem 3.1, $H = C_H = (Z(S))N_H(J_e(S)) = C_H(\Omega_1 Z\hat{J}(S))N_H(J_e(S)) = C_H(\Omega_1 Z(S))N_H(\hat{J}(S))$. Since H is S-irreducible, we get (h).

(i) Let T be a characteristic subgroup of S maximal subject to satisfying $T \lhd H$. By Theorem 3.1, $\bar{H} = C_{\bar{H}}(Z(\bar{S}))N_{\bar{H}}(J_e(\bar{S}))(O(\bar{H})\bar{S})$. Then we conclude $\bar{H} = O(\bar{H})\bar{S}$ by (a) and the maximality of T.

LEMMA 3.12. *Let r be a prime with $r \geqq 5$. Suppose D is a finite group, $\{D_i; 1 \leqq i \leqq n\}$ are normal subgroups of D, S is a 2-subgroup of D, and H is a $\{2, r\}$-subgroup of D with a Sylow 2-subgroup S. Assume*:

(1) *S is normal in some Sylow 2-subgroup of D;*

(2) *H is an S-irreducible group which is not 2-closed;*

(3) *$O^2(H) \subseteqq D_1 D_2 \cdots D_n$;*

(4) *$[D_i, D_j] = 1$ for all $i; 1 \leqq i \neq j \leqq n$;*

(5) *$D_i \cap < D_j; 1 \leqq j \neq i \leqq n >$ is a 2-group for each $i; 1 \leqq i \leqq n$.*

Then there exist subgroups $\{K_i; 1 \leqq i \leqq n\}$ of D which satisfy the following conditions:

(a) *K_i is a $\{2, r\}$-group with a Sylow 2-subgroup S, $1 \leqq i \leqq n$;*

(b) *$O^2(K_i) \subseteqq D_i$ for all $i; 1 \leqq i \leqq n$;*

(c) *$\langle K_i; 1 \leqq i \leqq n \rangle$ is a $\{2, r\}$-group with a Sylow 2-subgroup S, and $H \subseteqq \langle K_i; 1 \leqq i \leqq n \rangle$;*

(d) *thhere exists a subgroup T of S which satisfies the following conditions*:

(d.1) *T is normal in $\langle N_D(S), K_i; 1 \leqq i \leqq n \rangle$; and*

(d.2) *$\langle K_i; 1 \leqq i \leqq n \rangle / T$ is r-closed.*

For any subgroup J of S,

(e) *if $O_2(H) \supseteqq J$, then $O_2(K_i) \supseteqq J$ for all $i; 1 \leqq i \leqq n$;*

(f) *if $O^2(H) \subseteqq [O_{2,r}(H), J]$, then $O^2(K_i) \subseteqq [O_{2,r}(K_i), J]$ for all $i; 1 \leqq i \leqq n$;*

(g) *if H is r-closed, then $\langle K_i; 1 \leqq i \leqq n \rangle$ is r-closed.*

Proof. In this proof, for any subgroup X of D, we let

$$X_i^* = \{x_i; x_1 x_2 \cdots x_n \in X \cap D_1 D_2 \cdots D_n \text{ and } x_i \in D_i, 1 \leqq i \leqq n\}, \quad 1 \leqq i \leqq n .$$

Since $D_i \lhd D$, it is easily verified that for any subgroup X of D, (3.12.1) $N_D(X_i^*) \supseteqq N_D(X)$ for all $i; 1 \leqq i \leqq n$.

For another subgroup Y of D, by (4),

(3.12.2) $[X \cap D_1D_2\cdots D_n, Y \cap D_1D_2\cdots D_n] \subseteqq [X_i^*, Y_i^*]; 1 \leqq i \leqq n$,

and

(3.12.3) $[X_i^*, Y \cap D_1D_2\cdots D_n] = [X_i^*, Y_i^*]$ for all $i; 1 \leqq i \leqq n$.

Suppose $O^2(H) \subseteqq D_i$ for some $i; 1 \leqq i \leqq n$. By Lemma 3.11(i), there exists a subgroup T of S such that T is normal in $\langle N_D(S), H\rangle$ and H/T is r-closed. Hence, setting $K_i = H$ and $K_j = S$ for other $j; 1 \leqq j \neq i \leqq n$, we can easily verify that all our assertions hold. Thus, we may assume that:

(3.12.4) $O^2(H) \nsubseteq D_i$ for all $i; 1 \leqq i \leqq n$.

So, by Lemma 3.11(e),

(3.12.5) $S \cap D_i \lhd H$ for all $i; 1 \leqq i \leqq n$.

By (3.12.1) and (5),

(3.12.6) $O_2(H)_i^*$ is a 2-group which is normalized by H for all $i; 1 \leqq i \leqq n$.

By (1), there exists a Sylow 2-subgroup U of D in which S is normal. Since $U \subseteqq N_D(S) \subseteqq N_D(S_i^*)$ and $O_2(H)_i^* \subseteqq S_i^*$, we obtain that $O_2(H)_i^* \subseteqq U$ for all $i; 1 \leqq i \leqq n$. Since $S \lhd U$, $[O_2(H)_i^*, S] \subseteqq S \cap D_i$ for all $i; 1 \leqq i \leqq n$. Hence, by (3.12.5) and (3.12.6), $S \subseteqq C_H(O_2(H)_i^*/(S \cap D_i)) \lhd H$ for all $i; 1 \leqq i \leqq n$. So, by (2) and Lemma 3.11(f),

(3.12.7) $H = C_H(O_2(H)_i^*/(S \cap D_i))$ for all $i; 1 \leqq i \leqq n$.

Let $T = \langle S \cap D_i; 1 \leqq i \leqq n\rangle$. By (3.12.5),

(3.12.8) $T \lhd \langle N_D(S), H\rangle$.

Let R be a Sylow r-subgroup of H. Then, by (3.12.2), (3.12.3) and (3.12.7),

(3.12.9)
$$\begin{aligned}[O_2(H), R] &= [O_2(H), R, R] \\ &\subseteqq [O_2(H) \cap (D_1D_2\cdots D_n), R] \\ &\subseteqq [\langle O_2(H)_i^*; 1 \leqq i \leqq n\rangle, R] \\ &\subseteqq \langle [O_2(H)_i^*, R]; 1 \leqq i \leqq n\rangle \\ &\subseteqq \langle S \cap D_i; 1 \leqq i \leqq n\rangle \\ &\subseteqq T .\end{aligned}$$

So, by (2) and Lemma 3.11(c).

(3.12.10) $$H/T \text{ is } r\text{-closed}.$$

By (3), (3.12.2), (3.12.3) and (3.12.5),$[T, R_i^*] = [S \cap D_i, R_i^*] = [S \cap D_i, R] \subseteq S \cap D_i \subseteq T$. Hence, R_i^* normalizes T, so that

(3.12.11) $$O_r(R_i^*) \text{ normalizes } T \text{ for all } i; 1 \leqq i \leqq n .$$

Then, by (3.12.1), (3.12.10) and the Frattini argument,

(3.12.12) $$S \subseteq N_H(RT) \subseteq N_H(R)T \subseteq N_H(R_i^*)T \subseteq N_H(O_r(R_i^*)T) \quad \text{for all } i; 1 \leqq i \leqq n .$$

It follows that $SO_r(R_i^*)$ is a $\{2, r\}$-group with a Sylow 2-subgroup S for all $i; 1 \leqq i \leqq n$. Let $K_i = SO_r(R_i^*), 1 \leqq i \leqq n$. Now, we will check that $K_i; 1 \leqq i \leqq n$ satisfy all the conclusions of this lemma.

(a) is proved in the above argument.

Since $O^2(K_i) = \langle O_r(R_i^*)^x; x \in K_i \rangle \subseteq D_i, 1 \leqq i \leqq n$, we get (b). By (4) and (5), R_i^* is r-closed, $1 \leqq i \leqq n$, so that $H = SR \subseteq S \ \langle O_r(R_i^*); 1 \leqq i \leqq n \rangle = \langle K_i; 1 \leqq i \leqq n \rangle$. Since $[O_r(R_i^*), O_r(R_j^*)] = 1, 1 \leqq i \neq j \leqq n$, it follows that $\{K_i; 1 \leqq i \leqq n\}$ is a $\{2, r\}$-group with a Sylow 2-subgroup S. Thus (c) is proved.

By (3.12.8), $T \lhd N_D(S)$. Then, (d, 1) follows from (3.12.11), and (d.2) follows from (3.12.12). Thus, (d) is proved.

Let J be a subgroup of S. Since J normalizes $O^2(H)$, we get:

(3.12.13) $$[O^2(H), J]_i^* = [O^2(H)_i^*, J] \text{ for all } i; 1 \leqq i \leqq n .$$

Suppose $J \subseteq O_2(H)$. Then by (d), $[O^2(H), J] \subseteq T$. By (d) and (3.12.13), $[O^2(H)_i^*, J] \subseteq T_i^*, 1 \leqq i \leqq n$. It follows that

$$[O_r(R_1^*), J][O_r(R_2^*), J] \cdots [O_r(R_n^*), J]$$

is a 2-group, which implies (e).

Suppose $O^2(H) \subseteq [O_{2,r}(H), J]$. Since $H = O_{2,r,2}(H)$, we have that $R \subseteq [O^2(H), J]$. Hence, $O_r(R_i^*) \subseteq [O_r(R_i^*), J] \subseteq [O_{2,r}(H), J], 1 \leqq i \leqq n$, which implies (f).

Finally, suppose H is r-closed. Then, $[O^2(H), T]=1$. By (3.12.2), $[O^2(K_i), T] = 1, 1 \leqq i \leqq n$. By (d), it follows that K_i is r-closed for all $i; 1 \leqq i \leqq n$, which proves (g). Hence this lemma is proved.

PROPOSITION 3.13. *Let r be a prime with $r \geqq 5$. Suppose D a finite group, V is a normal subgroup of D, $\{D_i; 1 \leqq i \leqq n\}$ are normal subgroups of D all of which contain V, S is a 2-subgroup of D, and H is a $\{2, r\}$-subgroup of D with a Sylow 2-subgroup S. Assume*:

(1) *S is normal in some Sylow 2-subgroup of D;*

(2) *H is an S-irreducible group which is not 2-closed;*

(3) $O^2(H) \subseteqq D_1D_2\cdots D_n$;

(4) *V is 2-closed with $O_2(V) = S \cap V$.*

Let $\bar{X} = XV/V$ for any subgroup X of D.

(5) $[\bar{D}_i, \bar{D}_j] = 1$ *for all* $i, j; 1 \leqq i \neq j \leqq n$;

(6) $\bar{D}_i \cap \langle \bar{D}_j; 1 \leqq j \neq i \leqq n\rangle$ *is a 2-group for each* $i; 1 \leqq i \leqq n$.

Then there exist subgroups $\{L_i; 1 \leqq i \leqq t\}$ of D which satisfy the following conditions:

(a) *L_i is an S-irreducible $\{2, r\}$-group with a Sylow 2-subgroup S for all $i; 1 \leqq i \leqq t$;*

(b) *$\langle L_i; 1 \leqq i \leqq t\rangle$ is a $\{2, r\}$-group with a Sylow 2-subgroup S, and $H \subseteqq \langle L_i; 1 \leqq i \leqq t\rangle$;*

(c) *there exists a subgroup T of S which is normal in $\langle N_D(S), L_i; 1 \leqq i \leqq t\rangle$, and $\langle L_i; 1 \leqq i \leqq t\rangle/T$ is r-closed.*

(d) *for each L_i, $1 \leqq i \leqq t$, $O^2(L_i)$ is contained in some D_j, $1 \leqq j \leqq n$;*

(e) *if $H = N_H(J_e(S))$, then $L_i = N_{L_i}(J_e(S))$ for all $i; 1 \leqq i \leqq t$;*

(f) *if $H \neq N_H(J_e(S))$, then $L_i \neq N_{L_i}(J_e(S))$, $1 \leqq i \leqq t$; and*

(g) *if $H/V \cap S$ is r-closed, then $L_i/V \cap S$ is r-closed, $1 \leqq i \leqq t$.*

Proof. In this proof, let $\bar{X} = XV/V$ for any subgroup X of D. By Lemma 3.12, there exist subgroups $\{\bar{K}_i; 1 \leqq i \leqq n\}$ of $\bar{D}$ which satisfy all the conclusions of Lemma 3.12. Let F_1 be a Hall $\{2, r\}$-subgroup of the pre-image of $\{\bar{K}_i; 1 \leqq i \leqq n\}$ in D which contains H. Then, by (4) and Lemma 3.12(c),

(3.13.1) F_1 is a $\{2, r\}$-group with a Sylow 2-subgroup S, $H \subseteqq F_1$ and $\bar{F}_1 = \langle \bar{K}_i; 1 \leqq i \leqq n\rangle$.

By Lemma 3.12(d), there exists a subgroup $\bar{T}$ of $\bar{S}$ such that:

(3.13.2) $$\bar{T} \lhd \langle N_{\bar{D}}(\bar{S}), \bar{F}_1\rangle \text{ and } \bar{F}_1/\bar{T} \text{ is } r\text{-closed.}$$

Let T be the intersection of S and the pre-image of $\bar{T}$ if $\bar{H}$ is not r-closed; otherwise, $T = S \cap V$. Then, by (2) and Lemma 3.11(e),

(3.13.3) $$H = N_H(T) .$$

Let $F_2 = N_{F_1}(T)$. Then, by (3.13.3), Lemma 3.12(c) and (g),

(3.13.4) $H \subseteqq F_2$, $T \lhd F_2$, F_2/T is r-closed, and $\bar{F}_2 = \langle \bar{K}_i; 1 \leqq i \leqq n\rangle$.

Finally, let $F = N_{F_2}(J_e(S))$ if $H = N_H(J_e(S))$; otherwise, let $F = [O_{2,r}(F_2), J_e(S)]S$. By (2) and Lemma 3.11(d), if $H \neq N_H(J_e(S))$, then $O^2(H) \subseteqq [O_{2,r}(H), J_e(S)]$. So, in either case that $H = N_H(J_e(S))$ or $H \neq N_H(J_e(S))$, (3.13.5) $$H \subseteq F .$$

By (3.13.4), Lemma 3.12(e) and (f),

(3.13.6) $T \lhd F$ and F/T is r-closed; and

(3.13.7) $$\bar{F} = \langle \bar{K}_i; 1 \leqq i \leqq n \rangle .$$

By Lemma 3.12(g),

(3.13.8) if $H/V \cap S$ is r-closed, then $F/V \cap S$ is r-closed.

By (3.13.7),

(3.13.9) $$F = \langle (F \cap D_i)S; 1 \leqq i \leqq n \rangle .$$

Since S is a Sylow 2-subgroup of $(F \cap D_i)S$, by Lemma 3.10,

(3.13.10) each $(F \cap D_i)S$ is generated by S-irreducible subgroups with a Sylow 2-subgroup $S, 1 \leqq i \leqq n$.

Therefore, there exist subgroups $\{L_i; 1 \leqq i \leqq t\}$ of F which satisfy the following conditions:

(3.13.11) L_i is an S-irreducible group with a Sylow 2-subgroup $S, 1 \leqq i \leqq t$;

(3.13.12) for each L_i, $1 \leqq i \leqq t$, $O^2(L_i)$ is contained in some D_j, $1 \leqq j \leqq n$.

Furthermore, by (3.13.9),

(3.13.13) $$\langle L_i; 1 \leqq i \leqq t \rangle = F .$$

Then, we may assume that:

(3.13.14) $F \supset \langle L_i; 1 \leqq i \neq j \leqq t \rangle$ for any $j; 1 \leqq j \leqq t$.

Now, we check that $\{L_i; 1 \leqq i \leqq t\}$ satisfy all the conclusions of this lemma. Since F is a $\{2, r\}$-group, (a) follows from (3.13.11), and (b) follows from (3.13.5) and (3.13.13). Since $T \lhd N_D(S)$, (c) follows from (3.13.6). (d) follows from (3.13.12). And, (g) follows from (3.13.8).

To prove (e), suppose $H = N_H(J_e(S))$. Then $F = N_F(J_e(S))$, so that $L_i = N_{L_i}(J_e(S))$ for all $i; 1 \leqq i \leqq t$, which proves (e). Next, suppose $H \neq N_H(J_e(S))$. Then by definition,

(3.13.15) $$F = [O_{2,r}(F), J_e(S)]S .$$

Let $L_i \neq N_{L_i}(J_e(S)), 1 \leqq i \leqq s$, and $L_j = N_{L_j}(J_e(S)), s+1 \leqq j \leqq t$. Then, $F = \langle N_F(J_e(S)), L_i; 1 \leqq i \leqq s \rangle$. By (3.13.15), $F = [O_{2,r}(F), J_e(S)]S =$

$\langle L_i; 1 \leqq i \leqq s\rangle$. So, by (3.13.14), $s = t$. Thus, we conclude that $L_i \neq N_{L_i}(J_e(S))$ for all $i; 1 \leqq i \leqq t$, which proves (f). Hence this lemma is proved.

