A SHAPE FIBRATION WITH FIBERS OF DIFFERENT SHAPE

JAMES KEESLING AND SIBE MARDEŠIĆ*

Let E and B be metric continua. Let $p: E \to B$ be a shape fibration in the sense of Mardešić and Rushing. If B is arcwise connected, then all the fibers of p have the same shape. This is also true if B is connected by shape paths in the sense of Krasinkiewicz and Minc. It had been asked whether that this would be true without any assumptions other than B being a continuum. In this paper an example is given of a shape fibration $p: E \to B$ with E and B metric continua such that p has fibers of different shape.

1. **Introductions.** Let $\underline{E} = (E_n, h_n)$ and $\underline{B} = (B_n, q_n)$ be inverse sequences of metric compacta. A *level map* $\underline{p}: \underline{E} \to \underline{B}$ is a sequence of maps $p_n: E_n \to B_n$ such that for each n the following diagram commutes.

(1)
$$
\begin{array}{ccc}
E_n & \xleftarrow{\ h_n\ } & E_{n+1} \\
{\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p_{n+1}} \\
B_n & \xleftarrow{\ q_n\ } & B_{n+1}
\end{array}
$$

If each p_n is a fibration, we say that $\underline{p}$ is a *level map of fibrations*. Each level map $\underline{p}: \underline{E} \to \underline{B}$ *induces* a map $p: E \to B$ between the inverse limits $E = \varprojlim \underline{E}$ and $B = \varprojlim \underline{B}$.

It is well-known that all the fibers of a fibration $p: E \to B$ have the same homotopy type provided B is pathwise connected. If $p: E \to B$ is induced by a level map of fibrations (or more generally by a shape fibration in the sense of S. Mardešić and T. B. Rushing ([8] and [7])), and if B is pathwise connected, then any two fibers have the same shape. This is actually true more generally when B is connected by shape paths in the sense of J. Krasinkiewicz and P. Minc [6]. In this paper we show that one cannot strengthen this by assuming only that B is connected. More precisely we prove the following theorem.

THEOREM 1. *There exist continua E and B and a map $p: E \to B$ induced by a level map of fibrations $\{p_n\}$ such that p admits two fibers of different shape.*

From the construction proving Theorem 1 we actually have that each $p_n: E_n \to B_n$ is a fiber bundle. Moreover, for each n $B_n = S^1$

and $E_n = M$ is a closed 3-manifold, all the p_n's are the same, and all the fibers of each p_n are homeomorphic to the 2-torus T^2. The space B is the dyadic solenoid.

Before proving Theorem 1 we first discuss some implications of it and its proof.

COROLLARY 1.   *There exists a shape fibration $p: E \to B$ with connected base-space B such that p has two fibers of different shape.*

Corollary 1 follows immediately from Theorem 1 and answers a question posed in [8].

REMARK 1.   One might at first assume that connectedness in shape theory is the natural analogue of path connectedness in homotopy theory.   There are many instances when this is true, but not in general.   Connectedness by shape paths (which is equivalent to pointed 1-movability [6]) seems to be the more accurate analogue. Theorem 1 and Corollary 1 support this thesis.   Another argument supporting this thesis is the example by the first author of a metric continuum whose shape groups and homotopy pro-groups depend on the choice of base point [4].   Neither the example proving Theorem 1 in this paper nor the example in [4] can exist if the space is connected by shape paths.

REMARK 2.   Recently M. Jani [1] has defined a notion of fiber shape equivalence of shape fibrations.   He has shown that the pullbacks of a shape fibration by strongly shape equivalent maps are fiber shape equivalent shape fibrations.   Now any two maps of a point into a continuum B are shape equivalent (but not necessarily strongly shape equivalent).   However, the pull-backs in this case are just the fibers over the two image points.   Consequently, Corollary 1 shows that one cannot strengthen Jani's result by replacing strong shape equivalence by shape equivalence.

REMARK 3.   In a recent paper [7] the second author has introduced a new equivalence relation between metric compacta called *S-type*.   It was easy to show that shape equivalent compacta have the same S-type and it was asked whether there were S-equivalent compacta of different shape.   In [7] it was shown that for shape fibrations over connected spaces, the fibers have the same S-type. Consequently our theorem also has the following corollary.

COROLLARY 2.   *There exist two 2-torus-like metric continua which have the same S-type, but which have different shape.*

In the proof of Theorem 1, it is shown that the two fibers of different shape actually have different 1-dimensional Čech cohomology groups. Thus we have the following corollary.

