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A DENSITY RESULT IN SPACES OF SILVA HOLOMORPHIC MAPPINGS

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A modification of an idea of Aron and Schottenloher is used to show: For any finitely Runge open subset Ω in certain locally convex spaces E (including the class of quasi-complete dualnuclear spaces) the space $(H(\Omega, F), \tau_0)$ of holomorphic functions on Ω with values in a locally convex space F is a dense topological subspace of $(H_S(\Omega, F), \tau_{so})$, the space of Silva holomorphic functions endowed with the strictly compact open topology. This is used to give a certain bidual interpretation for $(H_S(\Omega, F), \tau_{so})$, if F is a complete Schwartz locally convex space.

Preface. The aim of the present article is to prove the result announced above. In order to do this we first recall some definitions and results and fix the notation; mainly for vector spaces with bornology. Then we show how the notion of Silva holomorphy offers the possibility to carry over—in a natural way—many topologies on spaces of holomorphic functions on Banach spaces to such spaces on bornological vector spaces. However, we prove that on Schwartz bornological vector spaces the natural generalizations of the compact open and of Nachbin's ported topology coincide. Since our main result will need the Schwartz property anyway, we work with the strictly compact open topology which is easier to handle. This becomes clear in the proof of the density lemma 7, where an idea of Aron and Schottenloher is used to show that the continuous finite polynomials are dense in the space of all Silva holomorphic functions on finitely Runge open subsets of certain bornological vector spaces. The result announced in the abstract is then an easy consequence of the lemma. Observing that $(H_S(\Omega, F), \tau_{so})$ is a complete Schwartz l.c. space for any complete Schwartz l.c. space F, we can use the density result to describe $(H_S(\Omega, F), \tau_{so})$ as a mixed (topological-bornological) bidual of $(H(\Omega, F), \tau_{so})$ for finitely Runge open subsets of certain l.c. spaces E and any complete Schwartz locally convex space F.

Part of the results of this article have been announced in [5], [6].

1. Notation. For our notation from the theory of locally convex (l.c.) spaces we refer to Horváth [8] or Schaefer [10], while we shall refer to Hogbe-Nlend [7] for notations from the theory of

bornological spaces. Throughout this article all l.c. spaces are assumed to be Hausdorff and complex vector spaces. A bornological vector (b.v.) space will always denote a complex, convex, separated, and complete bornological vector space in the terminology of Hogbe-Nlend [7].

The following notations, definitions, and results will be used without any further comment: Let E be a b.v. space. We define $\mathfrak{B} = \mathfrak{B}(E)$ as the set of all convex, balanced, and bounded subsets of E. For $B \in \mathfrak{B}$ E_B denotes the linear hull of B, normed by the gauge of B. A set $B \in \mathfrak{B}$ is called a *Banach disc*, if E_B is a Banach space. E is called a *Schwartz* b.v. space, if for any $A \in \mathfrak{B}$ there is $B \in \mathfrak{B}$ such that A is relatively compact in E_B. A subset Ω of E is called *M-open*, if for any $B \in \mathfrak{B}$ the set $\Omega \cap E_B$ is open in E_B. A subset K of E is cally *strictly compact*, if there exists $B \in \mathfrak{B}$ such that K is compact in E_B. We say that E has *property* (a), if for any strictly compact set K in E, there is a Banach disc $B \in \mathfrak{B}$ such that K is compact in E_B and such that the identity on E_B can be approximated uniformly on K by continuous linear operators on E_B, which have finite rank. Obviously E has property (a), if for any strictly compact set K in E there is a Banach disc $B \in \mathfrak{B}$ such that K is compact in E_B and such that E_B has the approximation property. The vector space of all bounded linear forms on E is denoted by $E^{\times}$. E is called *t-separated or separated by its (bornological) dual*, if $E^{\times}$ separates the points of E. It is an easy consequence of the theorem of Hahn-Banach that E is t-separated iff the locally convex inductive limit $\underrightarrow{\mathrm{ind}}_{B \in \mathfrak{B}} E_B$ is Hausdorff or iff $\sigma(E, E^{\times})$ is Hausdorff.