HYPOTHESIS B.I. Suppose M is a finite group, W is a normal subgroup of M, and Y is a subgroup of W. Assume:

(B.1) $W = E(W) = E_1 \times \cdots \times E_r$, where E_k is a non-Abelian simple group with $E_k \simeq E_1$, $1 \leqq k \leqq r$; and

(B.2) $E_k \not\subseteq Y$ for all $k; 1 \leqq k \leqq r$.

Let $\Lambda = \{1, 2, \cdots, r\}$. We identity E_k with the element k of Λ, $1 \leqq k \leqq r$. Let Γ be the following family of subsets of Λ:

$\Gamma = \{\{k_1, \cdots, k_s\}; \{k_1, \cdots, k_s\}$ is minimal under inclusion such that $Y \cap (E_{k_1} \times \cdots \times E_{k_s}) \neq 1\}$. We say that s is the length of $\{k_1, \cdots, k_s\}$. Let π_k be the projection mapping from W to E_k, $1 \leqq k \leqq r$.

PROPOSITION 3.14. *Assume Hypothesis* B.I. *Assume also that*:

(B.0) *whenever* $1 \leqq k \leqq r$, $\pi_k(Y) = E_k$.

Then

(a) *for any distinct elements* $\{k_1, \cdots, k_s\}$ *and* $\{j_1, \cdots, j_t\}$ *of* Γ, $\{k_1, \cdots, k_s\} \cap \{j_1, \cdots, j_t\} = \emptyset$;

(b) $|Y| = |E_1|^{|\Gamma|}$;

(c) $|Y|^2 \leqq |W|$;

(d) *if equality holds in* (c), *then*

(d.1) r *is even*;

for suitable renumbering Λ,

(d.2) $\Gamma = \{\{k, k + r/2\}; 1 \leqq k \leqq r/2\}$;

(d.3) $Y = \prod_{k=1}^{r/2} (Y \cap (E_k \times E_{k+r/2}))$, *and* $Y \cap (E_k \times E_{k+r/2}) \simeq E_1$, $1 \leqq k \leqq r/2$.

(e) *conversely, if the lengths of all elements of* Γ *are equal to* 2, *then equality holds in* (c);

(f) W *centralizes* $(\bigcap_{k=1}^{r} N_M(E_k)) \cap C_M(Y)$.

Proof. Take an arbitrary element $\{k_1, \cdots, k_s\}$ of Γ. We may assume that $\{k_1, \cdots, k_s\} = \{1, \cdots, s\}$, renumbering if necessary.

Let $Y_0 = Y \cap (\prod_{k=1}^{s} E_k)$, and let ψ_k be the projection mapping from Y_0 to E_k, $1 \leqq k \leqq s$.

By minimal nature of $\{1, \cdots, s\}$,

$$\text{(3.14.1)} \qquad \operatorname{Ker} \psi_k = 1, \text{ and } \operatorname{Im} \psi_k \neq 1, 1 \leqq k \leqq s .$$

Since $Y_0 \lhd Y$, by (3.14.1), (B.1) and (B.0),

$$\text{(3.14.2)} \qquad \operatorname{Im} \psi_k = \langle (\operatorname{Im} \psi_k)^Y \rangle = \langle (\operatorname{Im} \psi_k)^{E_k} \rangle = E_k, 1 \leqq k \leqq s .$$

By (3.14.1) and (3.14.2), we conclude that ψ_k is an isomorphism for all $k; 1 \leqq k \leqq s$.

Therefore,

$$Y_0 \simeq E_1 . \tag{3.14.3}$$

To prove (a), take distinct elements $\{k_1, \cdots, k_s\}$ and $\{j_1, \cdots, j_t\}$ of Γ.

Let $Y_1 = Y \cap (\prod_{i=1}^{s} E_{k_i})$, and $Y_2 = Y \cap (\prod_{i=1}^{t} E_{j_i})$. By minimality of $\{k_1, \cdots, k_s\}$ and $\{j_1, \cdots, j_t\}$, $Y_1 \cap Y_2 = 1$. Since $Y_1 \lhd Y$ and $Y_2 \lhd Y$,

$$[Y_1, Y_2] = 1 . \tag{3.14.4}$$

Suppose that $\{k_1, \cdots, k_s\} \cap \{j_1, \cdots, j_t\} \neq \emptyset$, and take h in this intersection. By (3.14.4) and (B.0), $1 = [\Psi_h(Y_1), \Psi_h(Y_2)] = [E_h, E_h]$, which contradicts (B.1), and (a) is proved. Then (B.0) implies that $\Lambda = \cup \gamma$ (disjoint union), where γ ranges over all the elements of Γ. Then (3.14.3) yields (b), and (c) follows from (b) and (B.2).

Assume equality holds in (c). By (c), r is even, and we may assume that $\Gamma = \{\{k, k + r/2\}; 1 \leqq k \leqq r/2\}$. By (a) and (3.14.3),

$$Y = \prod_{k=1}^{r/2} (Y \cap (E_k \times E_{k+r/2})) \simeq \prod_{k=1}^{r/2} E_k ,$$

which proves (d).

(e) follows from (b) and (B.0). Finally, to prove (f), take an arbitrary subscript k and an arbitrary element x_k of E_k. By (B.0), Y possesses an element y such that $y = x_1 \cdots x_k \cdots x_r$, where $x_i \in E_i$, $1 \leqq i \leqq r$. Let $K = (\bigcap_{k=1}^{r} N_M(E_k)) \cap C_M(Y)$. Then $1 = [K, y] = [K, x_1] \cdots [K, x_k] \cdots [K, x_r]$, and $[K, x_i] \subseteqq E_i$, $1 \leqq i \leqq r$. Therefore $[K, x_k] = 1$, which implies (f).

HYPOTHESIS B.II: Assume Hypothesis B.I. Let q be an odd prime. Suppose N is a subgroup of M which normalizes Y, H is a subgroup of N, and S is a Sylow 2-subgroup of H. Assume also that:

(B.3) $N = \langle H, N_N(S) \rangle$;

(B.4) there exists a subgroup T of S such that

(B.4.1) $T(S \cap O_2(N)) = S$, and

(B.4.2) $[R, T] \subseteqq Y$ for any S-invariant q-subgroup R of W;

(B.5) there exists an S-invariant q-subgroup Q of W such that $\langle \pi_1(Q)^x; x \in N \rangle = W$.

Only for convenience to state and prove the following Proposition 3.15, we need one more hypothesis:

HYPOTHESIS B.III: Assume Hypothesis B.II. Assume also that:

(B.0) whenever $1 \leqq k \leqq r$, $\pi_k(Y) = E_k$; and

(B.6) $|Y|^2 = |W|$.

PROPOSITION 3.15. *Assume Hypothesis* B.II. *Then* $O^2(O^{2'}(H))$ *normalizes each* E_k, $1 \leqq k \leqq r$. *Moreover, at least one of the following* (a) *or* (b) *holds:*

(a) T *normalizes each* E_k, $1 \leqq k \leqq r$; *or*

(b) *Hypothesis* B.III *is satisfied, and* T *fixes each element of* Γ.

Proof. The proof is separated into seven steps. In the first step, we will note down the matters which will be needed for the development of the proof.

Step 1. (1.a) $O_2(N) \cap S \lhd N$;

(1.b) $\langle \pi_1(Q)^x; x \in N_N(E_1) \rangle = E_1$;

(1.c) N acts transitively on Λ;

(1.d) N induces a permutation of Γ, and if Hypothesis (B.0) is satisfied, N acts transitively on Γ; and

(1.e) if Hypothesis (B.0) is satisfied and the length of some element of Γ is equal to 2, then Hypothesis B.III is satisfied.

Proof of Step 1. Since S is a Sylow 2-subgroup of H, $O_2(N) \cap S \lhd H$. Since $O_2(N) \cap S \lhd N_N(S)$, (B.3) yields (1.a).

By (B.5), $E_1 = E_1 \cap \langle \pi_1(Q)^x; x \in N \rangle = \langle \pi_1(Q)^x; x \in N_N(E_1) \rangle$, which proves (1.b). Again, by (B.5), (1.c) follows from (1.b).

Since N leaves invariant Y, N induces a permutation of Γ. By Proposition 3.14(a) and (1.c), $\Gamma = \{\gamma^x; x \in N\}$, which proves (1.d). Suppose Hypothesis (B.0) is satisfied and the length of some element of Γ is equal to 2. Then by (1.d), the lengths of all elements of Γ are equal to 2. (1.e) follows from (1.d) and Proposition 3.14(e).

Step 2. Fix $k = 1, 2, \cdots, r$. Let R_k be a q-subgroup of E_k. Assume:

(1) $N_S(E_k)$ normalizes R_k;

(2) $\langle R_k^x; x \in N_N(E_k) \rangle = E_k$; and

(3) T does not normalize E_k.

Then

(2.a) Hypothesis B.III is satisfied; and

(2.b) whenever $x \in T$ and $k^x \neq k$, $\{k, k^x\}$ lies in Γ.

Proof of Step 2. Let $R = \langle R_k^y; y \in S \rangle$. By (1), R is an S-invariant q-subgroup of W. By (3), there exists an element x of T such that $E_k^x \neq E_k$. By (B.4.2), (3.15.1) $\pi_k(Y) \supseteqq \pi_k([R, T]) \supseteqq \pi_k([R, x]) = \pi_k(R_k) = R_k$,

and

(3.15.2) $Y \cap (E_k \times E_k^x) \supseteqq Y \cap [R_k, x] \neq 1$.

Since Y is N-invariant, by (2) and (3.15.1), $\pi_k(Y) \supseteqq \langle R_k^y; y \in N_N(E_k)\rangle = E_k$. By (1.c),

(3.15.3) $\pi_j(Y) = E_j, 1 \leqq j \leqq r$.

By (3.15.2),

(3.15.4) $\{k, k^x\}$ lies in Γ .

Then (2.a) follows from (3.15.3), (3.15.4) and (1.e). Since x is an arbitrary element of T such that $k^x \neq k$, (2.b) follows from (3.15.4). Hence this lemma is proved.

Let $\{1^x; x \in S\} = \{1, 2, \cdots, s\}$, renumbering if necessary.

Step 3. Assume T does not normalize some E_k, $1 \leqq k \leqq s$. Then
(3.a) Hypothesis B.III is satisfied; and
(3.b) whenever $x \in T$ and $k^x \neq k$, $\{k, k^x\}$ lies in Γ.

Proof of Step 3. By (B.5) and (1.b), $N_S(E_k)$ normalizes $\pi_k(Q)$ and $\langle \pi_k(Q)^y; y \in N_N(E_k)\rangle = E_k$. Thus, this step follows from Step 2.

Let $\{1^x; x \in O_2(N) \cap S\} = \{1, 2, \cdots, w\}$, and
$\{1^x; x \in H\} = \{1, 2, \cdots, h\}$, renumbering if necessary.

Step 4. Assume $w \neq s$. Then
(4.a) Hypothesis B.III is satisfied;
(4.b) $2w = s$;
(4.c) h/w is even.

Proof of Step 4. Since $O_2(N) \cap S \lhd H$ by (1.a), for any element x of H,

(3.15.5) $\{1, 2, \cdots, w\}^x$ is an $(O_2(N) \cap S)$-orbit; and

(3.15.6) $\{1, 2, \cdots, w\} \cap \{1, 2, \cdots, w\}^x = \emptyset$ or $\{1, 2, \cdots, w\}$.

Since $w \neq s$, by (B.4.1), there exists an element t of T such that $\{1, 2, \cdots, w\} \cap \{1, 2, \cdots, w\}^t = \emptyset$. Let $k^t = k + w, 1 \leqq k \leqq w$. Then we get (4.a) from (3.a). Then by (3.b),

(3.15.7) $\{k, k + w\}$ lies in Γ for all $k; 1 \leqq k \leqq w$.

Now, to prove (4.b), by (B.4.1) and (3.15.5), we may only show that:

(3.15.8) for any element x of T, $\{1, 2, \cdots, w\}^x \subseteqq \{1, 2, \cdots, 2w\}$.

Assume that $\{1, 2, \cdots, w\}^x \neq \{1, 2, \cdots, w\}$ for some element x of T. Since t is an arbitrary element of T with $\{1, 2, \cdots, w\}^t \neq \{1, 2, \cdots, w\}$ in (3.15.7), we obtain that $\{k, k^x\}$ lies in Γ for all $k; 1 \leqq k \leqq w$. Then by Proposition 3.14(a), $k^x = k^t = k + w$ for all $k; 1 \leqq k \leqq w$, which implies (3.15.8). Hence, (4.b) is proved. To prove (4.c), it is enough to show that:

$$(3.15.9) \quad \begin{aligned} &\{1, 2, \cdots, 2w\} \cap \{1, 2, \cdots, 2w\}^x = \emptyset \text{ or } \{1, 2, \cdots, 2w\} \\ &\qquad\qquad \text{for any element } x \text{ of } H . \end{aligned}$$

Assume that for some element x of H, $\{1, 2, \cdots, 2w\} \cap \{1, 2, \cdots, 2w\}^x \neq \emptyset$, and take j in this intersection. Then by (3.15.7), $\{j, j + w^*\}$ lies in Γ, where $w^* = w$ if $1 \leqq j \leqq w$, and $w^* = -w$ if $w + 1 \leqq j \leqq 2w$. So, by (1.d), $\{j^{x^{-1}}, (j + w^*)^{x^{-1}}\}$ lies in Γ. Since $j^{x^{-1}}$ lies in $\{1, 2, \cdots, 2w\}$, by (3.15.7) and Proposition 3.14(a), $(j + w^*)^{x^{-1}}$ lies in $\{1, 2, \cdots, 2w\}$. It follows that $\{1, 2, \cdots, w\} \cap \{1, 2, \cdots, 2w\}^x \neq \emptyset$ and $\{w + 1, w + 2, \cdots, 2w\} \cap \{1, 2, \cdots, 2w\}^x \neq \emptyset$. Then, (3.15.9) follows from (3.15.6). Hence (4.c) is proved.

Step 5. $N_S(E_1)$ is a Sylow 2-subgroup of $N_H(E_1)$.

Proof of Step 5. In this proof, for any natural number n, let n_2 be the highest power of 2 which divides n.

Let $v = h/w$.

By Step 4, we have that either $s = w$ or $s = 2w$ and v is even. Thus,

$$(3.15.10) \qquad s \leqq wv_2 .$$

Since $|N_H(E_1)|_2 = |H|_2/h_2 = |S|/wv_2 = |N_S(E_1)|(s/wv_2)$, (3.15.10) implies that $|N_H(E_1)|_2 \leqq |N_S(E_1)|$, which proves this step.

Step 6. Assume T does not normalize some E_k, $1 \leqq k \leqq h$. Then

(6.a) Hypothesis B.III is satisfied;

(6.b) h is even; and

(6.c) for some suitable renumbering, whenever $1 \leqq k \leqq h/2$, then $\{k, k + h/2\}$ lies in Γ, and $\{k, k + h/2\}^t = \{k, k + h/2\}$ for any element t of T.

Proof of Step 6. Let V be a Sylow 2-subgroup of $N_H(E_k)$ which contains $N_S(E_k)$. By Step 5, there exists an element z of H such that $N_S(E_1)^z = V$ and $E_1^z = E_k$. Then by (1.b), $\langle \pi_1(Q^z)^y; y \in N_N(E_k)\rangle = E_k$. By (B.5), V normalizes $\pi_1(Q)^z = \pi_k(Q^z)$, so that $N_S(E_k)$ normalizes $\pi_k(Q^z)$. Hence, (3.a) yields (6.a).

Let x be an element of T such that $k^x \neq k$. By (3.b), $\{k, k^x\}$ lies in Γ. By (1.d) and Proposition 3.14(a),

(3.15.11) $\{k, k^x\}^y$ lies in Γ for any element y of H, and $\{1, 2, \cdots, h\} = \bigcup_y \{k, k^x\}^y$ (disjoint union), where y ranges over all the elements of H.

This implies (6.b). Moreover, renumbering if necessary, we may assume: $\{k, k + h/2\}$ lies in Γ, $1 \leqq k \leqq h/2$. To prove the final assertion of this step, take an element t of T and an element $\{k, k + h/2\}$ of Γ, $1 \leqq k \leqq h/2$. Then by the above, $k = k^t$ or $\{k, k^t\}$ lies in Γ, so that $\{k, k + h/2\} \cap \{k, k + h/2\}^t \neq \varnothing$. Then by (3.15.11), $\{k, k + h/2\}^t = \{k, k + h/2\}$, as required.

Step 7. $O^2(O^{2'}(H))$ normalizes each E_k, $1 \leqq k \leqq h$.