COROLLARY 3. *The Čech cohomology groups with integer coefficients are not invariants of S-type.*

REMARK 4. The fibers in our example are compact connected abelian topological groups. This was a definite aid in developing the example. The basic theory of shape for such groups was developed by the first author in [2]. Some other applications of the theory are given in [3] and [5].

2. Integral matrices and maps between tori. In the construction of the example $p: E \to B$ establishing Theorem 1 we shall be considering maps of the 2-torus T^2 into T^2. In this section we review briefly some facts concerning maps from the n-torus T^n into T^n.

We think of S^1 as the multiplicative group of complex numbers of norm one. Then $T^n = S^1 \times \cdots \times S^1$ becomes a compact connected abelian topological group. From [9] every continuous maps $f: T^n \to T^n$ is homotopic to a unique continuous homomorphism. Every homomorphism $f: T^n \to T^n$ can be represented by an $n \times n$ matrix with integer entries in the following manner. Let $e: R \to S^1$ be the standard covering may defined by $e(t) = e^{2\pi i t}$. Then $e^n = e \times \cdots \times e: R^n \to T^n$ is a homomorphism of topological groups. Every homomorphism $f: T^n \to T^n$ lifts to a unique homomorphism $\tilde{f}: R^n \to R^n$ so that the following diagram commutes.

(1)
$$
\begin{array}{ccc}
R^n & \xrightarrow{\tilde{f}} & R^n \\
e^n \downarrow & & \downarrow e^n \\
T^n & \xrightarrow{f} & T^n
\end{array}
$$

It follows that $\tilde{f}(\ker e^n) \subset \ker e^n$. But $\ker e^n = Z^n$ and thus $g = \tilde{f}|Z^n$ is a homomorphism from Z^n to Z^n. We associate with f the unique $n \times n$ matrix M such that $(y_1, \cdots, y_n) = g(x_1, \cdots, x_n)$ implies that

(2)
$$
M\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.
$$

Clearly M is an integral matrix. Conversely, if M is any integral $n \times n$ matrix we obtain by (2) a continuous homomorphism $\tilde{f}: R^n \to R^n$. Because M is integral $\tilde{f}(Z^n) \subset Z^n$. Therefore $\tilde{f}$ defines

a homomorphism f on T^n to T^n so that (1) commutes. The homomorphism f is unique.

Note that $f: T^n \to T^n$ is an isomorphism if and only if the associated matrix M has det $M = \pm 1$. Actually $|\det M|$ is the number of points in each point inverse set of f if det $M \neq 0$, but we shall not use this fact.

3. The construction of p. Let $g: T^2 \to T^2$ be the continuous map determined by the integral matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since the determinant of this matrix is one, g is a homomorphism. Consider $T^2 \times R$ and define an action of the group Z on $T^2 \times R$ by $n(x, t) = (g^n(x), t - n)$. Let M be the quotient space of this action and let $r: T^2 \times R \to M$ be the quotient map. Then r is a covering map and M is a closed 3-manifold. Note that $M = r(T^2 \times [0, 1])$. Note also that M can also be described as the mapping torus of $g: T^2 \to T^2$. Let $\pi: T^2 \times R \to R$ denote projection onto the second coordinate and let $e: R \to S^1$ denote the exponential map $t \to e^{2\pi i t}$. Then there is a unique map $s: M \to S^1$ such that the following diagram commutes.

$$(1) \qquad \begin{array}{ccc} M & \xleftarrow{\ r\ } & T^2 \times R \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \pi} \\ S^1 & \xleftarrow{\ e\ } & R \end{array}$$

Indeed, if $r(x, t) = r(x', t')$, then there is an n in Z such that $(x', t') = (g^n(x), t - n)$. Hence, $e\pi(x', t') = e\pi(g^n(x), t - n) = e(t - n) = e(t) = e\pi(x, t)$. Clearly, s is a fiber bundle and thus a fibration.

Similarly, let $2: R \to R$ denote multiplication by 2. Then there is a unique map $q: S^1 \to S^1$ such that the following diagram commutes.

$$(2) \qquad \begin{array}{ccc} R & \xleftarrow{\ 2\ } & R \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ S^1 & \xleftarrow{\ q\ } & S^1 \end{array}$$

Of course, q is just z^2 on the complex numbers of norm one.

