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A DUAL RELATIONSHIP BETWEEN GENERALIZED ABEL-GONČAROV BASES AND CERTAIN PINCHERLE BASES

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Recent results on Abel-Gončarov polynomial expansions are applied to study the representability of holomorphic functions as infinite series in a given Pincherle sequence. As a generalization of the ordinary derivative we consider the so-called Gel'fond-Leont'ev derivative $\mathscr{D}$. We take the exponential function with respect to the derivative $\mathscr{D}$ and use a duality principle in order to investigate the completeness of the system $E_n(z) = z^n E(\lambda_n z)$ in the space $\mathscr{F}_r$ of functions holomorphic on the interior of the disc of radius $r \leqq \infty$. Finally we study the uniqueness of the representability of holomorphic functions as infinite series in the system E_n.

1. Basic facts and definitions. Let $0 < r \leqq \infty$. We shall be interested in the nuclear Fréchet space $\mathscr{F}_r$ consisting of all functions holomorphic on the open disk of radius r, equipped with the topology of uniform convergence on compact sets (see [15]). For the topology in the space $\mathscr{F}_r$, we can take the norms $\|\cdot\|_{r'}$, $0 < r' < r$ given by $\|f\|_{r'} = \max\{|f(z)|: |z| = r'\}$, $f \in \mathscr{F}_r$. It is easily seen that by Cauchy's estimates that system of norms $\{\|\cdot\|_{r'}, 0 < r' < r\}$ is equivalent to the system of norms $\{|||\cdot|||_{r'}, 0 < r' < r\}$, where

$$|||f|||_{r'} = \sup_{0 \leqq k < \infty} |a_k| r'^k ,$$

for $f \in \mathscr{F}_r$ with Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k .$$

We recall that two systems of seminorms $\{\|\cdot\|_p, p \in P\}$ and $\{|||\cdot|||_p, p \in P\}$ are equivalent, if for each $p \in P$ there exists a constant K_p depending on p and $q \in P$ such that $\|\cdot\|_p \leqq K_p |||\cdot|||_q$, and if for each $p' \in P$ there exists a constant $K_{p'}$ depending on p' and $q' \in P$ such that $|||\cdot|||_{p'} \leqq K_{p'} \|\cdot\|_{q'}$.

A sequence $(f_n)_{n=0}^{\infty}$ in $\mathscr{F}_r$ is complete if the set of all finite linear combinations of the functions f_n is dense in $\mathscr{F}_r$. And $(f_n)_{n=0}^{\infty}$ is a basis in $\mathscr{F}_r$ if each $f \in \mathscr{F}_r$ has a representation

$$f = \sum_{n=0}^{\infty} c_n f_n ,$$

where $(c_n)_{n=0}^{\infty}$ is a sequence of scalars uniquely determined by f and the infinite series converges in the topology of $\mathscr{F}_r$. Two bases $(f_n)_{n=0}^\infty$ and $(g_n)_{n=0}^\infty$ are equivalent if $\sum_{n=0}^\infty c_n f_n$ converges in $\mathscr{F}_r$ if and only if $\sum_{n=0}^\infty c_n g_n$ converges in $\mathscr{F}_r$. As is well known, the sequence of the functions $(z^n)_{n=0}^\infty$ constitutes a basis for any space $\mathscr{F}_r(0 < r \leqq \infty)$. A basis $(f_n)_{n=0}^\infty$ is called proper if it is equivalent to $(z^n)_{n=0}^\infty$ (see [1], [4]).

M. Arsove [1], in a series of papers, has considered Pincherle sequences $(f_n)_{n=0}^\infty$ in which f_n has the form

$$f_n(z) = z^n \psi_n(z) , \qquad n = 0, 1, 2, \cdots$$

where each function $\psi_n \in \mathscr{F}_r$ and $\psi_n(0) = 1$. Recently, in [1] Arsove and in [4] Dubinsky studied linear Pincherle sequences (see also [9])

$$f_n(z) = z^n\left(1 - \frac{z}{z_n}\right) , \qquad n = 0, 1, 2, \cdots .$$

In this paper we investigate the problem of determining when a system

$$E_n(z) = z^n E(\lambda_n z) , \qquad n = 0, 1, 2, \cdots$$

is complete in $\mathscr{F}_r$, when it is not complete and when it is a basis, even a proper basis in $\mathscr{F}_r$. Here $(\lambda_n)_{n=0}^\infty$ is a sequence of scalars and E is a generalized exponential function corresponding to a so-called Gel'fond-Leont'ev derivative $\mathscr{D}$ (see [8]).