Proof of Step 7. First, assume that T normalizes each E_k, $1 \leqq k \leqq h$. Let $H_0 = \bigcap_{k=1}^h N_H(E_k)$. By (B.4.1) and (1.a), $S = (O_2(N) \cap S)T \subseteqq (O_2(N) \cap S)H_0 \lhd H$. Thus, $O^2(O^{2'}(H)) \subseteqq H_0$, as required. So, we may assume that T does not normalize some E_k, $1 \leqq k \leqq h$. Then, Step 6 shows that Hypothesis B.III is satisfied, and we may assume that $\{k, k + h/2\}$ lies in $\Gamma, 1 \leqq k \leqq h/2$. Let $H_1 = \{x \in H; \{k, k + h/2\}^x = \{k, k + h/2\}$ for all $k; 1 \leqq k \leqq h/2\}$. Then by (B.4.1), (1.a) and (6.c), $S = (O_2(N) \cap S)T \subseteqq (O_2(N) \cap S)H_1 \lhd H$. Thus,

$$O^2(O^{2'}(H)) \subseteqq H_1 .$$

Since the lengths of all elements of Γ are equal to 2 and $O^2(O^{2'}(H))$ is generated by elements of odd order, $O^2(O^{2'}(H))$ must fix each E_k, $1 \leqq k \leqq h$. Hence this step is proved.

Proof of Proposition 3.15. By Lemma 3.3(d),

$$H = O^2(O^{2'}(H))N_H(S) .$$

So, by (B.3),

(3.15.12) $$N = \langle O^2(O^{2'}(H)), N_N(S)\rangle .$$

Take an element x of $N_N(S)$. Let $Q^* = \langle \pi_1(Q)^{xy}; y \in S\rangle$, $k = 1^x$, and γ be an element of Γ which contains k. By (B.5), Q^* is an

S-invariant q-subgroup of W, and by (1.b), $\langle \pi_k(Q^*)^y; y \in N_N(E_k)\rangle = E_k$. Then, replacing E_k by E_1, all the assumptions of Hypothesis B.II are satisfied. Then Step 7 and (6.c) show that: $O^2(O^{2'}(H))$ normalizes E_k; and T fixes k, or Hypothesis B.III is satisfied and T fixes γ. Since x is an arbitrary element of $N_N(S)$, by (1.c) and (3.15.12), $N_N(S)$ acts transitively on Λ. Then, the above argument implies this lemma.

LEMMA 3.16. *Let p be a prime with $p \geqq 5$. Suppose M is a finite group, W is a normal subgroup of M, and H is a subgroup of M with a Sylow 2-subgroup S. Assume*:

(1) $W = E(W) = E_1 \times E_2 \times \cdots \times E_r$, *where E_k is non-Abelian simple*, $1 \leqq k \leqq r$;

(2) *H is an S-irreducible $\{2, p\}$-group which is not 2-closed*;

(3) $H = O_2(H)(\bigcap_{k=1}^r N_H(E_k))$; *and*

(4) $\mathscr{D}(WH: p) = \varnothing$.

Then

$$O^2(H) \subseteqq WC_M(W) .$$

Proof. Let $H_0 = \bigcap_{k=1}^r N_H(E_k)$.

Then, $H_0 \lhd H$.

By (3),

$$(3.16.1)\qquad \begin{gathered} H_0/O_2(H_0) \text{ is isomorphic to a homomorphic image of} \\ H, \text{ and } O^2(H) \subseteqq H_0 . \end{gathered}$$

Suppose $O^2(H_0) \nsubseteqq E_kC_M(E_k)$ for some $k; 1 \leqq k \leqq r$. Then by Lemma 3.11(g), $H_0/(H_0 \cap E_kC_M(E_k))$ involves a dihedral group of order $2p$, which contradicts (4). Hence, $O^2(H) \subseteqq O^2(H_0) \subseteqq \bigcap_{k=1}^r E_kC_M(E_k) \subseteqq WC_M(W)$, as required.

PROPOSITION 3.17. *Let p be a prime with $p \geqq 5$. Assume Hypothesis* B.II. *Further assume that*:

(1) *H is an S-irreducible $\{2, p\}$-group which is not 2-closed*;

(2) $\mathscr{D}(WH: p) = \varnothing$.

Then, $O^2(H) \subseteqq WC_M(W)$.

Proof. By (1) and Lemma 3.11(f), $H = O^2(H)$. Then by Proposition 3.15 and 3.14(d), $O^2(H)$ normalizes each E_k, $1 \leqq k \leqq r$. Moreover, at least one of the following holds:

(α) T normalizes each E_k, $1 \leqq k \leqq r$; or

(β) for some suitable renumbering of $\{1, 2, \cdots, r\}$, T normalizes each $Y \cap (E_k \times E_{k+r/2})$, and $Y \cap (E_k \times E_{k+r/2}) \simeq E_1$ for all $k; 1 \leqq k \leqq r/2$.

Assume (α) holds. Then by (B.4.1), $S = (S \cap O_2(N))T \subseteqq O_2(H)T \subseteqq$

$O_2(H)(\bigcap_{k=1}^{r} N_H(E_k)) \lhd H$. By (1) and Lemma 3.11(f),

$$O_2(H)(\bigcap_{k=1}^{r} N_H(E_k)) = H .$$

Then by Lemma 3.16, $O^2(H) \subseteqq WC_M(W)$, as required. Next, assume (β) holds. Then by the same argument as above, we have that $O_2(H)(\bigcap_{k=1}^{r/2} N_H(Y \cap (E_k \times E_{k+r/2}))) = H$. Then by Lemma 3.16 (with Y in place of W), $O^2(H) \subseteqq YC_M(Y)$. Let x be any element of $O^2(H)$. Then, we may let $x = x_1x_2$, where $x_1 \in Y$ and $x_2 \in C_M(Y)$. By Lemma 3.14(f),

$$x_2 = x_1^{-1}x \in \left(\bigcap_{k=1}^{r} N_M(E_k) \right) \cap C_M(Y) \subseteqq C_M(W) .$$

Since x is an arbitrary element of $O^2(H)$, we conclude that $O^2(H) \subseteqq WC_M(W)$. Hence this lemma is proved.

4. **Preliminaries in the minimal situation.** From now on, we shall prove the theorem by way of contradiction. Let π be a set of primes. Suppose that G is a finite group, and S is a nonidentity 2-subgroup of G. Assume that (π, G, S) satisfies all the assumptions of the theorem, but violates the conclusions of the theorem. Take G of minimal order and, subject to this condition, take S of minimal order.

LEMMA 4.1. *Let T and T_0 be subgroups of S. Assume:*

(1) $1 \subset T_0 \subseteqq T \subset S$;

(2) $T \lhd N_G(S)$; *and*

(3) $T_0 \lhd N_G(T)$.

Then

(a) $T_0 \lhd N_G(S)$.

So by the assumption (1) *of the theorem, there exists a Sylow* 2-*subgroup U of $N_G(T_0)$ in which S is normal. Then,*

(b) $T \lhd U$;

(c) $\mathscr{D}(N_G(T_0)/C_G(T_0)\colon \pi) = \varnothing$;

(d) *there exists a unique nonidentity subgroup $W(T)$ of T maximal subject to satisfying the following condition:*

(d.1) $W(T) \lhd N_G(T)$; *and*

(d.2) $W(T)O(H) \lhd H$ *for any π-subgroup H of G such that the pair (H, T) satisfies Hypothesis A.*

Let $N = N_G(T)$. Suppose that S_0/T is a nonidentity subgroup of S/T which is normal in $N_{N/T}(S/T)$. Then,

(e) $S_0/T \lhd U/T$;

(f) $\mathscr{D}(N_{N/T}(S_0/T)/C_{N/T}(S_0/T)\colon \pi) = \varnothing$;

(g) *there exists a unique nonidentity subgroup $W(S/T)$ of S/T*

maximal subject to satisfying condition:

(g.1) $W(S/T) \lhd N_{N/T}(S/T)$;

(g.2) $W(S/T)O(H/T) \lhd H/T$ *for any* π*-subgroup* H/T *of* N/T *such that the pair* $(H/T, S/T)$ *satisfies Hypothesis A.*

Proof. Obvious.

Notation. In the following discussion, without notice, we shall use the following notation: for any subgroup T of S such that $1 \subset T \subset S$ and $T \lhd N_G(S)$, we denote by $W(T)$ (or $W(S/T)$) the subject to satisfying the conclusions of (d) (or (g)) of Lemma 4.1.

DEFINITION. Let:

$$\mathfrak{C}_G(S) = \{H \subseteqq G;\ \text{the pair } (H, S) \text{ satisfies Hypothesis } A\}\,,$$

and

$$\mathfrak{F}_G(S) = \{H \in \mathfrak{C}_G(S);\ H \text{ is } S\text{-irreducible}\}\,.$$

For a subfamily $\mathfrak{K}$ of $\mathfrak{F}_G(S)$, we define $O_S(\mathfrak{K})$ to be the largest subgroup of S that satisfies the following conditions (a) and (b):

(a) $O_S(\mathfrak{K}) \lhd N_G(S)$; and

(b) $K = N_K(O_S(\mathfrak{K}))O(K)$ for any element K of $\mathfrak{K}$.

For simplicity, let $O_S(K) = O_S(\{K\})$ for any element K of $\mathfrak{F}_G(S)$. Whenever $1 \subset O_S(\mathfrak{K})$, we define: $W(\mathfrak{K}: 0) = 1$, $W(\mathfrak{K}: 1) = O_S(\mathfrak{K})$, and let $W(\mathfrak{K}: \nu + 1)$ be the pre-image of $W(S/W(\mathfrak{K}: \nu))$ in S, $\nu = 1, 2, \cdots$.

Then, we introduce the function $f_{\mathfrak{K}}$ which is defined on $\mathfrak{K}$ as the following: for any element K of $\mathfrak{K}$,

$$f_{\mathfrak{K}}(K) = \max.\ \{\nu;\ K \rhd W(\mathfrak{K}: \nu)\,,\quad 0 \leqq \nu \leqq \rho\}\,,$$

where ρ denotes the nonnegative integer such that $W(\mathfrak{K}: \rho) = S$ and $W(\mathfrak{K}: \rho - 1) \subset S$.

REMARK 4.1. Whenever $W(\mathfrak{K}: \nu) \subset S$, $W(\mathfrak{K}: \nu) \subset W(\mathfrak{K}: \nu + 1)$, because $W(S/W(\mathfrak{K}: \nu)) \neq 1$.

LEMMA 4.2. *Let K be an element of $\mathfrak{F}_G(S)$. Then the following* (a) *and* (b) *are equivalent:*

(a) *K is 2-closed or 2′-closed;*

(b) $O_S(K) = S$.

Proof. Assume (a). Then, $K = N_K(S)O(K)$, which shows (b). Conversely, assume (b). Then, $K = N_K(S)O(K) = N_K(S)(O(K)S)$. Since K is S-irreducible, it follows that $K = N_K(S)$ or $K = O(K)S$, which proves (a).

LEMMA 4.3. *Suppose* $\mathfrak{K}$ *is a subfamily of* $\mathfrak{F}_G(S)$, *none of whose elements are* $2'$-*closed. Let* K *be an element of* $\mathfrak{K}$. *Assume that* $1 \subset O_S(\mathfrak{K})$. *Let* $\nu = f_{\mathfrak{K}}(K)$. *Then*

(a) $O_S(\mathfrak{K}) \lhd \langle N_G(S), H \in \mathfrak{K}\rangle$, *and* $O_S(\mathfrak{K}) = S \cap O_2(\langle N_G(S), H \in \mathfrak{K}\rangle)$;

(b) $\nu \geqq 1$, *and* $f_{\mathfrak{K}}(L) = 1$ *for some element* L *of* $\mathfrak{K}$;

(c) $K/W(\mathfrak{K}: \nu)$ *is* $2'$-*closed*;

(d) $K/O_S(K)$ *is* $2'$-*closed*;

(e) $O_S(K) \supseteqq V$, *whenever* $V \lhd N_G(S)$ *and* $V \subseteqq O_2(K)$;

(f) $[O^2(K), O_S(K)] \subseteqq O_S(L)$, *whenever* $L \in \mathfrak{K}$ *and* $f_{\mathfrak{K}}(L) \geqq \nu$.

Proof. (a) Let H be an arbitrary element of $\mathfrak{K}$. Since H is an S-irreducible group which is not $2'$-closed,

$$H = N_H(O_S(\mathfrak{K}))O(H) = N_H(O_S(\mathfrak{K}))(O(H)S) = N_H(O_S(\mathfrak{K})) ,$$

which implies the former part of (a).

Let $T = S \cap O_2(\langle N_G(S), H \in \mathfrak{K}\rangle)$. By the former part of (a), $O_S(\mathfrak{K}) \subseteqq T$. On the other hand, for any element L of $\mathfrak{K}$, $L \rhd L \cap O_2(\langle N_G(S), H \in \mathfrak{K}\rangle) = T$, which shows $T \subseteqq O_S(\mathfrak{K})$. Hence, (a) is proved.

By (a) and the definition of $f_{\mathfrak{K}}$, we obtain that $\nu \geqq 1$. Suppose that $f_{\mathfrak{K}}(L) > 1$ for any element L of $\mathfrak{K}$. Then by definition, $W(\mathfrak{K}: 2) \subset W(\mathfrak{K}: 1)$, which is a contradiction. Hence (b) is proved. To prove (c), we may assume $S \supset W(\mathfrak{K}: \nu)$; otherwise, (c) is obvious. By (a) and definition, $K \rhd W(\mathfrak{K}: \nu)$. Let $\bar{X} = XW(\mathfrak{K}: \nu)/W(\mathfrak{K}: \nu)$ for any subgroup X of K. By Lemma 3.11(a), $\bar{K}$ is $\bar{S}$-irreducible. Suppose $\bar{K}$ is not $2'$-closed. By induction,

$$\bar{K} = N_{\bar{K}}(W(\bar{S}))O(\bar{K}) = N_{\bar{K}}(W(\bar{S}))(O(\bar{K})\bar{S}) = N_{\bar{K}}(W(\bar{S})) .$$

Hence, $K \rhd W(\mathfrak{K}: \nu + 1) \supset W(\mathfrak{K}: \nu)$, which contradicts $\nu = f_{\mathfrak{K}}(K)$. Thus, (c) is proved.

(d): By (a), $K \rhd O_S(K)$. By maximality of $O_S(K)$, $O_S(K) \supseteqq W(\mathfrak{K}: \nu)$. Then, (c) yields (d).

(e) By (d), $[K, VO_S(K)] \subseteqq VO_S(K)$. Since $VO_S(K) \lhd N_G(S)$, $V \subseteqq VO_S(K) \subseteqq O_S(K)$ by maximality of $O_S(K)$, which proves (e).

(f) Let $\mu = f_{\mathfrak{K}}(L)$. By (c), $[O^2(K), O_S(K)] \subseteqq W(\mathfrak{K}: \nu) \subseteqq W(\mathfrak{K}: \mu) \subseteqq O_S(L)$, which proves (f), and this lemma.

LEMMA 4.4. *Let* $\mathfrak{K}$ *be a subfamily of* $\mathfrak{F}_G(S)$. *Assume that there exists a subgroup* T *of* S *which satisfies the following conditions*:

(1) $1 \subset T \subset S$;

(2) $T \lhd N_G(S)$; *and*

(3) $[O^2(K), O_S(K)] \subseteqq T$ *for any element* K *of* $\mathfrak{K}$.
Then

$$O_S(\mathfrak{K}) \supset 1 .$$

Proof. Let $\mathfrak{K} = \{K_i; 1 \leqq i \leqq t\}$, and $L_i = O^2(K_i)T, 1 \leqq i \leqq t$. By (3) and Lemma 4.3(d), L_i is a group with a Sylow 2-subgroup T, and $O(L_i) \subseteqq O(K_i)$ for all $i; 1 \leqq i \leqq t$. By induction, $W(T) \supset 1$. Since $L_i = N_{L_i}(W(T))O(L_i)$, $K_i = N_{K_i}(W(T))O(K_i)$ for all $i; 1 \leqq i \leqq t$. Therefore, we conclude that $1 \subset W(T) \subseteqq O_S(\mathfrak{K})$, as required. Hence this lemma is proved.

LEMMA 4.5. $O_S(\mathfrak{C}_G(S)) = O_S(\mathfrak{F}_G(S)) = 1$.

Proof. Since (G, S) is a counterexample to the theorem,

$$O_S(\mathfrak{C}_G(S)) = 1 .$$

So, we may show only that $O_S(\mathfrak{C}_G(S)) = O_S(\mathfrak{F}_G(S))$. To prove this equation, let $T = O_S(\mathfrak{F}_G(S))$. Obviously, $O_S(\mathfrak{C}_G(S)) \subseteqq T$. To prove the opposite inclusion, take an element K of $\mathfrak{C}_G(S)$. By Lemma 3.10, $K = \langle K_i; 1 \leqq i \leqq t\rangle$ for some suitable elements $\{K_i; 1 \leqq i \leqq t\}$ of $\mathfrak{F}_G(S)$. Then $K_i = N_{K_i}(T)O(K_i)$, and $O(K_i) \subseteqq O(K)$ for all i; $1 \leqq i \leqq t$. Therefore, $K = N_K(T)O(K)$. Since K is an arbitrary element of $\mathfrak{C}_G(S)$, it follows that $T \subseteqq O_S(\mathfrak{C}_G(S))$, which proves this lemma.