Now let $f: T^2 \to T^2$ be the continuous map induced by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We claim that there exists a unique map $h: M \to M$ such that the following diagram commutes.

$$(3) \qquad \begin{array}{ccc} T^2 \times R & \xleftarrow{\ f \times 2\ } & T^2 \times R \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ M & \xleftarrow{\ \ h\ \ } & M \end{array}$$

Moreover, h makes the following diagram commutative also.

$$(4) \qquad \begin{array}{ccc} M & \xleftarrow{\ h\ } & M \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ S^1 & \xleftarrow{\ q\ } & S^1 \end{array}$$

The last claim follows from the preceeding one because r is onto and for (x, t) in $T^2 \times R$ we have by (1), (2), and (3) the following equalities.

$$qsr(x, t) = qe\pi(x, t) = e(2t) = e\pi(f(x), 2t)$$
$$= sr(f(x), 2t) = sr(f \times 2)(x, t) = shr(x, t) .$$

In order to prove the existence of h we have to show that:

$$(6) \qquad\qquad r(x, t) = r(x', t')$$

implies that

$$(7) \qquad\qquad r(f \times 2)(x, t) = r(f \times 2)(x', t') .$$

Now (6) implies that there is an n in Z such that $x' = g^n(x)$ and $t' = t - n$. But then $r(f \times 2)(x', t') = r(fg^n(x), 2t - 2n)$. It thus suffices to show that:

$$(8) \qquad\qquad fg^n = g^{2n}f$$

because then one has:

$$(9) \qquad\qquad \begin{aligned} r(f \times 2)(x', t') &= r(fg^n(x), 2t - 2n) \\ &= r(g^{2n}f(x), 2t - 2n) \\ &= r(f(x), 2t) \\ &= r(f \times 2)(x, t) . \end{aligned}$$

The third equality in (9) holds because $2n(f(x), 2t) = (g^{2n}f(x), 2t - 2n)$. Thus we need to establish (8) to verify that h is well-defined. But (8) is verified by the following matrix calculation.

$$(10) \qquad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2n & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{2n}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2n & 2 \end{pmatrix} .$$

Thus the matrices associated with fg^n and $g^{2n}f$ are the same and the maps fg^n and $g^{2n}f$ must coincide. Thus h is well-defined.

We now define the inverse sequences $\underline{E} = (E_n, h_n)$ and $\underline{B} = (B_n, q_n)$ and the level map of fibrations $\underline{p} = (p_n)$. We let $E_n = M$ for all n

and $h_n = h$ for all n. We let $B_n = S^1$ for all n and $q_n = q$ for all n. The we let $p_n = s$ for all n. Then $\underline{p} = (p_n)$ is a level map by (4) and since each p_n is a fibration, $\underline{p}$ is a level map of fibrations. Finally, we let $E = \lim E$, $B = \lim B$, and $p = \lim \underline{p}$. The example has now been completely described. We need to show now that the example has the desired properties to prove Theorem 1 and its corollaries.

We observe that B is clearly the dyadic solenoid. Moreover, all the fibers of p_n are 2-tori from (1). This implies that the fibers of p are inverse limits of 2-tori and are thus 2-torus-like continua. In [3] it is shown that the shape of a torus-like continuum is determined by its 1-dimensional Čech cohomology group. The only way two fibers F and F' of p can have different shape is if $H^1(F)$ and $H^1(F')$ are groups which are not isomorphic. In the next section of the paper we exhibit the fibers F and F'. Then in § 5 we show that the fibers F and F' are not shape equivalent by showing that $H^1(F)$ and $H^1(F')$ are not isomorphic groups.

4. Choosing the fibers F and F'. In this section we show how to choose the fibers F and F' of p. For the construction of F and F' we need two lemmas.

LEMMA 1. *Let $s_n: T^2 \to T^2$ be a sequence of maps such that for each n $s_n = f$ or $s_n = gf$. Then there is a point b in B such that the fiber $F_b = p^{-1}(b)$ is homeomorphic to the inverse limit of the inverse sequence (T^2, s_n).*

Proof. We shall define points b_n in B_n and homeomorphisms μ_n so that

$$(1) \qquad\qquad q_n(b_{n+1}) = b_n$$

and so that the following diagram commutes:

$$(2) \qquad
\begin{array}{ccccccc}
T^2 & \xleftarrow{\; s_1 \;} & T^2 & \xrightarrow{\; s_2 \;} & T^2 & \xrightarrow{\; s_3 \;} & \cdots \\
{\scriptstyle \mu_1}\downarrow & & {\scriptstyle \mu_2}\downarrow & & {\scriptstyle \mu_3}\downarrow & & \\
F_1 & \xleftarrow[\; h_1 \;]{} & F_2 & \xleftarrow[\; h_2 \;]{} & F_3 & \xleftarrow[\; h_3 \;]{} & \cdots
\end{array}
$$

where $F_i = p_i^{-1}(b_i)$. Then letting $b = \lim (b_n)$ and $F_b = \lim (F_n, h_n)$ we will have the desired fiber. We will define b_n, F_n, and the homeomorphisms μ_n by induction on n.