2. Definition. Let F be a l.c. space.

(a) If Ω is an open subset of a l.c. space E, a function $f: \Omega \to F$ is called holomorphic, if f is continuous and Gâteaux holomorphic. $H(\Omega, F)$ denotes the linear space of holomorphic F-valued functions on Ω, $H(\Omega)$ stands for $H(\Omega, \boldsymbol{C})$. The compact open topology of uniform convergence on the compact subsets of Ω is denoted by τ_0, while τ_ω denotes the ported topology introduced by Nachbin. It is defined by the system of all ported semi-norms p on $H(\Omega, F)$, where a semi-norm p on $H(\Omega, F)$ is called K-ported, if there exists a compact K in Ω and a continuous semi-norm q on F such that for any open neighborhood $V \subset \Omega$ of K there is a $c(V) > 0$ such that

$$p(f) \leqq c(V) \sup_{x \in V} q(f(x)) \quad \text{for any} \quad f \in H(\Omega, F) .$$

(b) If Ω is an M-open subset of a b.v. space E, a function

$f: \Omega \to F$ is called Silva holomorphic, if for any $B \in \mathfrak{B}$ the restriction of f to $\Omega_B := \Omega \cap E_B$ is holomorphic on the open subset Ω_B of E_B. $H_S(\Omega, F)$ denotes the vector space of all Silva holomorphic functions on Ω with values in F, $H_S(\Omega)$ stands for $H_S(\Omega, C)$.

From the definition of Silva holomorphy the proof of the following lemma is obvious.

3. **Lemma.** *Let Ω be an M-open subset of a* b.v. *space E and let F be a l.c. space. By using the inclusion as an ordering of $\mathfrak{B}$ and using the restrictions as linking maps one has (algebraically)* $H_S(\Omega, F) = \underset{\longleftarrow}{\mathrm{proj}}_{B \in \mathfrak{B}} H(\Omega_B, F)$.

This lemma indicates that there is a natural way to extend topologies on spaces of holomorphic functions on normed spaces to spaces of Silva holomorphic functions. We shall do this, however, only for the compact open and the ported topology.

4. **Definition.** Let Ω be an M-open subset of a b.v. space E and let F be a l.c. space. The strictly compact open topology τ_{so} (resp. the strictly ported topology $\tau_{s\omega}$) on $H_S(\Omega, F)$ is defined as the (topological) projective limit $\underset{\longleftarrow}{\mathrm{proj}}_{B \in \mathfrak{B}}(H(\Omega_B, F), \tau_0)$ resp. $\underset{\longleftarrow}{\mathrm{proj}}_{B \in \mathfrak{B}}(H(\Omega_B, F), \tau_\omega)$.

REMARK. (a) The topology $\tau_{s\omega}$ was introduced for $H(\Omega, F)$ by Bianchini, Paques and Zaine [3].

(b) It is easy to see that τ_{so} is just the topology of uniform convergence on the strictly compact subset of the M-open set Ω.

On Banach spaces τ_ω is known to be strictly finer than τ_0. Hence one might guess that $\tau_{s\omega}$ is a better topology than τ_{so}. However, it turns out that $\tau_{s\omega}$ coincides with τ_{so} if E is a Schwartz b.v. space.