5. Properties of elements of $\mathscr{C}\mathscr{L}(G)$.

DEFINITION. Let m be the natural number defined as follows:

$$O_S(\mathfrak{K}) \supset 1 \text{ for any subfamily } \mathfrak{K} \text{ of } \mathfrak{F}_G(S) \text{ with } |\mathfrak{K}| < m ;$$

and

$$O_S(\mathfrak{K}) = 1 \text{ for some subfamily } \mathfrak{K} \text{ of } \mathfrak{F}_G(S) \text{ with } |\mathfrak{K}| = m .$$

REMARK 5.1. Such a natural number m exists by Lemma 4.5.

Define the class of subfamilies $\mathscr{C}\mathscr{L}(G)$ of $\mathfrak{F}_G(S)$ as follows:

$$\mathscr{C}\mathscr{L}(G) = \{\mathfrak{K} \subseteqq \mathfrak{F}_G(S); |\mathfrak{K}| = m \text{ and } O_S(\mathfrak{K}) = 1\} .$$

Take an element $\mathfrak{H}$ of $\mathscr{C}\mathscr{L}(G)$. And let:

$$\mathfrak{H} = \{H_k; 1 \leqq k \leqq m\} ;$$

and

$$\mathfrak{H}_i = \{H_k; 1 \leqq k \neq i \leqq m\},\ 1 \leqq i \leqq m .$$

In this section, we shall investigate the relations among the elements of $\mathfrak{H}$, and also the relations between the elements of $\mathfrak{H}$ and the other elements of $\mathfrak{F}_G(S)$.

We shall employ the above notation throughout this paper.

LEMMA 5.1. *Let $\mathfrak{K}$ be a subfamily of $\mathfrak{F}_G(S)$.*
Then

(a) *whenever* $|\mathfrak{K}| \leqq 2$, $O_S(\mathfrak{K}) \supset 1$; *in particular,*

(b) $m \geqq 3$.

Proof. Take a subfamily $\mathfrak{K}$ of $\mathfrak{F}_G(S)$ with $|\mathfrak{K}| \leqq 2$. Then, Lemma 3.11(h) implies that $\Omega_1 Z(S)$, $\Omega_1 Z\hat{J}(S)$ or $J_e(S)$ is contained in $O_S(\mathfrak{K})$, which proves this lemma.

LEMMA 5.2. *Fix* $k = 1, 2, \cdots, m$. *Then*

(a) H_k *is neither* 2-*closed, nor* 2′-*closed;*

(b) $1 \subset O_S(H_k) \subset S$;

(c) *for some prime* $p_k \geqq 5$, H_k *is a* $\{2, p_k\}$-*group with a nonidentity Sylow* p_k-*subgroup.*

Proof. Suppose that H_k is 2-closed or 2′-closed. Let $T = O_S(\mathfrak{H}_k)$. Since $|\mathfrak{H}_k| < m$, $T \supset 1$. Obviously, $H_k = N_{H_k}(T)O(H_k)$ and $H_i = N_{H_i}(T)O(H_i)$ for all $i; 1 \leqq i \neq k \leqq m$. Therefore, $1 \subset T \subseteqq O_S(\mathfrak{H})$, which is a contradiction. Hence, (a) is proved.

By Lemma 5.1, $1 \subset O_S(H_k)$, and by (a) and Lemma 4.2, $O_S(H_k) \subset S$, which proves (b).

By Lemma 3.11(b), H_k is a $\{2, p_k\}$-group with a nonidentity Sylow p_k-subgroup for some odd prime p_k. Then by (a) and Lemma 3.9, $p_k \neq 3$, which proves (c).

The next lemma follows from Lemma 5.2(a) and 4.3(b):

LEMMA 5.3. *Fix* $k = 1, 2, \cdots, m$. *Then*

(a) $f_{\mathfrak{H}_k}(H_i) \geqq 1$ *for all* $i; 1 \leqq i \neq k \leqq m$; *and*

(b) $f_{\mathfrak{H}_k}(H_j) = 1$ *for some* $j; 1 \leqq j \neq k \leqq m$.

Since $O_S(\mathfrak{H}) = 1$ by definition, the next lemma follows from Lemma 4.4.

LEMMA 5.4. *There does not exist a subgroup T of S with the following properties:*

(1) $1 \subset T \subset S$;
(2) $T \lhd N_G(S)$; *and*
(3) $[O^2(H_i), O_S(H_i)] \subseteqq T$ *for all* $i; 1 \leqq i \leqq m$.

LEMMA 5.5. *There does not exist* H_k *such that*

$$[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i) \text{ for all } i; 1 \leqq i \leqq m .$$

Proof. Suppose that there exists an element H_k of $\mathfrak{H}$ such that $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$ for all $i; 1 \leqq i \leqq m$. Let H_j be an element of $\mathfrak{H}_k$ such that $f_{\mathfrak{H}_k}(H_j) = \max. \{f_{\mathfrak{H}_k}(H_i); 1 \leqq i \neq k \leqq m\}$. Let $T = O_S(H_j)$. By Lemma 5.2(b),

$$(5.5.1) \qquad 1 \subset T \subset S, \text{ and } T \lhd N_G(S) .$$

By Lemma 4.3(f),

$$(5.5.2) \qquad [O^2(H_i), O_S(H_i)] \subseteqq T \text{ for all } i; 1 \leqq i \neq k \leqq m .$$

On the other hand, by the assumption,

$$(5.5.3) \qquad [O^2(H_k), O_S(H_k)] \subseteqq T .$$

By (5.5.2) and (5.5.3),

$$(5.5.4) \qquad [O^2(H_i), O_S(H_i)] \subseteqq T \text{ for all } i; 1 \leqq i \leqq m .$$

Then, (5.5.1) and (5.5.4) contradict the preceding lemma. Hence this lemma is proved.

DEFINITION. Define the mapping σ from $\mathfrak{H}$ to the family of all subsets of $\mathfrak{H}$ as follows: for any element H_k of $\mathfrak{H}$,

$$\sigma(H_k) = \{H_i \in \mathfrak{H}_k; f_{\mathfrak{H}_k}(H_i) = 1\} .$$

We call σ the eigen -mapping of $\mathfrak{H}$.

LEMMA 5.6. (a) $\sigma(H_j) \neq \varnothing$ *for any element* H_j of $\mathfrak{H}$;
(b) $\sigma(H_j) \cap \sigma(H_h) = \varnothing$ *for any distinct elements* H_j *and* H_h *of* $\mathfrak{H}$; *and*
(c) σ *induces a permutation of* $\mathfrak{H}$.

Proof. Since $\mathfrak{H}_k \neq \varnothing$ by Lemma 5.1, $1 \leqq k \leqq m$, (a) follows from Lemma 5.3(b). To prove (b), we assume that $\sigma(H_j) \cap \sigma(H_h) \neq \varnothing$ for some distinct elements H_j and E_h of $\mathfrak{H}$. Take H_k in $\sigma(H_j) \cap \sigma(H_h)$.
As $H_k \in \sigma(H_j)$, by Lemma 5.2(a) and Lemma 4.3(f),

$$(5.6.1) \qquad [O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i) \text{ for all } i; 1 \leqq i \neq j \leqq m .$$

Similarly, as $H_k \in \sigma(H_h)$,

(5.6.2) $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$ for all $i; 1 \leqq i \neq h \leqq m$.

By (5.6.1) and (5.6.2),

$$[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i) \text{ for all } i; 1 \leqq i \leqq m ,$$

which contradicts the preceding lemma. Hence (b) is proved.

Since $|\mathfrak{H}|$ is finite, (c) follows from (a) and (b). Thus, this lemma is proved.

According to the preceding lemma, we may consider σ as an element of a permutation group on $\mathfrak{H}$.

LEMMA 5.7. *$\langle\sigma\rangle$ acts transitively on $\mathfrak{H}$.*

Proof. Suppose that $\langle\sigma\rangle$ acts intransitively on $\mathfrak{H}$. Let $\mathfrak{K}$ be a $\langle\sigma\rangle$-orbit of $\mathfrak{H}$. Then $\mathfrak{K} \subset \mathfrak{H}$. So, $O_S(\mathfrak{K}) \supset 1$. Let H_k be an element of $\mathfrak{K}$ such that $f_{\mathfrak{K}}(H_k) = \min. \{f_{\mathfrak{K}}(H); H \in \mathfrak{K}\}$. By Lemma 4.3(f),

(5.7.1) $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$ for any element H_i of $\mathfrak{K}$.

Let H_j be an element of $\mathfrak{H}$ such that $\sigma(H_j) = H_k$. Then

(5.7.2) H_j lies in $\mathfrak{K}$.

As $f_{\mathfrak{H}_j}(H_k) = 1$, Lemma 4.3(f) shows that

(5.7.3) $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$ for all $i; 1 \leqq i \neq j \leqq m$.

By (5.7.1), (5.7.2) and (5.7.3), $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$ for all $i; 1 \leqq i \leqq m$, which contradicts Lemma 5.5. Hence this lemma is proved.

By the preceding lemma, we may assume:

$$\sigma = \begin{pmatrix} H_1 & H_2 & H_3 \cdots H_{m-1} & H_m \\ H_2 & H_3 & H_4 \cdots H_m & H_1 \end{pmatrix}, \text{ renumbering if necessary .}$$

REMARK 5.2. In the following, without notice, all the suffixes of elements of $\mathfrak{H}$ will be used modulo m, if necessary. For instance, $H_{m+1} = H_1$, $H_0 = H_m$, and so on.

LEMMA 5.8. *Fix $k = 1, 2, \cdots, m$. Then*
(a) $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$, *if* $i \not\equiv k - 1$ (modulo m);
(b) $[O^2(H_k), O_S(H_k)] \not\subseteqq O_S(H_i)$, *if* $i \equiv k - 1$ (modulo m).

Proof. By definition, $f_{\mathfrak{H}_{k-1}}(H_k) = 1$. So, by Lemma 5.3(a), $f_{\mathfrak{H}_{k-1}}(H_k) = \min. \{f_{\mathfrak{H}_{k-1}}(H_i); 1 \leqq i \neq k - 1 \leqq m\}$. Then (a) follows from

Lemma 4.3(f).

Suppose $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_{k-1})$. Then by (a), $[O^2(H_k), O_S(H_k)] \subseteqq O_S(H_i)$ for all $i; 1 \leqq i \leqq m$, which contradicts Lemma 5.5. Hence (b) holds. This lemma is proved.

LEMMA 5.9. *Fix* $k = 1, 2, \cdots, m$. *Then*

$$1 = f_{\mathfrak{H}_k}(H_{k+1}) < f_{\mathfrak{H}_k}(H_{k+2}) < \cdots < f_{\mathfrak{H}_k}(H_m) < f_{\mathfrak{H}_k}(H_1) < \cdots < f_{\mathfrak{H}_k}(H_{k-1}) .$$

Proof. By the definition of σ, $f_{\mathfrak{H}_k}(H_{k+1}) = 1$. Suppose $f_{\mathfrak{H}_k}(H_i) \geqq f_{\mathfrak{H}_k}(H_i) \geqq f_{\mathfrak{H}_k}(H_{i+1})$ for some $i; 1 \leqq i \neq k-1, k \leqq m$. By Lemma 4.3(f), $[O^2(H_{i+1}), O_S(H_{i+1})] \subseteqq O_S(H_i)$, which contradicts Lemma 5.8(b). So, $f_{\mathfrak{H}_k}(H_i) < f_{\mathfrak{H}_k}(H_{i+1})$ for all $i; 1 \leqq i \neq k \leqq m$, which implies this lemma.

LEMMA 5.10. *Fix* $k = 1, 2, \cdots, m$. *Let* $\{H_i, H_{i+1}, \cdots, H_{i+s}\}$ *be a subset of* $\mathfrak{H}$. *Let* $\mathfrak{K}$ *be a subfamily of* $\mathfrak{H}_G(S)$ *which contains* $\{H_i, H_{i+1}, \cdots, H_{i+s}\}$. *Assume* $O_S(\mathfrak{K}) \supset 1$. *Then*

(a) $f_{\mathfrak{K}}(H_i) < f_{\mathfrak{K}}(H_{i+1}) < \cdots < f_{\mathfrak{K}}(H_{i+s})$; *and*

(b) *if* $\mathfrak{K} = \{H_i, H_{i+1}, \cdots, H_{i+s}\}$, *then* $f_{\mathfrak{L}}(H_i) = 1$.

Proof. Suppose $f_{\mathfrak{K}}(H_{i+u}) \geqq f_{\mathfrak{K}}(H_{i+u+1})$ for some $u; 0 \leqq u \leqq s-1$. Then by Lemma 4.3(f), $[O^2(H_{i+u+1}), O_S(H_{i+u+1})] \subseteqq O_S(H_{i+u})$, which contradicts Lemma 5.8(b). So, $f_{\mathfrak{K}}(H_{i+u}) < f_{\mathfrak{K}}(H_{i+u+1})$ for all $u; 0 \leqq u \leqq s-1$, which implies (a).

(b) follows from Lemma 5.2(a) and Lemma 4.3(b).

LEMMA 5.11. *Fix* $k = 1, 2, \cdots, m$. *Suppose* M *is a subgroup of* G *which contains* $N_G(S)$, H_k *and* H_{k+1}. *Let* V *be a normal subgroup of* M. *Then*

(a) *if* $O^2(H_k) \not\subseteqq V$, *then* $S \cap V \lhd H_k$ *and* $S \cap V \subseteqq O_S(H_k)$;

(b) *if* $O^2(H_{k+1}) \not\subseteqq V$, *then* $S \cap V \lhd H_{k+1}$ *and* $S \cap V \subseteqq O_S(H_{k+1})$;

(c) *if* $O^2(H_{k+1}) \subseteqq V$, *then* $O^2(H_k) \subseteqq V$;

(d) *if* $M = \langle N_G(S), H_k, H_{k+1}\rangle$ *and* $O^2(H_k) \not\subseteqq V$, *then* $S \cap V \subseteqq O_S(\{H_k, H_{k+1}\})$ *and* $S \cap V \lhd M$.

Proof. (a) Since $O^2(H_k) \not\subseteqq V$, by Lemma 5.2(a) and 3.11(e), $S \cap V \lhd H_k$. Since $S \cap V \lhd N_G(S)$, by Lemma 4.3(e), $S \cap V \subseteqq O_S(H_k)$, which proves (a). Similarly, we obtain (b).

Suppose $O^2(H_{k+1}) \subseteqq V$ and $O^2(H_k) \not\subseteqq V$. Then by (a), $[O^2(H_{k+1}), O_S(H_{k+1})] \subseteqq V \cap S \subseteqq O_S(H_k)$, which contradicts Lemma 5.8(b). Hence (c) is proved. (d) follows from (a), (b) and (c). Thus, this lemma is proved.

REMARK 5.3. We note that all the statements from Lemma 5.1 to 5.11 are valid for any element of $\mathscr{C}\mathscr{L}(G)$. So, we can define the eigen-mapping for any element of $\mathscr{C}\mathscr{L}(G)$.

LEMMA 5.12. *Fix* $k=1, 2, \cdots, m$. *Let* K *be an element of* $\mathfrak{F}_G(S)$, *and let* $\mathfrak{K} = \mathfrak{H}_{\mathfrak{K}} \cup \{K\}$. *Assume that* $\mathfrak{K}$ *lies in* $\mathscr{C}\mathscr{L}(G)$. *Let* τ *be the eigen-mapping of* $\mathfrak{K}$. *Then*

(a) $\tau(H_i) = H_{i+1}$, *if* $i \neq k-1, k$;

(b) $\tau(K) = H_{k+1}$; *and*

(c) $\tau(H_{k-1}) = K$.

Proof. Lemma 5.8 implies that the following (α), (β) and (γ) are equivalent: for any element L and M of $\mathfrak{K}$,

(α) $\tau(L) = M$;

(β) $[O^2(M), O_S(M)] \not\subseteqq O_S(L)$; and

(γ) $[O^2(M), O_S(M)] \subseteqq O_S(N)$ for any element N of $\mathfrak{K}$ with $N \neq L$.

Therefore, (a) follows from the equivalence (α) to (β). (b) follows from the equivalence (α) to (γ). Then (c) follows from (a) and (b). By the preceding lemma and Lemma 5.8, we obtain:

LEMMA 5.13. *Fix* $k = 1, 2, \cdots, m$. *Let* K *be an element of* $\mathfrak{F}_G(S)$. *Assume that* $\mathfrak{H}_{\mathfrak{K}} \cup \{K\}$ *lies in* $\mathscr{C}\mathscr{L}(G)$. *Then*

(a) $[O^2(H_{k+1}), O_S(H_{k+1})] \not\subseteqq O_S(K)$;

(b) $[O^2(H_i), O_S(H_i)] \subseteqq O_S(K)$ *for all* $i; 1 \leqq i \neq k, k+1 \leqq m$;

(c) $[O^2(K), O_S(K)] \not\subseteqq O_S(H_{k-1})$; *and*

(d) $[O^2(K), O_S(K)] \subseteqq O_S(H_i)$ *for all* $i; 1 \leqq i \neq k-1, k \leqq m$.