Let $(d_k)_{k=1}^\infty$ denote a nondecreasing sequence of positive numbers. The Gel'fond-Leont'ev derivative $\mathscr{D}$ is defined by

$$\mathscr{D} f(z) = \sum_{k=1}^\infty d_k a_k z^{k-1} ,$$

where

$$f(z) = \sum_{k=0}^\infty a_k z^k .$$

As in [2] or [7] we suppose that the sequence $(d_k)_{k=1}^\infty$ satisfies the following condition

(1.1) $\qquad (d_{k+1}/d_k)_{k=1}^\infty$ is nonincreasing and has limit 1 .

Then it follows

$$\lim_{k\to\infty} d_k^{1/k} = 1 .$$

Thus if f has radius of convergence $c(f)$ then

$$\mathscr{D}f(z) = \sum_{k=1}^{\infty} d_k a_k z^{k-1}$$

has also radius of convergence $c(f)$.

The operator $\mathscr{D}$ corresponds to the ordinary derivative when $d_k = k$ $(k = 1, 2, \cdots)$ and to the shift operator $\mathscr{S}$ when $d_k = 1$ $(k = 1, 2, , \cdots)$. $\mathscr{S}$ is defined by

$$\mathscr{S}f(z) = \sum_{k=1}^{\infty} a_k z^{k-1} .$$

The operators $\mathscr{D}^n$ $(n = 1, 2, \cdots)$ are the successive iterates of $\mathscr{D}$ and we have

$$\mathscr{D}^n f(z) = \sum_{k=n}^{\infty} \frac{e_{k-n}}{e_k} a_k z^{k-n} ,$$

where $e_0 = d_0 = 1$ and $e_n = (d_1 d_2 \cdots d_n)^{-1}$ for $n \geqq 1$.

We write

$$E(z) = \sum_{k=0}^{\infty} e_k z^k$$

and note that this function bears the same relationship to the operator $\mathscr{D}$ that the exponential function bears to the ordinary differentiation. This means

$$E(0) = 1 \quad \text{and} \quad \mathscr{D}E(z) = E(z) .$$

Let $R = c(E)$, then, by the monotonicity of the sequence $(d_k)_{k=1}^{\infty}$ we have (see [2])

$$R = \lim_{k\to\infty} d_k = \sup_{1\leq k<\infty} d_k .$$

The E-type of a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is the number

$$\tau_E(f) = \limsup_{k\to\infty} |a_k/e_k|^{1/k} .$$

If $R < \infty$ then

$$\tau_E(f) = \frac{R}{c(f)} , \tag{1.2}$$

(see [2], [7]).

Now we define for a sequence $(\lambda_k)_{k=0}^{\infty}$ of scalars the polynomials $Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$ by $Q_0(z) \equiv 1$ and

$$Q_n(z; \lambda_0, \cdots, \lambda_{n-1}) = e_n z^n - \sum_{k=0}^{n-1} e_{n-k} \lambda_k^{n-k} Q_k(z; \lambda_0, \cdots, \lambda_{k-1}) .$$

It is easily seen that

$$e_n z^n = \sum_{k=0}^{n} e_{n-k} \lambda_k^{n-k} Q_k(z; \lambda_0, \cdots, \lambda_{k-1}) . \tag{1.3}$$

The polynomials $Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$ are called the Gončarov polynomials belonging to the operator $\mathscr{D}$ (see [2]). They reduce to the ordinary Gončarov polynomials if $d_k = k$ $(k = 1, 2, \cdots)$ and the remainder polynomials if $d_k = 1$ $(k = 1, 2, \cdots)$.