5. **Proposition.** *Let E be a Schwartz b.v. space and F a l.c. space. Then we have $(H_S(\Omega, F), \tau_{s\omega}) = H_S(\Omega, F), \tau_{so})$ for any M-open subset Ω of E.*

Proof. It is easy to see that $\tau_{s\omega}$ is finer than τ_{so}. Let us show that the converse holds, too. If any $B \in \mathfrak{B}$ is given, there exists $C \in \mathfrak{B}$ such that the canonical map $i_{BC}: E_B \to E_C$ is compact, since E is a Schwartz b.v. space. Then the mapping $j_{BC}: (H(\Omega_C, F), \tau_o) \to (H(\Omega_B, F), \tau_\omega)$, $j_{BC}(f) = f \circ i_{BC}$, is continuous. In order to see this, let p be any ported semi-norm on $H(\Omega_B, F)$, and let K and q be as in 2.(a). Since i_{BC} is compact, we can find an open subset V in Ω_B

with $K \subset V$ and a compact subset Q of Ω_C such that $i_{BC}(V) \subset Q$. Then we have for any $f \in H(\Omega_C, F)$:

$$p(j_{BC}(f)) \leqq c(V) \sup_{x \in V} q(j_{BC}(f)(x)) = c(V) \sup_{x \in V} q(f(i_{BC}(x)))$$
$$\leqq c(V) \sup_{y \in Q} q(f(y)) .$$

This shows the continuity of j_{BC}. By the definition of the topologies $\tau_{s\omega}$ and τ_{so} this implies that τ_{so} is finer than $\tau_{s\omega}$.

REMARK. (a) In Barroso, Matos and Nachbin [2] it is shown that on Silva spaces τ_0 and τ_ω coincide. Since $\tau_0 = \tau_{so}$ on such spaces, it follows that $\tau_\omega = \tau_{s\omega} = \tau_{so} = \tau_0$ in Silva spaces.

(b) If one wants to work only with the topology τ_{so}, then one may assume always that the underlying b.v. space is a Schwartz b.v. space because of the following reason: If E is a b.v. space, then we may define a new b.v. space E_s which is the vector space E with the bornology of the strictly compact subsets of the old b.v. space E. It is a consequence of the theorem of Banach-Dieudonné that E_s is a Schwartz b.v. space, having the same strictly compact sets as E. If Ω is M-open in E, then it is M-open in E_s and one can show easily that Silva holomorphy of a function defined on Ω does not depend on the fact whether Ω is regarded as a subset of E or E_s. This means that $(H_S(\Omega, F), \tau_{so}) = (H_S(\Omega_s, F), \tau_{so})$, where Ω_s denotes the M-open subset Ω of E_s.

In order to state the density result announced above we need some notation.

6. Definition. (a) Let E be a complex vector space. A set $\Omega \subset E$ is called finitely Runge, if for any finite dimensional linear subspace E_0 in E the set $\Omega \cap E_0$ is a (nonempty) Runge open subset of E_0 (i.e., $H(E_0)$ is dense in $(H(\Omega \cap E_0), \tau_0)$).

(b) Let $(E, \mathfrak{F})$ be a l.c. space. A function p on E is called a finite monomial if it is a constant or if there exist $n \in N$ and

$$y_1, \cdots, y_n \in (E, \mathfrak{F})' \quad \text{such that} \quad p(x) = \prod_{j=1}^{n} \langle y_j, x\rangle \quad \text{for any} \quad x \in E .$$

A finite polynomial is a finite sum of finite monomials. The space of all finite polynomials on E is denoted by $\mathscr{P}_f(E, \mathfrak{F})$.

REMARK. It follows from the convergence properties of the Taylor series that any translate of a balanced open set Ω in a l.c. space E is a finitely Runge subset of E.

Let us recall that it follows from a remark in 1. that on a t-separated b.v. space E there exists at least one l.c. topology $\mathfrak{F}$ such that any bounded subset of the b.v. space E is also bounded in the l.c. space $(E, \mathfrak{F})$.