LEMMA 5.14. *Fix* $k = 1, 2, \cdots, m$. *Suppose* $\{K_i; 1 \leqq i \leqq t\}$ *is a subfamily of* $\mathfrak{F}_G(S)$. *Let* T *be a subgroup of* S. *Assume*:

(1) $T \lhd N_G(S)$;

(2) $T \lhd K_i$ *and* K_i/T *is* 2'*-closed; and*

(3) $\mathfrak{H}_k \cup \{K_i\}$ *lies in* $\mathscr{C}\mathscr{L}(G)$ *for all* $i; 1 \leqq i \leqq t$.

Then

$$S = O_S(H_{k+1})\Big(\bigcap_{i=1}^{t} O_S(K_i)\Big).$$

Proof. Let $V = \bigcap_{i=1}^{t} O_S(K_i)$. By (1), (2) and Lemma 4.3(e), $T \subseteqq V$. By (2),

(5.14.1) $[O^2(K_i), O_S(K_i)] \subseteqq T \subseteqq V \subseteqq VO_S(H_{k+1})$ for all $i; 1 \leqq i \leqq t$.

By (3) and Lemma 5.13(b), $[O^2(H_i), O_S(H_i)] \subseteqq O_S(K_i)$ for all $i, j; 1 \leqq j \neq k-1, k \leqq m$ and $1 \leqq i \leqq t$. Therefore,

(5.14.2) $[O^2(H_j), O_S(H_j)] \subseteqq V \subseteqq VO_S(H_{k+1})$ for all $j; 1 \leqq j \neq k-1, k \leqq m$.

on the other hand, obviously,

$$(5.14.3) \qquad [O^2(H_{k+1}), O_S(H_{k+1})] \subseteqq O_S(H_{k+1}) \subseteqq VO_S(H_{k+1}) .$$

Suppose $VO_S(H_{k+1}) \subset S$. Then by (5.14.1), (5.14.2), (5.14.3) and Lemma 4.4, $O_S(\mathfrak{H}_k \cup \{K_i\}) \supset 1$ for all $i; 1 \leqq i \leqq t$, which contradicts (3). Hence $S = VO_S(H_{k+1})$, as required.

Next, we shall prove the dual statement of the preceding lemma in some sense.

LEMMA 5.15. *Fix* $k = 1, 2, \cdots, m$. *Suppose* $\mathfrak{K} = \{K_i; 1 \leqq i \leqq t\}$ *is a subfamily of* $\mathfrak{F}_G(S)$. *Let* T *be a subgroup of* S. *Assume*:

(1) $T \lhd N_G(S)$;

(2) $K_i \rhd T$, *and* K_i/T *is 2'-closed; and*

(3) $O_S(\mathfrak{H}_{\mathfrak{K}} \cup \{K_i\}) \supset 1$ *for all* $i; 1 \leqq i \leqq t$.

Then

$$O_S(\mathfrak{H}_{\mathfrak{K}} \cup \mathfrak{K}) \supset 1 .$$

Proof. We shall use induction on t.

By (1), (2) and Lemma 4.3(e),

$$(5.15.1) \qquad T \subseteqq O_S(K_i) \text{ for all } i; 1 \leqq i \leqq t .$$

Let $\mathfrak{K}_i = \mathfrak{H}_k \cup \{K_i\}$, $1 \leqq i \leqq t$. Assume that $O_S(K_j) = S$ for some $j; 1 \leqq j \leqq t$. Let $V = O_S(\bigcup_{1 \leqq i \neq j \leqq t} \mathfrak{K}_i)$. By induction and definition,

$$(5.15.2) \qquad V \supset 1 \text{ and } L = N_L(V)O(L) \text{ for all } L \in \bigcup_{1 \leqq i \neq j \leqq t} \mathfrak{K}_i .$$

On the other hand, by Lemma 4.2,

$$(5.15.3) \qquad K_j = N_{K_j}(V)O(K_j) .$$

By (5.15.2) and (5.15.3), $1 \subset V \subseteqq O_S(\mathfrak{H}_{\mathfrak{K}} \cup \mathfrak{K})$, as required. Hence, we may assume that:

$$(5.15.4) \qquad 1 \subset O_S(K_i) \subset S \text{ for all } i; 1 \leqq i \leqq t .$$

Next, assume that: for some $j; 1 \leqq j \leqq t$, $f_{\mathfrak{K}_j}(K_j) = \max. \{f_{\mathfrak{K}_j}(L); L \in \mathfrak{K}_j\}$. Then by Lemma 4.3(f),

$$(5.15.5) \qquad [O^2(H_i), O_S(H_i)] \subseteqq O_S(K_j) \text{ for all } i; 1 \leqq i \neq k \leqq m .$$

By (2) and (5.15.1),

$$(5.15.6) \qquad [O^2(K_i), O_S(K_i)] \subseteqq T \subseteqq O_S(K_j) \text{ for all } i; 1 \leqq i \leqq t .$$

By (5.15.4), (5.15.5), (5.15.6) and Lemma 4.4, $O_S(\mathfrak{H}_{\mathfrak{K}} \cup \mathfrak{K}) \supset 1$, as required. Thus we may assume that:

$$f_{\mathfrak{K}_i}(K_i) < \max.\ \{f_{\mathfrak{L}_i}(L); L \in \mathfrak{K}_i\} \text{ for all } i; 1 \leqq i \leqq t\ .$$

By Lemma 5.10(a),

$$f_{\mathfrak{L}_i}(H_{k-1}) = \max.\ \{f_{\mathfrak{K}_i}(L); L \in \mathfrak{K}_i\} \text{ for all } i; 1 \leqq i \leqq t\ .$$

By Lemma 4.3(f),

(5.15.7) $[O^2(H_j), O_S(H_j)] \subseteqq O_S(H_{k-1})$ for all $j; 1 \leqq j \neq k \leqq m$,

and

(5.15.8) $[O^2(K_i), O_S(K_i)] \subseteqq O_S(H_{k-1})$ for all $i; 1 \leqq i \leqq t$.

Thus, by (5.15.7) and (5.15.8), Lemmas 4.4 and 5.2(b), $O_S(\mathfrak{H}_k \cup \mathfrak{K}) \supset 1$, which proves this lemma.

DEFINITION. Let $\mathfrak{K} = \{K_i; 1 \leqq i \leqq m\}$ be an element of $\mathscr{CL}(G)$, and τ be the eigen-mapping of $\mathfrak{K}$.

Then we may assume that: $\tau(K_i) = K_{i+1}$ for all $i; 1 \leqq i \leqq m-1$, and $\tau(K_m) = K_1$, renumbering if necessary.

Let $\mathscr{Y}(\mathfrak{K}) = \{K \in \mathfrak{K}; K \neq N_K(J_e(S))\}$. Then $\mathscr{Y}(\mathfrak{K}) \neq \emptyset$; otherwise $1 \subset J_e(S) \subseteqq O_S(\mathfrak{K})$, a contradiction to the fact that $\mathfrak{K}$ lies in $\mathscr{CL}(G)$.

So, we may assume that K_1 lies in $\mathscr{Y}(\mathfrak{K})$, by (cyclically) renumbering, if necessary.

Let $\mathscr{Y}(\mathfrak{K}) = \{K_1 = K_{i_1}, K_{i_2}, \cdots, K_{i_\lambda}\}$, where $1 = i_1 < i_2 < \cdots < i_\lambda$. For each element K_{i_ν} of $\mathscr{Y}(\mathfrak{K})$, $1 \leqq \nu \leqq \lambda$, we define:

if $|\mathscr{Y}(\mathfrak{K})| \geqq 2$, $\alpha(\mathfrak{K}: K_{i_\nu}) = \{K_t; i_\nu \leqq t < i_{\nu+1}\}$, $1 \leqq \nu \leqq \lambda$;

and

if $\mathscr{Y}(\mathfrak{K}) = \{K_1\}$, $\alpha(\mathfrak{K}: K_1) = \{K_t; 1 \leqq t \leqq m-1\}$.

Define $\beta(\mathfrak{K}: K_{i_\nu}) = \{K \in \alpha(\mathfrak{K}: L_{i_\nu}); [O^2(K), \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{K}: K_{i_\nu}))J_e(S))] = 1\}$. Finally, we define $\delta(\mathfrak{K}) = \sum_{\nu=1}^{\lambda} |\beta(\mathfrak{K}: K_{i_\nu})|$.

REMARK 5.4 (i). We can easily verify that for any element $\mathfrak{K}$ of $\mathscr{CL}(G)$ and any element K of $\mathscr{Y}(\mathfrak{K})$, $\alpha(\mathfrak{K}: K)$, $\beta(\mathfrak{K}: K)$ and $\delta(\mathfrak{K})$ are defined independently of numbering of the elements of $\mathfrak{K}$.

(ii) By definition, if $|\mathscr{Y}(\mathfrak{K})| \geqq 2$, then $\mathfrak{K} = \bigcup_{\nu=1}^{\lambda} \alpha(\mathfrak{K}: K_{i_\nu})$ (disjoint union).

LEMMA 5.16. *Suppose H_j is an element of $\mathscr{Y}(\mathfrak{H})$.* Let $\mathfrak{K} = \alpha(\mathfrak{K}: H_j)$, *$T = O_S(\mathfrak{K})J_e(S)$, and $A = \langle N_G(S), H \in \mathfrak{K}\rangle$. Then*

(a) $[\Omega_1 Z(S), O^2(H_j)] = [\Omega_1 Z\hat{J}(T), O^2(H_j)] = 1$;

(b) $\Omega_1 Z\hat{J}(T) \lhd A$; *and*

(c) $[\Omega_1 Z(S), H_{j-1}] = 1$.

Proof. By Lemma 3.2,

(5.16.1) $\Omega_1 Z(S) \subseteqq \Omega_1 Z\hat{J}(S) \subseteqq \Omega_1 Z\hat{J}(T)$, and $J_e(T) = J_e(S)$.

Since $\mathfrak{K} \subset \mathfrak{H}_{j-1}$, by Lemma 5.10(b), $f_{\mathfrak{K}}(H_j) = 1$. By Lemma 4.3(c), T is a Sylow 2-subgroup of $O^2(H_j)T$. Let $L = O^2(H_j)T$. By Theorem 3.1(b), $L = N_L(J_e(T))C_L(\Omega_1 Z\hat{J}(T))O(L)$. So, by (5.16.1),

$$H_j = N_{H_j}(J_e(S))(C_{H_j}(\Omega_1 Z\hat{J}(T))S)(O(L)S) .$$

Since $H_j \neq N_{H_j}(J_e(S))$ and $H_j \neq O(H_j)S$,

(5.16.2) $$H_j = C_{H_j}(\Omega_1 Z\hat{J}(T))S .$$

(a) follows from (5.16.1) and (5.16.2).

To prove (b), take any element H_k of $\mathfrak{K}$ with $H_k \neq H_j$, if possible. Since $H_k = N_{H_k}(J_e(S))$, $H_k = N_{H_k}(T)$. So,

(5.16.3) $$T \lhd H_k .$$

Therefore, (b) follows from (a). To prove (c), by Lemma 3.11(h), we may assume that:

(5.16.4) $$H_{j-1} = N_{H_{j-1}}(\hat{J}(S)) .$$

Let $B = \langle N_G(S), H_{j-1}, H_j \rangle$, and $B_0 = C_B(\Omega_1 Z\hat{J}(S))$. By (a) and (5.16.4), $O^2(H_j) \subseteqq B_0 \lhd B$. Then by Lemma 5.11(c), $O^2(H_{j-1}) \subseteqq B_0$, so that $[\Omega_1 Z(S), O^2(H_{j-1})] \subseteqq [\Omega_1 Z\hat{J}(S), O^2(H_{j-1})] = 1$, which proves (c), and this lemma.

LEMMA 5.17. *Let H_j be an element of $\mathscr{Y}(\mathfrak{H})$. Assume:*

(1) $H_{k+1} \in \beta(\mathfrak{H}: H_j)$; *and*

(2) $H_k \in \alpha(\mathfrak{H}: H_j)$,

then

$$H_k \in \beta(\mathfrak{H}: H_j) .$$

Proof. Let $T = O_S(\alpha(\mathfrak{H}: H_j))J_e(S)$, $B = \langle N_G(S), H_k, H_{k+1} \rangle$, and $B_0 = C_B(\Omega_1 Z\hat{J}(T))$. By Lemma 5.16(b), $B_0 \lhd B$. By (1), $O^2(H_{k+1}) \subseteqq B_0$. Then by Lemma 5.11(c), $O^2(H_k) \subseteqq B_0$, which shows that $H_k \in \beta(\mathfrak{H}: H_j)$, as required. Hence this lemma is proved.

LEMMA 5.18. *There exists a trio of elements H_j, H_k and H_{k+1} of $\mathfrak{H}$ which satisfies the following conditions:*

(a) $H_j \in \mathscr{Y}(\mathfrak{H})$;

(b) $H_k \in \beta(\mathfrak{H}: H_j)$; *and*

(c) $H_{k+1} \in \alpha(\mathfrak{H}: H_j) \backslash \beta(\mathfrak{H}: H_j)$.

(Here, it is admissible also that $H_j = H_k$.)

Proof. Suppose that for any element H_j of $\mathscr{Y}(\mathfrak{H})$, $\beta(\mathfrak{H}: H_j) = \alpha(\mathfrak{H}: H_j)$. Assume $|\mathscr{Y}(\mathfrak{H})| \geqq 2$. Then, from the definitions, we have that $\mathfrak{H} = \bigcup_{H_j} \beta(\mathfrak{H}: H_j)$ (disjoint union), where H_j ranges over all the elements of $\mathscr{Y}(\mathfrak{H})$. Hence by definition, $H_i = C_{H_i}(\Omega_1 Z(S))$ for all $i; 1 \leqq i \leqq m$, so that $1 \subset \Omega_1 Z(S) \subseteqq O_S(\mathfrak{H})$, a contradiction. Next, assume $|\mathscr{Y}(\mathfrak{H})| = 1$. Let $\mathscr{Y}(\mathfrak{H}) = \{H_j\}$. By definition and the assumption, $\mathfrak{H} = \alpha(\mathfrak{H}: H_j) \cup \{H_{j-1}\} = \beta(\mathfrak{H}: H_j) \cup \{H_{j-1}\}$. By Lemma 5.16(c) and definition, $1 \subset \Omega_1 Z(S) \subseteqq O_S(\mathfrak{H})$, a contradiction. Therefore, there exists an element H_j of $\mathscr{Y}(\mathfrak{H})$ such that $\beta(\mathfrak{H}: H_j) \subset \alpha(\mathfrak{H}: H_j)$. Let $\alpha(\mathfrak{H}: H_j) = \{H_j, H_{j+1}, \cdots, H_{j+t}\}$. By Lemma 5.17, we may assume that $\beta(\mathfrak{H}: H_j) = \{H_j H_{j+1}, \cdots, H_{j+s}\}$, where $s < t$. Setting $H_k = H_{j+s}$, we have this lemma.

LEMMA 5.19. *Suppose that H_j, H_k, and H_{k+1} are elements of $\mathfrak{H}$, and L is an element of $\mathfrak{F}_G(S)$. Let $\mathfrak{K} = \mathfrak{H}_k \cup \{L\}$. Assume:*

(1) $H_j \in \mathscr{Y}(\mathfrak{H})$;

(2) $H_k \in \beta(\mathfrak{H}: H_j)$;

(3) $H_{k+1} \in \alpha(\mathfrak{H}: H_j) \backslash \beta(\mathfrak{H}: H_j)$;

(4) $\mathfrak{K}$ lies in $\mathscr{C}\mathscr{L}(G)$;

(5) $\langle N_G(S), L, H_{k+1}\rangle \subseteqq \langle N_G(S), H_k, H_{k+1}\rangle$; *and*

(6) $L = N_L(J_e(S))$ *if and only if* $H_k = N_{H_k}(J_e(S))$.

Then

(a) *Assume* $L \neq N_L(J_e(S))$.

Then

(a.1) $H_{k+1} \in \alpha(\mathfrak{K}: L)$;

(a.2) $\delta(\mathfrak{K}) \geqq \delta(\mathfrak{H})$; *and*

(a.3) *if* $H_{k+1} \in \beta(\mathfrak{K}: L)$, $\delta(\mathfrak{K}) > \delta(\mathfrak{H})$.

(b) *Assume* $L = N_L(J_e(S))$ *and* $[O^2(L), \Omega_1 Z \hat{J}(O_S(\alpha(\mathfrak{H}: H_j)) J_e(S))] = 1$.

Then

(b.1) $H_j \in \mathscr{Y}(\mathfrak{K})$ *and* $H_{k+1} \in \alpha(\mathfrak{K}: H_j)$;

(b.2) $\delta(\mathfrak{K}) \geqq \delta(\mathfrak{H})$; *and*

(b.3) *if* $H_{k+1} \in \beta(\mathfrak{K}: H_j)$, $\delta(\mathfrak{K}) > \delta(\mathfrak{H})$.

Proof. In this proof, we shall quote the results of Lemma 5.12 without notice.