One verifies easily that

$$\mathscr{D}^k Q_n(\lambda_k; \lambda_0, \cdots, \lambda_{n-1}) = \delta_{nk} \qquad \text{(see [2])} . \tag{1.4}$$

Therefore the polynomials $Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$ are biorthogonal to the linear functionals

$$\mathscr{L}_n(f) = \mathscr{D}^n f(\lambda_n) .$$

Now we consider the problem under which conditions the polynomials $Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$ constitute a basis in $\mathscr{F}_r$, i.e.,

$$f(z) = \sum_{n=0}^{\infty} \mathscr{D}^n f(\lambda_n) Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$$

for each $f \in \mathscr{F}_r$ and the infinite series converges in the topology of $\mathscr{F}_r$.

In this connection the Whittaker constant $W(\mathscr{D})$ belonging to the operator $\mathscr{D}$ plays an important role. We can introduce the Whittaker constant $W(\mathscr{D})$ by

$$W(\mathscr{D}) = \left(\sup_{1 \leqq n < \infty} H_n^{1/n} \right)^{-1} ,$$

where

$$H_n = \max | Q_n(0; \lambda_0, \cdots, \lambda_{n-1}) | \qquad (n = 1, 2, \cdots)$$

and the maximum is taken over all sequences $(\lambda_k)_{k=0}^{n-1}$ whose terms lie on the unit circle (see Buckholtz and Frank [2]).

The Whittaker constant satisfies the inequality (see [2])

$$0 < \frac{d_1}{2} \leqq W(\mathscr{D}) < d_1 . \tag{1.5}$$

In [7], Frank and Shaw investigated the above problem and the following theorem is an easy consequence of their Theorem A in [7]:

THEOREM A. *Let $(\lambda_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that*

$$|\lambda_n| \leqq \frac{e_{n+1}}{e_n} s , \qquad n = 0, 1, 2, \cdots$$

for a real number $s > 0$. Then the Gončarov polynomials constitute a basis in any space $\mathscr{F}_r$ for

$$r > \frac{s}{W(\mathscr{D})} .$$

The following theorem, which is again an easy consequence of a theorem due to Buckholtz and Frank [3], shows that Theorem A is sharp in a certain sense:

THEOREM B. *Let r and s be positive numbers such that*

$$\frac{s}{W(\mathscr{D})} > r .$$

Then there exists a holomorphic function F of radius of convergence r such that $\mathscr{D}^n F$ has a zero in $|z| \leqq (e_{n+1}/e_n)\, s$ for all but finitely many n.

In the following we will use Theorem A and Theorem B and two duality principles for $\mathscr{F}_r$ in order to investigate the behavior of the Pincherle sequences

$$E_n(z) = z^n E(\lambda_n z) \qquad n = 0, 1, 2, \cdots ,$$

where E is the exponential function belonging to the operator $\mathscr{D}$ and $(\lambda_n)_{n=0}^{\infty}$ is a given sequence of scalars.

2. Completeness of the system $\{z^n E(\lambda_n z)\}$. Let $E \in \mathscr{F}_R$ $(0 < R < \infty)$ with the power series expansion

$$E(z) = \sum_{k=0}^{\infty} e_k z^k \quad \text{and} \quad \limsup_{k\to\infty} |e_k|^{1/k} = \frac{1}{R} .$$

We suppose that $e_0 = 1$ and $e_k > 0$ for $k = 1, 2, \cdots$.

In the sequel, we will always require that the sequence $(e_k)_{k=0}^{\infty}$ satisfies the following conditions:

(2.1a) $(e_{k-1}/e_k)_{k=1}^{\infty}$ is nondecreasing;

(2.1b) $(e_k^2/e_{k-1}e_{k+1})_{k=1}^{\infty}$ is nonincreasing and has limit 1 (compare (1.1)).

From condition (2.1a) we have

$$\lim_{k\to\infty} (e_k/e_{k-1}) = \frac{1}{R}$$

since $E \in \mathscr{F}_R$.

THEOREM 1. *Let* $(\lambda_n)_{n=0}^{\infty}$ *be a sequence of complex numbers such that*

$$|\lambda_n| \leqq \frac{e_{n+1}}{e_n} s \qquad n = 0, 1, 2, \cdots$$

for a real number $s > 0$. *Then the system* $\{z^n E(\lambda_n z)\}_{n=0}^{\infty}$ *is complete in any space* $\mathscr{F}_r$ *for* $R/r \geqq s/W(\mathscr{D})$.