7. **Lemma.** *Let E be a t-separated* b.v. *space with property* (a), *and let $\mathfrak{F}$ be a* l.c. *topology on E for which any bounded subset of the* b.v. *space E is bounded in the* l.c. *space $(E, \mathfrak{F})$. Then $\mathscr{P}_f(E, \mathfrak{F}) \otimes F$ is dense in $(H_S(\Omega, F), \tau_{so})$ for any finitely Runge M-open subset of E and any* l.c. *space F.*

Proof. By a standard reduction argument we may assume that F is a Banach space. Let K be a strictly compact subset of Ω and let $f \in H_S(\Omega, F)$ and $\varepsilon > 0$ be given. Since E has property (a) by hypothesis, there exists a Banach disc B in E, such that K is compact in E_B and such that the identity on E_B can be approximated uniformly on K by continuous linear operators on E_B, which have finite rank. Then $\Omega_B := \Omega \cap E_B$ is a finitely Runge open subset of E_B and $f|\Omega_B$ is in $H(\Omega_B, F)$. Using these facts one proves in the same way as in Theorem 2.2 of Aron and Schottenloher [1], that for any $\varepsilon > 0$ there is $p \in \mathscr{P}_f(E_B) \otimes F$ with

$$\sup_{x \in K} \|p(x) - f(x)\| \leqq \frac{\varepsilon}{2} .$$

Since the canonical inclusion $j_B: E_B \to (E, \mathfrak{F})$ is continuous and injective its transpose $t_{j_B}: (E, \mathfrak{F})' \to (E_B)'$ has $\sigma(E_B', E_B)$-dense range. Now observe that the topology $\lambda(E_B', E_B)$ of uniform convergence on the pre-compact subsets of E_B is consistent with the duality between E_B and E_B'. Hence the range of t_{j_B} is even $\lambda(E_B', E_B)$-dense. By the definition of a finite polynomial it follows that there exists $q \in \mathscr{P}_f(E, \mathfrak{F}) \otimes F$ with

$$\sup_{x \in K} \|q(x) - f(x)\| \leqq \varepsilon .$$

Hence the lemma is proved.

REMARK. Under the hypothesis of Lemma 7 $(H_S(\Omega), \tau_{so})$ has the approximation property. This follows from Lemma 7, Remark (b) after Proposition 5, Proposition 3 in [4] and a characterization of the approximation property by means of the ε-product (Schwartz [11]). We do not give any further detail because a more general result can be obtained by translating the notion of Silva approximation property introduced by Paques [9] to the bornological setting.

Our density result will now be a consequence of the preceeding lemma. Remember that a l.c. E is called a co-Schwartz space if E endowed with its von Neumann bornology is a Schwartz b.v. space.

8. Theorem. *Let $(E, \mathfrak{F})$ be a* l.c. *space and assume that E endowed with its von Neumann bornology is a* b.v. *space with property* (a). *Then for any* l.c. *space F and any finitely Runge open subset Ω of $(E, \mathfrak{F})$ the space $H(\Omega)\otimes F$ and a fortiori $H(\Omega, F)$ is dense in $(H_S(\Omega, F), \tau_{so})$. If $(E, \mathfrak{F})$ is a co-Schwartz space moreover, then $(H(\Omega, F), \tau_o)$ is a dense topological linear subspace of $(H_S(\Omega, F), \tau_{so}) = (H_S(\Omega, F), \tau_{s\omega})$.*

Proof. E endowed with its von Neumann bornology is obviously a t-separated b.v. space, which has property (a) by hypothesis. Hence it follows from Lemma 7, that $\mathscr{P}_f(E, \mathfrak{F})\otimes F$ is dense in $(H_S(\Omega, F), \tau_{so})$. Clearly this implies the density of $H(\Omega)\otimes F$ and $H(\Omega, F)$ in $(H_S(\Omega, F), \tau_{so})$.

If $(E, \mathfrak{F})$ is a co-Schwartz-space, then a subset K of E is compact in $(E, \mathfrak{F})$ iff it is strictly compact in the b.v. space E introduced above. Hence $(H(\Omega, F), \tau_o)$ is a topological subspace of $H_S(\Omega, F)$, $\tau_{so})$ which is equal to $(H_S(\Omega, F), \tau_{s\omega})$ by Proposition 5.

REMARK. Any quasi-complete dual-nuclear l.c. space E satisfies all the hypotheses of Theorem 8.