Let $\mathscr{Y}(\mathfrak{H}) = \{H_j = H_{j_1}, H_{j_2}, \cdots, H_{j_t}\}$.

To prove (a), assume $L \neq N_L(J_e(S))$. By (1), (2) and (6), $H_j = H_k$. So, by (3) and Lemma 5.17,

$$\beta(\mathfrak{H}: H_j) = \{H_j\} . \tag{5.19.1}$$

Then it is easily verified that

$$\begin{gathered}\mathscr{Y}(\mathfrak{K}) = \{L, H_{j_2}, H_{j_3}, \cdots, H_{j_t}\},\ \alpha(\mathfrak{K}: L) = \{L\} \cup \alpha(\mathfrak{H}: H_j) \backslash \{H_j\}, \\ \text{and whenever } 2 \leqq i \leqq t,\ \beta(\mathfrak{K}: H_{j_i}) = \beta(\mathfrak{H}: H_{j_i}) .\end{gathered} \tag{5.19.2}$$

Then (a.1) follows from this fact.

By (5.19.1) and (5.19.2),

$$\begin{aligned}\delta(\mathfrak{K}) &= \sum_{i=2}^{t} |\beta(\mathfrak{K}\colon H_{j_i})| + |\beta(\mathfrak{K}\colon L)| \\ &\geqq \sum_{i=2}^{t} |\beta(\mathfrak{K}\colon H_{j_i})| + 1 \\ &= \sum_{i=2}^{t} |\beta(\mathfrak{H}\colon H_{j_i})| + |\beta(\mathfrak{H}\colon H_j)| \\ &= \delta(\mathfrak{H}) ,\end{aligned}$$

which proves (a.2). Then we observe that equality holds in the above if and only if $\beta(\mathfrak{K}\colon L) = \{L\}$, which proves (a.3).

Next, assume $L = N_L(J_e(S))$. By (1), (2), and (6), $H_k \neq H_j$. Then, it is easily verified that:

(5.19.3) $$\mathscr{Y}(\mathfrak{K}) = \mathscr{Y}(\mathfrak{H}),\ \text{and whenever } 2 \leqq i \leqq t,\ \beta(\mathfrak{K}\colon H_{j_i}) = \beta(\mathfrak{H}\colon H_{j_i});\ \text{moreover,}$$

(5.19.4) $$\alpha(\mathfrak{K}\colon H_j) = \{L\} \cup \alpha(\mathfrak{H}\colon H_j)\backslash\{H_k\} .$$

Then, (b.1) follows from (5.19.3) and (5.19.4). By (5) and (5.19.4), $\langle N_G(S),\ K \in \alpha(\mathfrak{K}\colon H_j)\rangle \subseteqq \langle N_G(S),\ K \in \alpha(\mathfrak{H}\colon H_j)\rangle$. By Lemma 4.3(a), $O_S(\alpha(\mathfrak{K}\colon H_j))J_e(S) \supseteqq O_S(\alpha(\mathfrak{H}\colon H_j))J_e(S)$. Thus, by Lemma 3.2(c),

(5.19.5) $$\Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{K}\colon H_j))J_e(S)) \subseteqq \Omega_1 Z\hat{J}((\alpha(\mathfrak{H}\colon H_j))J_e(S)) .$$

By (5.19.4) and (5.19.5),

(5.19.6) $$\beta(\mathfrak{H}\colon H_j)\backslash\{H_k\} \subseteqq \beta(\mathfrak{K}\colon H_j) .$$

Since $L \in \beta(\mathfrak{K}\colon H_j)$ by the assumption of (b) and (5.19.5),

(5.19.7) $$|\beta(\mathfrak{K}\colon H_j)| \geqq |\beta(\mathfrak{H}\colon H_j)| .$$

By (5.19.3) and (5.19.7),

$$\begin{aligned}\delta(\mathfrak{H}) &= \sum_{i=2}^{t} |\beta(\mathfrak{H}\colon H_{j_i})| + |\beta(\mathfrak{H}\colon H_j)| \\ &\leqq \sum_{i=2}^{t} |\beta(\mathfrak{K}\colon H_{j_i})| + |\beta(\mathfrak{K}\colon H_j)| \\ &= \delta(\mathfrak{K}) ,\end{aligned}$$

which proves (b.2).

By (5.19.6), equality holds in the above if and only if $\beta(\mathfrak{K}\colon H_j) = \{L\} \cup \beta(\mathfrak{H}\colon H_j)\backslash\{H_k\}$. Then (3) yields (b.3). Hence this lemma is proved.

LEMMA 5.20. *Suppose that H_j, H_k and H_{k+1} are elements of $\mathfrak{H}$,* *and* L *is an element of* $\mathfrak{F}_G(S)$. *Let* $\mathfrak{K} = \mathfrak{H}_{k+1} \cup \{L\}$. *Assume*:

(1) $H_j \in \mathscr{Y}(\mathfrak{H})$;

(2) $H_k \in \beta(\mathfrak{H}: H_j)$;

(3) $H_{k+1} \in \alpha(\mathfrak{H}: H_j) \backslash \beta(\mathfrak{H}: H_j)$;

(4) $\mathfrak{K}$ *lies in* $\mathscr{C}\mathscr{L}(G)$;

(5) $\langle N_G(S), H_k, L \rangle \subseteqq \langle N_G(S), H_k, H_{k+1} \rangle$; *and*

(6) $[O^2(L), \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{H}: H_j))J_e(S))] = 1$.

Then

$$\delta(\mathfrak{K}) > \delta(\mathfrak{H}) .$$

Proof. Let $\mathscr{Y}(\mathfrak{H}) = \{H_j = H_{j_1}, H_{j_2}, \cdots, H_{j_t}\}$.

It is easily verified that

(5.20.1) $$\text{whenever } 2 \leqq i \leqq t,\ \beta(\mathfrak{K}: H_{j_i}) = \beta(\mathfrak{H}: H_{j_i}) .$$

First, assume $L \neq N_L(J_e(S))$. Then

(5.20.2) $$\mathscr{Y}(\mathfrak{K}) = \{L, H_{j_1}, H_{j_2}, \cdots, H_{j_t}\} .$$

By (2), (3) and Lemma 5.17,

(5.20.3) $$\beta(\mathfrak{K}: H_j) = \beta(\mathfrak{H}: H_j) .$$

By (5.20.1), (5.20.2) and (5.20.3),

$$\begin{aligned}\delta(\mathfrak{K}) &= \sum_{i=2}^{t} |\beta(\mathfrak{K}: H_{j_i})| + |\beta(\mathfrak{K}: H_j)| + |\beta(\mathfrak{K}: L)| \\ &> \sum_{i=2}^{t} |\beta(\mathfrak{H}: H_{j_i})| + |\beta(\mathfrak{H}: H_j)| \\ &= \delta(\mathfrak{H}) .\end{aligned}$$

Next, assume $L = N_L(J_e(S))$. Then it is easily verified that

(5.20.4) $$\mathscr{Y}(\mathfrak{K}) = \mathscr{Y}(\mathfrak{H}); \text{ and}$$

(5.20.5) $$\alpha(\mathfrak{K}: H_j) = \alpha(\mathfrak{H}: H_j) \cup \{L\} \backslash \{H_{k+1}\} .$$

By (5) and (5.20.5),

$$\langle N_G(S), K \in \alpha(\mathfrak{K}: H_j) \rangle \subseteqq \langle N_G(S), K \in \alpha(\mathfrak{H}: H_j) \rangle .$$

Then by Lemma 4.3(a) and 3.2(c),

(5.20.6) $$\Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{K}: H_j))J_e(S)) \subseteqq \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{H}: H_j))J_e(S)) .$$

By (3), (6) and (5.20.6), $L \in \beta(\mathfrak{K}: H_j)$ and $\beta(\mathfrak{K}: H_j) \supset \beta(\mathfrak{H}: H_j)$, so,

(5.20.7) $$|\beta(\mathfrak{K}: H_j)| > |\beta(\mathfrak{H}: H_j)| .$$

By (5.20.1) and (5.20.7), $\delta(\mathfrak{K}) > \delta(\mathfrak{H})$, which completes the proof of this lemma.

6. On some groups involved in the minimal situation.

NOTATION. Fix a trio of elements H_j, H_k and H_{k+1} of $\mathfrak{H}$ which satisfies the conclusions (a), (b) and (c) of Lemma 5.18.

Let

$$M = \langle N_G(S), H_k, H_{k+1} \rangle ;$$
$$N = \langle N_G(S), H_{k+1} \rangle ;$$
$$W = \langle O^2(H_k)^M \rangle ;$$

and

$$S_0 = W \cap O_S(\{H_k, H_{k+1}\}) .$$

By Lemma 5.2(c), we recall that for some odd prime $q \geqq 5$, H_k is a $\{2, q\}$-group with a nonidentity Sylow q-subgroup. From now on, q denotes the prime such that H_k is a $\{2, q\}$-group.

LEMMA 6.1. (a) $W = \langle O^2(H_k)^N \rangle \lhd M$;
(b) $[W, \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{H}: H_j))J_e(S))] = 1$;
(c) $[O^2(H_{k+1}), \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{H}: H_j))J_e(S))] \neq 1$, *in particular*

$$O^2(H_{k+1}) \nsubseteq W .$$

Proof. $W = \langle O^2(H_k)^N \rangle \lhd \langle O^2(H_k), N \rangle = M$, which proves (a). (b) follows from (a) and Lemma 5.16(a).

Since the trio of elements H_j, H_k and H_{k+1} satisfies the conclusions of Lemma 5.18, the former part of (c) follows from Lemma 5.18(c). Then the latter part of (c) follows from (b).

LEMMA 6.2. (a) H_k/S_0 is *q-closed*;
(b) $1 \subset S_0 \lhd M$, *in particular* $S_0 \lhd N_G(S)$;
(c) $1 \subset O_S(H_{k+1}) = S \cap O_2(N) \lhd N$;
(d) *for any S-invariant subnormal subgroup V of M, S is normal in some Sylow 2-subgroup of VS.*

Proof. Let $\mathfrak{K} = \{H_k, H_{k+1}\}$. By Lemma 5.1(a), $1 \subset O_S(\mathfrak{K})$. Then by Lemma 5.10(b), $f_{\mathfrak{K}}(H_k) = 1$. Hy Lemma 4.3(c),

(6.2.1) $$H_k/W(\mathfrak{K}: 1) \text{ is } q\text{-closed}.$$

Since $O^2(H_k) \subseteq W$ and $W(\mathfrak{K}: 1) = O_S(\mathfrak{K})$,

(6.2.2) $$[O^2(H_k), O_S(\mathfrak{K})] \subseteq W \cap O_S(\mathfrak{K}) = S_0 .$$

Then, (a) follows from (6.2.1) and (6.2.2).

Suppose $S_0 = 1$. Then by (a), H_k is q-closed, which contradicts Lemma 5.2(a). Hence, we obtain that $1 \subset S_0$. Since $O_S(\mathfrak{K}) \lhd M$, by Lemma 6.1(a),

$$S_0 = W \cap O_S(\mathfrak{K}) \lhd M , \tag{6.2.3}$$

which proves the former part of (b).

Since $N_G(S) \subseteqq M$, the latter part of (b) follows from (6.2.3), By Lemma 5.2(b), $1 \subset O_S(H_{k+1})$. Since $N = \langle N_G(S), H_{k+1}\rangle$, by Lemma 4.3(a), $N \rhd O_S(H_{k+1}) = S \cap O_2(N)$, which shows (c). Since $1 \subset S_0 \lhd N_G(S)$ by (b), the assumption (1) of the theorem implies that S is normal in some Sylow 2-subgroup U of $N_G(S_0)$. Since $U \subseteqq N_G(S) \subseteqq M \subseteqq N_G(S_0)$, U is a Sylow 2-subgroup of M. Since V is subnormal in M, by Lemma 3.3(a), $V \cap U$ is a Sylow 2-subgroup of V. So, $S(V \cap U)$ is a Sylow 2-subgroup of VS. Since $S(V \cap U) \subseteqq U$, (d) is proved.

NOTATION. Let K be the pre-image of $O(W/S_0)$ in W. And by Lemma 6.2(d), let U be a Sylow 2-subgroup of M in which S is normal.

LEMMA 6.3. *Let $\bar{X} = XK/K$ for any subgroup X of M. Then*
(a) $O^2(\bar{H}_k) = O(\bar{H}_k) = O_q(\bar{H}_k)$;
(b) $O_2(\bar{W}) \cap \bar{S} = 1$;
(c) $O_2(\bar{W}) = Z(\bar{W}) = F_\infty(\bar{W}) \subseteqq Z(\bar{W}\bar{H}_{k+1})$.

Proof. (a) follows from Lemma 6.2(a). To prove (b) and (c), we may assume that $\bar{W} \neq 1$, so that

$$O^2(\bar{H}_k) \not\subseteqq O_2(\bar{W}) . \tag{6.3.1}$$

Let S_1 be the intersection of S and the pre-image of $O_2(\bar{W})$ in M. Then by (6.3.1) and Lemma 5.11(d), $S_1 \subseteqq O_S(\{H_k, H_{k+1}\}) \cap W = S_0$. So, $O_2(\bar{W}) \cap \bar{S} = \bar{S}_1 \subseteqq \bar{S}_0 = 1$, which proves (b).

Since $O_2(\bar{W}) \lhd \bar{U}$ and $\bar{S} \lhd \bar{U}$, by (b), $[O_2(\bar{W}), \bar{S}] \subseteqq O_2(\bar{W}) \cap \bar{S} = 1$. Thus, $\bar{S} \subseteqq C_{\bar{M}}(O_2(\bar{W})) \lhd \bar{M}$. Then by Lemma 3.11(f),

$$\bar{H}_k, \bar{H}_{k+1} \subseteqq C_{\bar{M}}(O_2(\bar{W})) . \tag{6.3.2}$$

Since $O(\bar{W}) = 1$, by (6.3.2),

$$F_\infty(\bar{W}) \subseteqq O_2(\bar{W}) \subseteqq Z(\bar{W}\bar{H}_{k+1}) . \tag{6.3.3}$$

On the other hand, generally,

$$Z(\bar{W}) \subseteqq O_2(\bar{W}) \subseteqq F_\infty(\bar{W}), \text{ and } \bar{W} \cap Z(\bar{W}\bar{H}_{k+1}) \subseteqq Z(\bar{W}) . \tag{6.3.4}$$

Then, (c) follows from (6.3.3) and (6.3.4). Hence, this lemma is proved.

DEFINITION. Let $\mathfrak{L}$ be the family of all the subgroups L of M that satisfy the following conditions:

(α) $L \in \mathfrak{F}_G(S)$;

(β) L is a $\{2, q\}$-group;

(γ) $O^2(L) \subseteqq W$; and

(δ) L/S_0 is q-closed.

Let

$$\mathfrak{L}_1 = \{L \in \mathfrak{L}; O_S(\mathfrak{H}_k \cup \{L\}) \supset 1\} ;$$

and

$$\mathfrak{L}_2 = \{L \in \mathfrak{L}; O_S(\mathfrak{H}_k \cup \{L\}) = 1\} .$$

And let

$$Y = N_M(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))O(W) .$$

LEMMA 6.4. (a) *H_k lies in $\mathfrak{L}_2$;*

(b) *for any element L of $\mathfrak{L}_2$, $\mathfrak{H}_k \cup \{L\}$ lies in $\mathscr{CL}(G)$;*

(c) $O_S(\mathfrak{H}_k \cup \mathfrak{L}_1) \supset 1$.

Proof. Obviously, H_k is a $\{2, q\}$-group which lies in $\mathfrak{F}_G(S)$. By definition, $O^2(H_k) \subseteqq W$. By Lemma 6.2(a), H_k/S_0 is q-closed. Hence, H_k lies in $\mathfrak{L}$. Suppose H_k lies in $\mathfrak{L}_1$. Then by definition, $1 \subset O_S(\mathfrak{H}_k \cup \{H_k\}) = O_S(\mathfrak{H})$, which is a contradiction. Hence, (a) is proved.

(b) follows from the definitions.

(c) follows from Lemma 5.15.

LEMMA 6.5. *Suppose D is a normal subgroup of M. Assume:*

(1) $K \subseteqq D \subseteqq W$; *and*

(2) $S_0 \subset D \cap S$.

Then

$$D = W .$$

Proof. Suppose $D \subset W$. Since $D \lhd M$, it must be that $O^2(H_k) \nsubseteqq D$. Then by Lemma 5.11(d), $D \cap S \subseteqq O_S(\{H_k, H_{k+1}\}) \cap W = S_0$, which contradicts (2). Hence, this lemma is proved.

ASSUMPTION (A). In the following discussion, without loss of generality, we may assume that:

(A.1) $$\delta(\mathfrak{H}) = \max. \{\delta(\mathfrak{K}); \mathfrak{K} \in \mathscr{CL}(G)\} ;$$

and

(A.2) for any element L of $\mathfrak{L}_2$,

$\langle O^2(L)^x; x \in \langle N_G(S), H_{k+1}\rangle\rangle = W$, if L satisfies the following conditions:

(α) $\delta(\mathfrak{H}_k \cup \{L\}) = \delta(\mathfrak{H})$; and

(β) the trio of elements H_j, L and H_{k+1} satisfies the conclusions (a), (b), and (c) of Lemma 5.18 with H_k replaced by L.