Proof. Here we use the following well known form of the Hahn-Banach theorem: A subset $G \subseteqq \mathscr{F}_r$ is dense in $\mathscr{F}_r$ if and only if for each continuous linear functional L on $\mathscr{F}_r$ such that $L(g) = 0$ for each $g \in G$ it follows that $L = 0$.

Let $\{(e_k)_{k=0}^{\infty}, r\}$ denote the space of all holomorphic functions $h(z) = \sum_{k=0}^{\infty} h_k z^k$ with the property

$$\limsup_{k\to\infty} |h_k/e_k|^{1/k} < r .$$

This means that the functions $h \in \{(e_k)_{k=0}^{\infty}, r\}$ are holomorphic on the disk $|z| \leqq R/r$, since

$$\limsup_{k\to\infty} |h_k/e_k|^{1/k} = R \limsup_{k\to\infty} |h_k|^{1/k} < r ,$$

and on the other hand that the function

$$h_E(z) = \sum_{k=0}^{\infty} \frac{h_k}{e_k} z^{-k-1}$$

is holomorphic for $|z| \geqq r$.

A duality between $\mathscr{F}_r$ and $\{(e_k)_{k=0}^{\infty}, r\}$ is defined by the bilinear forms

$$\langle g, h\rangle = \frac{1}{2\pi i}\int_{\gamma} g(z) h_E(z) dz , \tag{2.2}$$

where $g \in \mathscr{F}_r$, $h \in \{(e_k)_{k=0}^{\infty}, r\}$ and γ is a circle contained in the intersection of the domain of holomorphy of g with the domain of holomorphy of h_E.

Formula (2.2) gives the general form of the continuous linear functionals on $\mathscr{F}_r$ (see [6] or [12]).

Now let $L \in \mathscr{F}_r'$ such that $L(E_n) = 0$ for $n = 0, 1, 2, \cdots$, where $E_n(z) = z^n E(\lambda_n z)$. Then there exists a function $h \in \{(e_k)_{k=0}^{\infty}, r\}$ such that

$$L(E_n) = \frac{1}{2\pi i}\int_{\gamma} z^n E(\lambda_n z) h_E(z) dz = \frac{1}{2\pi i}\int_{\gamma}\left(\sum_{k=0}^{\infty} e_k \lambda_n^k z^{k+n}\right)\left(\sum_{k=0}^{\infty}\frac{h_k}{e_k} z^{-k-1}\right) dz$$
$$= \sum_{k=n}^{\infty} \frac{e_{k-n}}{e_k} \lambda_n^{k-n} h_k = \mathscr{D}^n h(\lambda_n) .$$

By condition (2.1a) and inequality (1.5) we have

$$|\lambda_n| \leqq \frac{e_{n+1}}{e_n} s \leqq e_1 s < \frac{s}{W(\mathscr{D})} \leqq \frac{R}{r},$$

which implies that $E_n \in \mathscr{F}_r$ for $n = 0, 1, 2, \cdots$.

Since

$$H = \left(\limsup_{k\to\infty} |h_k|^{1/k}\right)^{-1} > \frac{R}{r}$$

the assumption $R/r \geqq s/W(\mathscr{D})$ implies that the corresponding Gončarov-polynomials constitute a basis in $\mathscr{F}_H$ (see Theorem A). By the uniqueness-property of a basis we have $h \equiv 0$ if $\mathscr{D}^n h(\lambda_n) = 0$ for $n=0, 1, 2, \cdots$. Now it follows $L=0$, which completes our proof.

In the next theorem we show that Theorem 1 is sharp in a certain sense:

THEOREM 2. *Let r and s be positive numbers such that $se_1 < R/r < s/W(\mathscr{D})$. Then there exists a sequence of complex numbers $(\lambda_n)_{n=0}^{\infty}$ with the property*

$$|\lambda_n| \leqq \frac{e_{n+1}}{e_n} s$$

such that the functions $E_n(z) = z^n E(\lambda_n z)$ are in $\mathscr{F}_r$ but are not complete in $\mathscr{F}_r$.