9. Corollary. *If $(E, \mathfrak{F})$ is a co-Schwartz* l.c. *space satisfying all the hypotheses of Theorem* 8, *and if F is a complete* l.c. *space, then $(H_S(\Omega, F), \tau_{so})$ is the completion of $(H(\Omega, F), \tau_o)$ for any finitely Runge open subset Ω of E.*

This follows from Theorem 8 and the observation that $(H_S(\Omega, F), \tau_{so})$ is complete, if F is complete. In [3] the completion of $(H(E), \tau_{s\omega})$ is studied; under the above assumptions on E it follows that it is exactly $(H_S(E)\tau_{so})$.

Now we want to use Theorem 8 to describe $(H_S(\Omega, F), \tau_{so})$ as a mixed (topological-bornological) bidual of $(H(\Omega, F), \tau_0)$, if F is a complete Schwartz l.c. space. Before we can do this we need the following result on the Schwartz property of $(H_S(\Omega, F), \tau_{so})$ which is a consequence of Cauchy's integral formula and the theorem of Arzela-Ascoli. The detailed proof is left to the reader.

10. Proposition. *Let E be a* b.v. *space, $\Omega \neq \emptyset$ an M-open subset of E and F a* l.c. *space. Then $(H_S(\Omega, F), \tau_{so})$ is a Schwartz* l.c. *space iff F is a Schwartz* l.c. *space.*

Let us recall that for a l.c. space G one can regard the locally convex dual G' as a b.v. space by taking the bornology of the equicontinuous subsets. The bornological dual $G'^{\times}$ of G' becomes a l.c. space under the topology of uniform convergence on the bounded subsets of this b.v. space G'. In Theorem 4. p. 95 in Hogbe-Nlend [7] it is shown that $G = G'^{\times}$ for any complete Schwartz l.c. space E. This together with Theorem 8 and Proposition 10 gives the description announced above:

11. Corollary 2. *Let* $(E, \mathfrak{T})$ *be a co-Schwartz* l.c. *space and assume that* E *endowed with its von Neumann bornology is a* b.v. *space with property* (a). *Then we have for any finitely Runge open subset* Ω *of* $(E, \mathfrak{T})$ *and any complete Schwartz* l.c. *space* F

$$(H(\Omega, F), \tau_0)'^{\times} = (H_S(\Omega, F), \tau_{so}) .$$

Proof. By Theorem 8 $(H(\Omega, F), \tau_0)$ is a dense topological linear subspace of $(H_S(\Omega, F), \tau_{so})$, hence we have $(H(\Omega, F), \tau_0)' = (H_S(\Omega, F), \tau_{so})'$ and both spaces have the same system of equicontinuous sets. By the completeness of F and Proposition 10 we get that $(H_S(\Omega, F), \tau_{so})$ is a complete Schwartz l.c. space. Hence it follows from the considerations above that we have

$$(H(\Omega, F), \tau_0)'^{\times} = (H_S(\Omega, F), \tau_{so})'^{\times} = (H_S(\Omega, F), \tau_{so}) .$$

Remark. In [5] where Corollary 2 was announced, it was unfortunately forgotten to assume that E has property (a).

Remark. Let us also notice that the proof of Theorem 8 gives actually the following more general formulation of Theorem 8:

Let $(E, \mathfrak{T})$ be a l.c. space in which any compact set is contained and compact in the normed space E_Q for some balanced convex compact set Q and such that E_c, the vector space E endowed with the bornology of the compact sets of $(E, \mathfrak{T})$ has property (a). Then for any finitely Runge open subset Ω of $(E, \mathfrak{T})$ and any l.c. space F, the space $(H(\Omega, F), \tau_0)$ is a dense topological subspace of $(H_S(\Omega_c, F), \tau_{so}) = (H_S(\Omega_c, F), \tau_{s\omega})$, where Ω_c denotes Ω as a subset of E_c.

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