LEMMA 6.6. *Let V be a normal subgroup of M such that $S_0 \subseteqq V \subseteqq K$, and let $D_1, D_2, \cdots, D_n$ be S-invariant normal subgroups of W all of which contain V. Assume:*

(1) $O^2(H_k) \subseteqq D_1D_2\cdots D_n$.

Let $\bar{X} = XV/V$ for any subgroup X of M.

(2) $\bar{D}_i \cap \langle \bar{D}_j; 1 \leqq j \neq i \leqq n\rangle$ *is a 2-group for all i; $1 \leqq i \leqq n$;*

(3) $[\bar{D}_i, \bar{D}_j] = 1$ *for all i, j: $1 \leqq i \neq j \leqq n$.*

Then there exists an element L of $\mathfrak{L}_2$ which satisfies the following conditions:

(a) *for some i; $1 \leqq i \leqq n$, $O^2(L) \subseteqq D_i$;*

(b) $\langle O^2(L)^x; x \in N\rangle = W$;

(c) $\mathfrak{H}_k \cup \{L\}$ *lies in* $\mathscr{CL}(G)$;

(d) $\delta(\mathfrak{H}_k \cup \{L\}) = \delta(\mathfrak{H})$; *and*

(e) *replacing H_k by L and $\mathfrak{H}$ by $\mathfrak{H}_k \cup \{L\}$, the conclusions of Lemma* 5.18(a), (b) *and* (c) *are satisfied.*

Proof. By Lemma 6.2(d), S is normal in some Sylow 2-subgroup of $(D_1D_2\cdots D_n)S$. Since $S_0 \subseteqq V \subseteqq K$, V is 2-closed, and $O_2(V) = S_0 = S \cap V$. Hence, all the assumptions of Proposition 3.13 are satisfied, for $q = r$, $(D_1D_2\cdots D_n)S = D$, $H_k = H$, and the other notation as is.

So, there exist S-irreducible subgroups $\{L_i; 1 \leqq i \leqq t\}$ of $(D_1D_2\cdots D_n)S$ which satisfy all the conclusions of Proposition 3.13.

Let $F = \langle L_i; 1 \leqq i \leqq t\rangle$. By Lemma 3.13(a) and (b),

(6.6.1) F is a solvable $\{2, q\}$-group with a Sylow 2-subgroup S, and $H_k \subseteqq F$.

Since $[O(H_k), O_2(F)] = 1 = [O(L_i), O_2(F)]$, (6.6.1) implies that:

(6.6.2) $O(H_k) \subseteqq O(F)$ and $O(L_i) \subseteqq O(F)$ for all i; $1 \leqq i \leqq t$.

Since H_k/S_0 is q-closed by Lemma 6.2(a), by Proposition 3.13(d) and (g), FV/V is q-closed, so that F/S_0 is q-closed. And $O^2(F) \subseteqq D_1D_2\cdots D_n \subseteqq W$. Hence, L_i lies in $\mathfrak{L}$ for all i; $1 \leqq i \leqq t$.

Suppose L_i lies in $\mathfrak{L}_1$ for all i; $1 \leqq i \leqq t$. Then by (6.6.2) and

Lemma 6.4(c), $F = N_F(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))$ and $O_S(\mathfrak{H}_k \cup \mathfrak{L}_1) \supset 1$. Since $H_k \subseteq F$ by (6.6.1), $H_k = N_{H_k}(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))O(H_k)$, which shows that $1 \subset O_S(\mathfrak{H}_k \cup \mathfrak{L}_1) \subseteq O_S(\mathfrak{H})$. This is a contradiction. Hence, we may assume that L_1 lies in $\mathfrak{L}_2$.

Let $L = L_1$ and $\mathfrak{K} = \mathfrak{H}_k \cup \{L\}$. By Lemma 6.4(b),

$$(6.6.3) \qquad \mathfrak{K} \text{ lies in } \mathscr{C}\mathscr{L}(G) ,$$

which proves (c).

By Proposition 3.13(d),

$$(6.6.4) \qquad L \text{ lies in } D_i \text{ for some } i; 1 \leqq i \leqq n ,$$

which proves (a). By (6.6.3) and Assumption (A.1),

$$(6.6.5) \qquad \delta(\mathfrak{H}) \geqq \delta(\mathfrak{K}) .$$

On the other hand, by Proposition 3.13(e) and (f),

$$(6.6.6) \qquad H_k = N_{H_k}(J_e(S)) \text{ if and only if } L = N_L(J_e(S)) .$$

Then all the assumptions of Lemma 5.19 are satisfied. Assume $H_k \neq N_{H_k}(J_e(S))$. Then by (6.6.6), $L \neq N_L(J_e(S))$. By (6.6.5) and Lemma 5.19(a),

$$(6.6.7) \qquad \delta(\mathfrak{H}) = \delta(\mathfrak{K}), L \in \mathscr{Y}(\mathfrak{K}) \text{ and } H_{k+1} \in \alpha(\mathfrak{K}: L)\backslash\beta(\mathfrak{K}: L) .$$

Next, assume $H_k = N_{H_k}(J_e(S))$. Then by (6.6.6), $L = N_L(J_e(S))$. Since $O^2(L) \subseteq W$, by Lemma 6.1(b), $[O^2(L), \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{H}: H_j))J_e(S))] = 1$. Then by (6.6.5) and Lemma 5.19(b),

$$(6.6.8) \qquad \delta(\mathfrak{K}) = \delta(\mathfrak{H}), H_j \in \mathscr{Y}(\mathfrak{K}) \text{ and } H_{k+1} \in \alpha(\mathfrak{K}: H_j)\backslash\beta(\mathfrak{K}: H_j) .$$

Hence, we obtain (d) and (e) by (6.6.7) and (6.6.8). Then, (b) follows from Assumption (A.2). Hence this lemma is proved.

LEMMA 6.7. *Suppose D_1 is a normal subgroup of M such that $K \subseteq D_1 \subseteq W$. Then*

$$D_1/K \subseteq Z(W/K) \text{ or } D_1 = W .$$

Proof. Let $D_2 = C_M(D_1/K) \cap W$. Then, $K \subseteq D_2 \lhd M$. Suppose $O^2(H_k) \subseteq D_1D_2$. Then, $D_1D_2 = W$. In this proof, let $\bar{X} = XK/K$ for any subgroup X of M. By Lemma 6.3(c), $\bar{D}_1 \cap \bar{D}_2 \subseteq Z(\bar{W}) = O_2(\bar{W})$, and $[\bar{D}_1, \bar{D}_2] = 1$. Then by Lemma 6.6(a) and (b), for some $i; 1 \leqq i \leqq 2$, $D_i = \langle D_i^x; x \in M \rangle = W$, which implies that $D_1 = W$ or $\bar{D}_1 = Z(\bar{W})$, as required. Hence, we may assume that $O^2(H_k) \nsubseteq D_1D_2$. Since $O^2(H_k) \subseteq W$, $O^2(H_k) \nsubseteq D_1C_M(D_1/K)$.

Then by Lemma 5.11(a),

(6.7.1) $$S \cap D_1C_M(D_1/K) \subseteq O_S(H_k) .$$

On the other hand, by Lemma 6.5, we may assume that

(6.7.2) $$\bar{D}_1 \cap \bar{S} \subseteq \bar{S}_0 = 1 .$$

By Lemma 3.3(a),

(6.7.3) $\bar{U} \cap \bar{D}_1$ is a Sylow 2-subgroup of $\bar{D}_1$.

By (6.7.2),

(6.7.4) $$[\bar{U} \cap \bar{D}_1, \bar{S}] \subseteq \bar{D}_1 \cap \bar{S} = 1 .$$

Let $\bar{F} = \bar{D}_1\bar{H}_{k+1}$ and $\bar{C} = \langle \bar{S}^x; x \in N_{\bar{F}}(\bar{U} \cap \bar{D}_1)\rangle$. Then, by (6.7.4),

(6.7.5) $$[\bar{U} \cap \bar{D}_1, \bar{C}] = 1 .$$

Since $O(\bar{D}_1) \subseteq O(\bar{W}) = 1$ by Lemma 6.3(c), (6.7.5) and a theorem of G. Glauberman [2] imply that

(6.7.6) $\bar{C}/C_{\bar{C}}(\bar{D}_1)$ has a normal 2-complement.

On the other hand, since $O^{2'}(\bar{H}_k) = \bar{H}_k$ by Lemma 5.2(a) and 3.11(f), the Frattini argument and the isomorphism theorem imply that:

(6.7.7) $$\bar{C}/(\bar{C} \cap \bar{D}_1C_{\bar{F}}(\bar{D}_1)) \simeq \bar{F}/\bar{D}_1C_{\bar{F}}(\bar{D}_1) \simeq \bar{H}_{k+1}/(\bar{H}_{k+1} \cap \bar{D}_1C_{\bar{F}}(\bar{D}_1)) .$$

By (6.7.6) and (6.7.7), $\bar{H}_{k+1}/(\bar{H}_{k+1} \cap \bar{D}_1C_{\bar{F}}(\bar{D}_1))$ has a normal 2-complement. So, $H_{k+1}/(H_{k+1} \cap D_1C_M(D_1/K))$ has a normal 2-complement. It follows that

(6.7.8) $$[O^2(H_{k+1}), O_S(H_{k+1})] \subseteq S \cap D_1C_M(D_1/K).$$

By (6.7.1) and (6.7.8), $[O^2(H_{k+1}), O_S(H_{k+1})] \subseteq O_S(H_k)$, which contradicts Lemma 5.8(b). Hence this lemma is proved.

LEMMA 6.8. $O^2(C_W(S_0)) = O(W)$.

Proof. Obviously, $O^2({}_W(S_0)) \supseteq O(W)$. To prove the opposite inclusion, first, we assume that $S_0C_W(S_0) \subseteq F_\infty(W)$. Then by Lemma 6.3(c), $S_0C_W(S_0)/S_0$ is $2'$-closed. It follows that $S_0C_W(S_0)$ has a normal 2-complement. Since $S_0C_W(S_0) \lhd W$, $O^2(C_W(S_0)) \subseteq O(W)$, as required. Hence, we may assume that

(6.8.1) $$S_0C_W(S_0) \not\subseteq F_\infty(W) .$$

By Lemma 6.3(c) and the preceding lemma, we have that $KC_W(S_0) = W$. For $D_1 = C_W(S_0)S_0$, $D_2 = K$ and $V = D_1 \cap D_2$, all the assumptions of Lemma 6.6 are satisfied. By Lemma 6.6(a) and (b), for some $i; 1 \leqq i \leqq 2$, $D_i = \langle D_i^x; x \in N\rangle = W$, that is, $K = W$ or $S_0C_W(S_0) = W$.

Suppose $K = W$. Then, $S_0C_W(S_0) \subseteqq K \subseteqq F_\infty(W)$, which contradicts (6.8.1). Thus, we have that $S_0C_W(S_0) = W$. It follows that $[O^2(H_k), S_0] \subseteqq [O^2(W), S_0] = 1$. Then by Lemma 6.2(a), H_k is $2'$-closed, which contradicts Lemma 5.2(a). Hence this lemma is proved.

LEMMA 6.9. *For any element L of $\mathfrak{L}$, L lies in $\mathfrak{L}_1$ if and only if $L \subseteqq Y$, in particular $H_k \nsubseteq Y$.*

Proof. Suppose L lies in $\mathfrak{L}_1$. Then, $L = N_L(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))O(L)$. Since $O(L) \subseteqq O^2(C_W(S_0))$, by Lemma 6.8, $O(L) \subseteqq O(W)$. Hence, $L \subseteqq N_M(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))O(W) = Y$, as required.

Conversely, suppose $L \subseteqq Y$. Then, $L = N_L(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))O(L)$. Hence by Lemma 6.4(c), $1 \subset O_S(\mathfrak{H}_k \cup \mathfrak{L}_1) \subseteqq O_S(\mathfrak{H}_k \cup \{L\})$, which shows that L lies in $\mathfrak{L}_1$. Since H_k lies in $\mathfrak{L}_2$ by Lemma 6.4(a), we have the latter part of this lemma. Hence this lemma is proved.

LEMMA 6.10. *Suppose V is a normal subgroup of M such that $S_0 \subseteqq V \subseteqq F_\infty(W)$. Let $\bar{X} = XV/V$ for any subgroup X of M, and $\bar{T} = \bigcap_{L \in \mathfrak{L}_2} C_{\bar{S}}(O_q(\bar{L}))$. Then*

(a) $\bar{S} = \overline{O_S(H_{k+1})}\bar{T}$;

(b) $O^2(\bar{L}) = O_q(\bar{L})$ *for any element L of $\mathfrak{L}$.*

For any $\bar{S}$-invariant q-subgroup $\bar{R}$ of $\bar{W}$,

(c) *there exist elements $\{L_i; 1 \leqq i \leqq t\}$ of $\mathfrak{L}$ such that $\bar{R} = \langle O_q(\bar{L}_i); 1 \leqq i \leqq t\rangle$; and*

(d) $[\bar{R}, \bar{T}] \subseteqq \bar{Y}$.

Proof. For any element L of $\mathfrak{L}$, L/S_0 is q-closed by definition, so that

(6.10.1) $$L/V \text{ is } q\text{-closed, that is, } O^2(\bar{L}) = O_q(\bar{L}) ,$$

which proves (b).

Let $T_0 = \bigcap_{L \in \mathfrak{L}_2} O_S(L)$. Then by Lemma 5.14 and 6.4(b),

(6.10.2) $$S = O_S(H_{k+1})T_0 .$$

By (6.10.1), if $L \in \mathfrak{L}_2$, $[O_q(\bar{L}), \bar{T}_0] \subseteqq O_q(\bar{L}) \cap \bar{T}_0 = 1$, which shows that $\bar{T}_0 \subseteqq \bar{T}$. Then, (a) follows from (6.10.2).

Let D^* be a Hall $\{2, q\}$-subgroup of the pre-image of $\bar{S}\bar{R}$ in M which contains S. Then,

(6.10.3) $$O_q(\bar{D}^*) = \bar{R} .$$

By Lemma 6.3(c),

(6.10.4) $$D^*/S_0 \text{ is } q\text{-closed.}$$

Let D be the pre-image of $O_q(D^*/S_0)(S/S_0)$ in M. By (6.10.3) and (6.10.4),

$$O_q(\bar{D}) = \bar{R} . \tag{6.10.5}$$

Since S is a Sylow 2-subgroup of D, by Lemma 3.10, there exist S-irreducible subgroups $\{L_i; 1 \leqq i \leqq t\}$ of D with a Sylow 2-subgroup S, and $D = \langle L_i; 1 \leqq i \leqq t\rangle$. Then, it is easily verified that L_i lies in $\mathfrak{L}$ for all $i; 1 \leqq i \leqq t$. Hence, (c) is proved. To prove (d), we may assume that $\bar{R} = [\bar{R}, \bar{T}]$. Then, by (a), $O_q(\bar{L}_i) \subseteqq \Phi(\bar{R})$ for all $L_i \in \mathfrak{L}_2$. By the preceding lemma, $\bar{R} = \langle O_q(\bar{L}_i), \Phi(\bar{R}); L_i \in \mathfrak{L}_1\rangle = \langle \bar{R} \cap \bar{Y}, \Phi(\bar{R})\rangle = \bar{R} \cap \bar{Y}$, which proves (d).

LEMMA 6.11. (a) $N_G(S) \subseteqq N \subseteqq Y$;

(b) *Y contains a Hall $\{2, q\}$-subgroup of KS which contains S.*

Proof. Since $O_S(\mathfrak{H}_k \cup \mathfrak{L}_1) \lhd \langle N_G(S), H_{k+1}\rangle = N$,

$$S \subseteqq N_G(S) \subseteqq N \subseteqq N_M(O_S(\mathfrak{H}_k \cup \mathfrak{L}_1))O(W) = Y , \tag{6.11.1}$$

which proves (a).

Since S is a Sylow 2-subgroup of KS, by (a), we need only show that Y contains a Sylow q-subgroup of K. Suppose that K does not contain any Sylow q-subgroup of K. By the main theorem of W. Feit and J. G. Thompson [1], K possesses a chief factor V_1/V_2 which satisfies the following conditions:

(α) both V_1 and V_2 are normal subgroups of M which contains S_0;

(β) V_1/V_2 is a nonidentity elementary Abelian q-group; and

(γ) Y contains a Sylow q-subgroup of V_2, but does not contain any Sylow q-subgroup of V_1.