Proof. We have to show that there exists a continuous linear functional $L_0 \neq 0$ on $\mathscr{F}_r$ such that $L_0(E_n) = 0$ for $n = 0, 1, 2, \cdots$, where

$$E_n(z) = z^n E(\lambda_n z) \quad \text{for} \quad n = 0, 1, 2, \cdots$$

and $(\lambda_n)_{n=0}^{\infty}$ is a suitable sequence of complex numbers. In view of the proof of Theorem 1 it suffices to show that there exists a function

$$h_0 \in \{(e_k)_{k=0}^{\infty}, r\}$$

such that $\mathscr{D}^n h_0(\lambda_n) = 0$ for $n = 0, 1, 2, \cdots$ and $h_0 \not\equiv 0$.

In order to find such a function h_0 we apply Theorem B: by our assumption

$$\frac{R}{r} < \frac{s}{W(\mathscr{D})}$$

we can find a number H_0 such that

$$\frac{R}{r} < H_0 < \frac{s}{W(\mathscr{D})} ,$$

and by Theorem B there exists a function $\tilde{h}_0$ with $c(\tilde{h}_0) = H_0$ such that $\mathscr{D}^n\tilde{h}_0$ has a zero in

$$|z| \leqq \frac{e_{n+1}}{e_n} s$$

for all but finitely many n.

This implies that we can find a sequence $(\lambda_n)_{n=0}^{\infty}$ of complex numbers with

$$|\lambda_n| \leqq \frac{e_{n+1}}{e_n} s$$

for $n = 0, 1, 2, \cdots$ such that $|\mathscr{D}^n\tilde{h}_0(\lambda_n)| < \infty$ for $0 \leqq n \leqq N$ (take for instance $\lambda_n = 0$ for $0 \leqq n \leqq N$) and $\mathscr{D}^n\tilde{h}_0(\lambda_n) = 0$ for $n > N$, where N is a suitable natural number. Since $e_1 s < R/r$, we have $|\lambda_n| \leqq (e_{n+1}/e_n)s \leqq e_1 s < R/r$, which implies that $E_n \in \mathscr{F}_r$ for $n = 0, 1, 2, \cdots$.

Now define a polynomial p_0 by

$$p_0(z) = \sum_{n=0}^{N} \mathscr{D}^n\tilde{h}_0(\lambda_n)Q_n(z; \lambda_0, \cdots, \lambda_{n-1}) .$$

Then p_0 is a polynomial of degree not greater than N and has the property

$$\mathscr{D}^n p_0(\lambda_n) = \mathscr{D}^n\tilde{h}_0(\lambda_n) \quad \text{for} \quad 0 \leqq n \leqq N ,$$

and $\mathscr{D}^n p_0(\lambda_n) = 0$ for $n > N$ (see part 1).

We set now

$$h_0 = \tilde{h}_0 - p_0 ,$$

then

$$\mathscr{D}^n h_0(\lambda_n) = 0 \quad \text{for} \quad n = 0, 1, 2, \cdots$$

and $c(h_0) = H_0$.

If we write

$$h_0(z) = \sum_{k=0}^{\infty} h_{0,k} z^k ,$$

then

$$\limsup_{k\to\infty} |h_{0,k}/e_k|^{1/k} < r ,$$

since $H_0 > R/r$. This means $h_0 \in \{(e_k)_{k=0}^{\infty}, r\}$. So if we set

$$L_0(g) = \langle g, h_0 \rangle = \frac{1}{2\pi i} \int_\Gamma g(z)(h_0)_E(z)dz ,$$

then $L_0(E_n) = 0$ for $n = 0, 1, 2, \cdots$ and $L_0 \neq 0$.

The desired conclusion now follows again from the Hahn-Banach theorem.

3. **Uniqueness of the representation by the system $\{z^n E(\lambda_n z)\}$.** The purpose of this part is to derive conditions under which the system $\{z^n E(\lambda_n z)\}$ constitutes a basis in its closed linear hull in a certain space $\mathscr{F}_r$. In order to do this we use a dual relationship between basis theory and interpolation theory developed by M. M. Dragilev, V. P. Zaharjuta and Ju. F. Korobeinik in 1974 (see [4]):

Let X be a nuclear Fréchet space with a topology given by a family of seminorms $\{\|\cdot\|_p, p \in P\}$; let X' be the strong dual space. We consider two sequence spaces generated by a sequence $\{x_n\}_{n=0}^\infty$ of nonzero elements of X:

$$\mathscr{E} = \left\{c = (c_n)_{n=0}^\infty : |c|_p := \sum_{n=0}^\infty |c_n| \|x_n\|_p < \infty \text{ , for each } p \in P\right\}$$

with the topology determined by the family of seminorms $\{|c|_p, p \in P\}$, and

$$\mathscr{E}' = \left\{c' = (c'_n)_{n=0}^\infty : \text{there exists a } p \in P \text{ with } |c'|'_p := \sup_n \frac{|c'_n|}{\|x_n\|_p} < \infty\right\}$$

with the topology of the strong dual with respect to duality, given by the formula

$$\langle c, c' \rangle = \sum_{n=0}^\infty c_n c'_n .$$

THEOREM C. (*See* [4], [12].) *Let X be a nuclear Fréchet space. A sequence $\{x_n\}_{n=0}^\infty$ constitutes a basis in its closed linear hull in X if and only if for each sequence $(t_n)_{n=0}^\infty \in \mathscr{E}'$ there exists $x' \in X'$ such that*

$$x'(x_n) = t_n \quad \textit{for} \quad n = 0, 1, 2, \cdots .$$

(*In this case one says that the interpolation problem $(X', \{x_n\}_{n=0}^\infty)$ is solvable*).

In the sequel we use Theorem C for the system $E_n(z) = z^n E(\lambda_n z)$ considered in part 2.

THEOREM 3. *Let $(\lambda_n)_{n=0}^\infty$ be a sequence of complex numbers such that*

$$|\lambda_n| \leqq \frac{e_{n+1}}{e_n} s \qquad n = 0, 1, 2, \cdots .$$

for a real number $s > 0$. Then the system $\{z^n E(\lambda_n z)\}_{n=0}^{\infty}$ constitutes a basis in $\mathscr{F}_r$ for any $r > 0$ with the property $R/r \geqq s/W(\mathscr{D})$.

Proof. In order to apply the above principle we remark that Theorem C says that $\{E_n\}_{n=0}^{\infty}$ constitutes a basis in its closed linear hull if and only if for each sequence $(t_n)_{n=0}^{\infty}$ with the property

$$(3.1) \qquad |t_n| \leqq K \|E_n\|_{r'} \quad n = 0, 1, 2, \cdots ,$$

where $r' < r$ and K is a constant only depending on r', there exists a continuous linear functional $L \in \mathscr{F}_r'$ represented by a function $h \in \{(e_k)_{k=0}^{\infty}, r\}$ such that

$$L(E_n) = \langle E_n, h \rangle = \mathscr{D}^n h(\lambda_n) = t_n ,$$

for $n = 0, 1, 2, \cdots$.

We take a sequence $(t_n)_{n=0}^{\infty}$ such that inequality (3.1) holds. Since the systems of norms $(\|\cdot\|_{r'}, r' < r)$ and $(|||\cdot|||_{r'}, r' < r)$ are equivalent in $\mathscr{F}_r$, inequality (3.1) can be replaced by

$$(3.2) \qquad |t_n| \leqq K r'^n \sup_k (e_k |\lambda_n|^k r'^k) \qquad n = 0, 1, 2, \cdots .$$

This follows from the fact that

$$E_n(z) = z^n \sum_{k=0}^{\infty} e_k \lambda_n^k z^k$$

and by the definition of the norms $|||\cdot|||_{r'} (r' < r)$. Now we obtain

$$\limsup_{n\to\infty} |t_n|^{1/n} \leqq r' \limsup_{n\to\infty} \left[\sup_k (e_k |\lambda_n|^k r'^k) \right]^{1/n} .$$

By inequality (1.5) and the assumption $R/r \geqq s/W(\mathscr{D})$ we have $|\lambda_n| < R/r$ for $n = 0, 1, 2, \cdots$. This means $E_n \in \mathscr{F}_r$ for $n = 0, 1, 2, \cdots$, and

$$\sup_k (e_k |\lambda_n|^k r'^k) \leqq \sup_k \left(e_k \left(\frac{R}{r} \right)^k r'^k \right) .$$

Since $(R/r) r' < R$, we have

$$\sup_k \left(e_k \left(\frac{R}{r} \right)^k r'^k \right) < K_E ,$$

where K_E is a constant depending on E.