Let $\bar{X} = XV_2/V_2$ for any subgroup X of M. By (β),

$$\bar{V}_1 \cap \bar{Y} \subset \bar{V}_1 . \tag{6.11.2}$$

By Lemmas 6.10(b), (c) and 6.9,

(6.11.3) there exist elements $\{L_i; 1 \leqq i \leqq t\}$ of $\mathfrak{L}_2$ such that

$$\bar{V}_1 = (\bar{V}_1 \cap \bar{Y})\langle O_q(\bar{L}_i); 1 \leqq i \leqq t\rangle \text{ and}$$
$$O_q(\bar{L}_i) \not\subseteqq \bar{V}_1 \cap \bar{Y} \text{ for all } i; 1 \leqq i \leqq t .$$

By Lemmas 3.11(g), 3.4, and 4.3(e),

$$C_S(V_1/V_1 \cap Y) \subseteqq O_S(L_i) \text{ for all } i; 1 \leqq i \leqq t . \tag{6.11.4}$$

Since $\mathfrak{H}_k \cup \{L_i\}$ lies in $\mathscr{C}\mathscr{L}(G)$, $1 \leqq i \leqq t$, by Lemma 5.13(a),

$$[O^2(H_{k+1}), O_S(H_{k+1})] \not\subseteqq O_S(L_i) \text{ for all } i; 1 \leqq i \leqq t . \tag{6.11.5}$$

On the other hand, by Lemma 6.10(a),

(6.11.6) $$S = O_S(H_{k+1})C_S(V_1/V_1 \cap Y) .$$

Since $H_{k+1} \subseteqq Y$ by (a), H_{k+1} leaves invariant V_1 and $V_1 \cap Y$. So,

(6.11.7) $$C_{H_{k+1}}(V_1/V_1 \cap Y) \lhd H_{k+1} .$$

By (6.11.6) and (6.11.7), $S \subseteqq O_S(H_{k+1})C_{H_{k+1}}(V_1/V_1 \cap Y) \lhd H_{k+1}$. By (6.11.7) and Lemma 3.11(f),

(6.11.8) $$O^2(H_{k+1}) \subseteqq C_{H_{k+1}}(V_1/V_1 \cap Y) .$$

By (6.11.4) and (6.11.8), $[O^2(H_{k+1}), O_S(H_{k+1})] \subseteqq C_S(V_1/V_1 \cap Y) \subseteqq O_S(L_i)$ for all $i; 1 \leqq i \leqq t$, which contradicts (6.11.5). Hence this lemma is proved.

LEMMA 6.12. *Let $\bar{X} = XK/K$ for any subgroup X of M. Then*
(a) $Z(\bar{W}) \subseteqq \bar{Y}$;
(b) $O_q(\bar{H}_k) \not\subseteqq \bar{Y} \cap \bar{W} \subset \bar{W} = E(\bar{W})$; *and*
(c) $\mathscr{D}(\bar{W}\bar{H}_{k+1}: \pi) = \emptyset$.

Proof. By Lemmas 6.3(c) and 6.11(a), $Z(\bar{W}) = O_2(\bar{W}) \subseteqq \bar{U} \subseteqq \overline{N_G(S)} \subseteqq \bar{N} \subseteqq \bar{Y}$, which proves (a).

Suppose $O_q(\bar{H}_k) \subseteqq \bar{Y}$. Since $S \subseteqq Y$ by Lemma 6.11(a), $\bar{H}_k \subseteqq \bar{Y}$. Let V be the pre-image of $\bar{H}_k$ in M, and let D_1 be a Hall $\{2, q\}$-subgroup of V which contains H_k. By Lemma 6.11(b), $Y \cap V$ contains a Hall $\{2, q\}$-subgroup D_2 of V which contains S. Since S is a Sylow 2-subgroup of V, S is a Sylow 2-subgroup of D_1 and D_2.

By Lemma 3.5(b), $N_V(S)$ possesses an element x such that $D_2^x = D_1$. Since $x \in N_V(S) \subseteqq Y$ by Lemma 6.11(a), $H_k \subseteqq D_1 = D_2^x \subseteqq Y$, which contradicts Lemma 6.9. Hence, we obtain that $O_q(\bar{H}_k) \not\subseteqq \bar{Y}$. Since $O_q(\bar{H}_k) \subseteqq \bar{W}$, it follows that

(6.12.1) $$O_q(\bar{H}_k) \not\subseteqq \bar{Y} \cap \bar{W} \subset \bar{W} .$$

Hence, by (a), (6.12.1) and Lemma 6.3(c), $Z(\bar{W}) = F_\infty(\bar{W}) \subset \bar{W}$. So, $E(\bar{W}) \neq 1$. Since $E(\bar{W}) \not\subseteqq Z(\bar{W})$, by Lemma 6.7,

(6.12.2) $$E(\bar{W}) = \bar{W} .$$

Then, (b) follows from (6.12.1) and (6.12.2). Since H_{k+1} is solvable, by Lemma 6.8,

(6.12.3) $C_{WH_{k+1}}(S_0)$ is solvable.

By (6.12.3) and the assumption (2) of the theorem,

$$\mathscr{D}(\bar{W}\bar{H}_{k+1}: \pi) = \mathscr{D}(WH_{k+1}/C_{WH_{k+1}}(S_0): \pi) \subseteqq \mathscr{D}(N_G(S_0)/C_G(S_0): \pi) = \emptyset ,$$

which proves (c). Hence this lemma is proved.

LEMMA 6.13. *Let* K^* *be the pre-image of* $Z(W/K)$ *in* W, *and* $\bar{X} = XK^*/K^*$ *for any subgroup* X *of* M. *Then*

(a) *for some natural number* r,

$\bar{W} = E(\bar{W}) = \bar{E}_1 \times \bar{E}_2 \times \cdots \times \bar{E}_r$, *where* $\bar{E}_k$ *is a non-Abelian simple group with* $\bar{E}_k \simeq \bar{E}_1$ *for all* $k; 1 \leqq k \leqq r$;

(b) $\bar{E}_k \nsubseteq \bar{Y} \cap \bar{W}$ *for all* $k; 1 \leqq k \leqq r$;

(c) $\bar{N} = \langle \bar{H}_{k+1}, N_{\bar{N}}(\bar{S}) \rangle$;

(d) *there exists a subgroup* $\bar{T}$ *of* $\bar{S}$ *such that*

(d.1) $\bar{S} = (\bar{S} \cap O_2(\bar{N}))\bar{T}$, *and*

(d.2) $[\bar{R}, \bar{T}] \subseteqq \bar{Y} \cap \bar{W}$ *for any* $\bar{S}$*-invariant* q*-subgroup* $\bar{R}$ *of* $\bar{W}$;

(e) *renumbering of* $\{1, 2, \cdots, r\}$, *if necessary, there exists an* S*-invariant* q*-subgroup* $\bar{Q}$ *of* $\bar{W}$ *such that* $\langle \pi_1(\bar{Q})^x; x \in \bar{N} \rangle = \bar{W}$, *where* π_1 *denotes the projection mapping from* $\bar{W}$ to $\bar{E}_1$.

Proof. By Lemma 6.12(b), for some natural number r,

(6.13.1) $\bar{W} = E(\bar{W}) = \bar{E}_1 \times \bar{E}_2 \times \cdots \times \bar{E}_r$, where $\bar{E}_k$ is a non-Abelian simple group, $1 \leqq k \leqq r$.

Let $\bar{W}_0 = \langle \bar{E}_1^x; x \in \bar{N} \rangle$. Since $\bar{M} = \langle \bar{W}, \bar{N} \rangle$, $\bar{W}_0 \lhd \bar{M}$. Since $\bar{W}_0 \nsubseteq Z(\bar{W})$, by Lemma 6.7, $\bar{W}_0 = \bar{W}$, so that

(6.13.2) $\bar{N}$ acts transitively on $\{\bar{E}_k; 1 \leqq k \leqq r\}$.

Hence,

(6.13.3) $\bar{E}_k \simeq \bar{E}_1$ for all $k; 1 \leqq k \leqq r$.

Thus, (a) follows from (6.13.1) and (6.13.3).

Suppose $\bar{E}_k \subseteqq \bar{Y}$ for some $k; 1 \leqq k \leqq r$. Since $\bar{Y}$ is $\bar{N}$-invariant by Lemma 6.11(a), (6.13.2) shows that $\bar{W} = \langle \bar{E}_k^x; x \in \bar{N} \rangle \subseteqq \bar{Y}$, which contradicts Lemma 6.12(a) and (b). Hence, $\bar{E}_k \nsubseteq \bar{Y}$ for all $k; 1 \leqq k \leqq r$, which proves (b).

Since $\bar{N} = \langle \bar{H}_{k+1}, \overline{N_G(S)} \rangle$, $\bar{N} = \langle \bar{H}_{k+1}, N_{\bar{N}}(\bar{S}) \rangle$, which proves (c).

By Lemma 6.2(c),

(6.13.4) $\overline{O_S(H_{k+1})} = \bar{S} \cap \overline{O_2(N)} \subseteqq \bar{S} \cap O_2(\bar{N})$.

Let $\bar{T} = \bigcap_{L \in \mathfrak{L}_2} C_{\bar{S}}(O_q(\bar{L}))$. Then by Lemma 6.10 and (6.13.4), $\bar{S} = \overline{O_S(H_{k+1})}\bar{T} \subseteqq (\bar{S} \cap O_2(\bar{N}))\bar{T}$, and $[\bar{R}, \bar{T}] \subseteqq \bar{Y}$ for any $\bar{S}$-invariant q-subgroup $\bar{R}$ of $\bar{W}$, which proves (d).

By Lemma 3.6, $\bar{S}$ induces a permutation of $\Lambda = \{1, 2, \cdots, r\}$

where we identity k with E_k, $1 \leqq k \leqq r$. Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be the set of all $\bar{S}$-orbits of Λ, $\bar{D}_i = \langle \bar{E}_\lambda; \lambda \in \Lambda_i \rangle$, and let D_i be the pre-image of $\bar{D}_i$ in M, $1 \leqq i \leqq n$. Then, for $V = K$ and $D = WS$, all the assumptions of Lemma 6.6 are satisfied. By Lemma 6.6(a), (b) and Lemma 6.10(b), there exists an element L of $\mathfrak{L}_2$, such that

(6.13.5) $\langle O_q(\bar{L})^x; x \in \bar{N} \rangle = \bar{W}$ and $O_q(\bar{L}) \subseteqq \bar{D}_i$ for some $i; 1 \leqq i \leqq n$.

Renumbering Λ, if necessary, we may assume that $\bar{D}_i = \langle \bar{E}_1^x; x \in \bar{S} \rangle$. Since $\bar{L}$ is $\bar{S}$-irreducible by Lemma 3.11(a),

(6.13.6) $O_q(\bar{L}) \subseteqq \langle \pi_1(O_q(\bar{L}))^x; x \in S \rangle$, where π_1 denotes the projection mapping from $\bar{W}$ to $\bar{E}_1$.

By (6.13.5) and (6.13.6), $\langle \pi_1(O_q(\bar{L}))^x; x \in \bar{N} \rangle = \bar{W}$, which proves (e). Hence this lemma is proved.

LEMMA 6.14. $O^2(H_{k+1}) \subseteqq WC_M(W/K)$.

Proof. Let K^* be the pre-image of $Z(W/K)$ in W. Then by Lemma 6.12(c), $\mathscr{D}(WH_{k+1}/K^*: \pi) \subseteqq \mathscr{D}(WH_{k+1}/K: \pi) = \emptyset$. So, by Lemma 6.13 and Proposition 3.17, we obtain that $O^2(H_{k+1}) \subseteqq WC_M(W/K^*)$. Hence, it is enough to show that $C_M(W/K) \supseteqq C_M(W/K^*)$. Let $\bar{X} = XK/K$ for any subgroup X of M. Then, $\bar{W}$ stabilizes a normal series: $\overline{C_M(W/K^*)} = C_{\bar{M}}(\bar{W}/Z(\bar{W})) \supseteqq Z(\bar{W}) \supseteqq 1$. By Lemma 6.12(b), $O^2(\bar{W}) = \bar{W}$ r, so that by Lemma 3.7 $[\bar{W}, \overline{C_M(W/K^*)}] = 1$. Therefore, $C_M(W/K^*) \subseteqq C_M(W/K)$, as required. Hence this lemma is proved.

NOTATION. By Lemma 5.2(c), H_{k+1} is a $\{2, p\}$-group with a non-identity Sylow p-subgroup. From now, p denotes the prime such that H_{k+1} is a $\{2, p\}$-group.

LEMMA 6.15. *All the assumptions of Proposition* 3.13 *are satisfied, for* $r = p$, $D = M$, $V = K$, $D_1 = W$, $D_2 = C_M(W/K)$, *and* $H = H_{k+1}$.

Proof. (1) follows from Lemma 6.1(e). (2) follows from Lemma 5.2(a). (3) follows from the preceding lemma. (4) and (5) follow from the definitions. (6) follows from Lemma 6.3(c). Hence this lemma is proved.

LEMMA 6.16. *A contradiction.*

Proof. By Lemma 6.15, and Proposition 3.13(a), (b), (c) and (d), there exists a subfamily $\mathfrak{K} = \{L_i; 1 \leqq i \leqq t\}$ of $\mathfrak{F}_G(S)$ which satisfies the following (6.16.1), (6.16.2) and (6.16.3):

(6.16.1) $\langle L \in \mathfrak{K} \rangle$ is a solvable $\{2, p\}$-group with a Sylow 2-subgroup S which contains H_{k+1};

(6.16.2) there exists a subgroup T of S which is normal in $\langle N_M(S), L \in \mathfrak{K} \rangle$, and H_{k+1}/T is p-closed, and L/T is p-closed for all $L \in \mathfrak{K}$;

(6.16.3) for each $L \in \mathfrak{K}$, $O^2(L) \subseteqq W$ or $O^2(L) \subseteqq C_M(W/K)$.

First, we shall show that:

(6.16.4) $$O_S(\mathfrak{H}_{k+1} \cup \{L\}) \supset 1 \text{ for all } L \in \mathfrak{K} .$$

Suppose that $O_S(\mathfrak{H}_{k+1} \cup \{L\}) = 1$ (namely, $\mathfrak{H}_{k+1} \cup \{L\}$ lies in $\mathscr{C}\mathscr{L}(G)$) for some $L \in \mathfrak{K}$. If $O^2(L) \subseteqq C_M(W/K)$, then $[O^2(L), O_S(L)] \subseteqq C_S(W/K) \subseteqq O_S(H_k)$, which contradicts Lemma 5.13(c). So, we get $O^2(L) \subseteqq W$ by (6.16.3). Then by Lemma 6.1(b), $[O^2(L), \Omega_1 Z\hat{J}(O_S(\alpha(\mathfrak{H}: H_j))J_e(S))] = 1$. Then by Lemma 5.20, $\delta(\mathfrak{H}_{k+1} \cup \{L\}) > \delta(\mathfrak{H})$, which contradicts Assumption (A.1). Hence, we obtain (6.16.4). Since $N_M(S) = N_G(S)$, by (6.16.4), (6.16.2) and Lemma 5.15,

(6.16.5) $$O_S(\mathfrak{H}_{k+1} \cup \mathfrak{K}) \supset 1 .$$

Let $F = \langle L \in \mathfrak{K} \rangle$. Then by (6.16.1), $O(L) \subseteqq O(F)$ for all $L \in \mathfrak{K}$. Therefore, $F = N_F(O_S(\mathfrak{H}_{k+1} \cup \mathfrak{K}))O(F)$. Since $H_{k+1} \subseteqq F$ by (6.16.1), it follows that

(6.16.6) $$H = N_{H_{k+1}}(O_S(H_{k+1} \cup \mathfrak{K}))O(H_{k+1}) .$$

By (6.16.5) and (6.16.6), $1 \subset O_S(\mathfrak{H}_{k+1} \cup \mathfrak{K}) \subseteqq O_S(\mathfrak{H})$, which is a final contradiction. This completes the proof of the theorem.

Added in proof. The author would like to give thanks to Professor George Glauberman who pointed out some errors in this paper, and to the *Journal* for fixing up the manuscript.

REFERENCES

1. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math., **13** (1963), 775-1029.
2. G. Glauberman, *On the automorphism group of a finite group having no nonidentity normal sub- groups of odd order*, Math. Z., **93** (1966), 154-160.
3. ———, *A characteristic subgroup of a p-stable groups*, Canad. J. Math., **20** (1968).
4. ———, *Global and local properties of finite groups*, "*Finite Simple Groups*, edited by M. B. Powell and G. Higman, Academic Press, 1971.

5. ———, *Factorizations for 2-constrained groups*, to appear.
6. ———, *Factorizations in Local Subgroups in Finite Groups*, Amer. Math. Soc. CBMS No. 33, 1978.
7. D. M. Goldschmidt, 2-*Fusion in finite groups*, Annals of Math., **99** (1974), 70–117.
8. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
9. M. Hayashi, 2-*Factorization in finite groups, II.*, in preparation.
10. J. G. Thompson, *Normal p-complements for finite groups*, J. Algebra, **1** (1964), 43–46.
11. ———, *Factorizations of p-solvable groups*, Pacific J. Math., **16** (1966), 371–372.

Received April 10, 1978 and in revised form December 14, 1978. This paper was written in partial fulfillment of the requirements of a doctorial thesis at the University of Hokkaido.

AICHI UNIVERSITY OF EDUCATION
HIROSAWA, IGAYA-CHO, KARIYA
JAPAN 448