This implies

$$\limsup_{n\to\infty} |t_n|^{1/n} \leqq r' .$$

Now we obtain

$$\limsup_{n\to\infty} |e_n t_n|^{1/n} \leqq (\limsup_{n\to\infty} e_n^{1/n})\left(\limsup_{n\to\infty} |t_n|^{1/n}\right) \leqq \frac{r'}{R} .$$

By Theorem A the Gončarov-polynomials $Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$ constitute a basis in $\mathscr{F}_{R/r'}$, since

$$\frac{R}{r'} > \frac{R}{r} \geqq \frac{s}{W(\mathscr{D})} .$$

Consider the polynomials

$$e_n^{-1} Q_n(z; \lambda_0, \cdots, \lambda_{n-1}) ;$$

since

$$e_n^{-1} Q_n(z; \lambda_0, \cdots, \lambda_{n-1}) = z^n - \sum_{k=0}^{n-1} \frac{e_{n-k}}{e_n} \lambda_k^{n-k} Q_k(z; \lambda_0, \cdots, \lambda_{k-1}) ,$$

(see 1.3) it follows that the bases $\{e_n^{-1} Q_n(z; \lambda_0, \cdots, \lambda_{n-1})\}_{n=0}^{\infty}$ and $\{z^n\}_{n=0}^{\infty}$ are equivalent in $\mathscr{F}_{R/r'}$, i.e.,

$$\sum_{n=0}^{\infty} c_n e_n^{-1} Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$$

converges in $\mathscr{F}_{R/r'}$ if and only if $\sum_{n=0}^{\infty} c_n z^n$ converges in $\mathscr{F}_{R/r'}$ (see [14], pg. 188).

Now since $\limsup_{n\to\infty} |e_n t_n|^{1/n} \leqq r'/R$, we have $\sum_{n=0}^{\infty} e_n t_n z^n$ converges in $\mathscr{F}_{R/r'}$, and therefore $\sum_{n=0}^{\infty} t_n Q_n(z; \lambda_0, \cdots, \lambda_{n-1})$ converges in $\mathscr{F}_{R/r'}$; in other words: there exists a function $h \in \mathscr{F}_{R/r'}$ such that

$$\mathscr{D}^n h(\lambda_n) = t_n \qquad n = 0, 1, 2, \cdots .$$

The fact that $h \in \mathscr{F}_{R/r'}$ implies that for

$$h(z) = \sum_{k=0}^{\infty} h_k z^k$$

we have

$$\limsup_{k\to\infty} |h_k|^{1/k} \leqq \frac{r'}{R} < \frac{r}{R}$$

and hence

$$\limsup_{k\to\infty} \left|\frac{h_k}{e_k}\right|^{1/k} < r .$$

This means $h \in \{(e_k)_{k=0}^{\infty}, r\}$; now by the representation of the continuous linear functionals on $\mathscr{F}_r$ we see that there exists a continuous linear functional $L \in \mathscr{F}_r'$ such that

$$L(E_n) = \langle E_n, h \rangle = \mathscr{D}^n h(\lambda_n) = t_n \qquad n = 0, 1, 2, \cdots .$$

Theorem C implies that the system $\{E_n\}_{n=0}^{\infty}$ constitutes a basis in its closed linear hull in $\mathscr{F}_r$, by Theorem 1 the system $\{E_n\}_{n=0}^{\infty}$ is complete in $\mathscr{F}_r$, so $\{E_n\}_{n=0}^{\infty}$ constitutes a basis in $\mathscr{F}_r$ and the proof of Theorem 3 is finished.

By [14] it follows that the system $\{z^n E(\lambda_n z)\}_{n=0}^{\infty}$ constitutes a basis in $\mathscr{F}_r$ which is equivalent to the canonical basis $\{z^n\}_{n=0}^{\infty}$. We remark that under the assumptions of Theorem 2 the system $\{z^n E(\lambda_n z)\}_{n=0}^{\infty}$ does not constitute a basis for $\mathscr{F}_r$, because the system $\{z^n E(\lambda_n z)\}_{n=0}^{\infty}$ is even not complete in $\mathscr{F}_r$.

Some other results of this kind can be found in [11], [12] or [13] (see also the references in [11]). But these are all sufficient conditions for a system $\{z^n f(\lambda_n z)\}_{n=0}^{\infty}$ to be a basis in $\mathscr{F}_r$ and there is no similar result to Theorem 2.

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