The points b_n and the homeomorphism $\mu_n: T^2 \to F_n$ will be defined by the formulas

(3)
$$b_n = e(t_n)$$

and

(4)
$$\mu_n(x) = r(x, t_n) \quad \text{for } x \text{ in } T^2$$

where t_n is in R and will depend only on n.

Note that by (1) of § 3

(5)
$$p_n\mu_n(x) = sr(x, t_n) = e\pi(x, t_n) = e(t_n) = b_n$$

so that μ_n does map T^2 into $F_n = p_n^{-1}(b_n)$. Moreover, μ_n will be a homeomorphism. Indeed, let y be in F_n. Then there is an (x, t) in $T^2 \times R$ such that $r(x, t) = y$ because r is onto. Then by (1) of § 3, $e(t) = e\pi(x, t) = sr(x, t) = s(y) = b_n$. Now there is a unique m in Z such that $r(g^m(x), t - m) = r(x, t) = y$ and $t - m = t_n$. Thus $x' = g^m(x) \in T^2$ is the only point such that $\mu_n(x') = y$.

We now show how to choose t_n in R. We start the induction by putting $t_1 = 0$. Now assume that $t_1, \cdots, t_k$ have been defined so that (1) holds for $n = 1, 2, \cdots, k - 1$ and the diagram (2) holds as far as it has been defined. We have to define t_{k+1} anp verify (1) for $n = k$ and verify the commutativity of the following diagram.

(6)
$$\begin{array}{ccc} T^2 & \xleftarrow{\ s_k\ } & T^2 \\ \mu_k \downarrow & & \downarrow \mu_{k+1} \\ F_k & \xleftarrow[h_k]{} & F_{k+1} \end{array}$$

We distinguish two cases.

Case (i): Let $s_k = f$.

In this case we put $t_{k+1} = \frac{1}{2}t_k$. Then for $b_{k+1} = e(t_{k+1})$ we have by § 3 (2):

(7)
$$q_k(b_{k+1}) = qe(t_{k+1}) = qe\left(\frac{1}{2}\,t_k\right) = e2\left(\frac{1}{2}\,t_k\right) = e(t_k) = b_k\,.$$

Also, by § 3 (3) we have:

(8)
$$h_k\mu_{k+1}(x) = hr(x, t_{k+1}) = hr\left(x, \frac{1}{2}\,t_k\right) = r(f \times 2)\left(x, \frac{1}{2}\,t_k\right)$$
$$= r(f(x), t_k) = \mu_k(f(x)) = \mu_k s_k(x)$$

as required by (6).

Case (ii): Let $s_k = gf$.

In this case we put $t_{k+1} = 1/2(t_k + 1)$. Then we have:

$$(9) \qquad q_k(b_{k+1}) = q_k e(t_{k+1}) = q e\left(\frac{1}{2}(t_k + 1)\right) = e(t_k + 1)$$

$$= e(t_k) = b_k .$$

Furthermore,

$$(10) \qquad h_k \mu_{k+1}(x) = hr(x, t_{k+1}) = hr\left(x, \frac{1}{2}(t_k + 1)\right)$$

$$= r(f \times 2)\left(x, \frac{1}{2}(t_k + 1)\right) = r(f(x), t_k + 1) .$$

Now by definition of r,

$$(11) \qquad r(f(x), t_k + 1) = r(gf(x), t_k) = \mu_k(gf(x)) = \mu_k s_k(x) ,$$

as required by (6). This completes the proof.

We now define the fibers F and F' using Lemma 1 so as to obtain the following lemma.

LEMMA 2. *Let $1 \leq n_1 < n_2 < n_3 < \cdots$ be a sequence of integers. Then there exist points b and b' in B such that the fiber $F = p^{-1}(b)$ is homeomorphic to the limit of the inverse sequence of 2-tori T^2 with all bonding maps equal to f and the fiber $F' = p^{-1}(b')$ is homeomorphic to the limit of the inverse sequence of 2-tori where one has first n_1 bonding maps gf, then n_2 bonding maps f, then n_3 bonding maps gf, etc.*

This is an immediate application of Lemma 1. Lemma 2 now defines the fibers F and F' which we claim are not shape equivalent. In the next two sections we show that $H^1(F)$ and $H^1(F')$ are not isomorphic which will prove our claim.

5. **Cohomology of the fibers F and F'.** Let H^1 denote the first Čech cohomology groups with coefficient group Z. Recall that there are natural isomorphisms $H^1(T^2) \cong \mathrm{Hom}(H_1(T^2), Z)$ (from the Universal Coefficient Theorem for cohomology) and $H_1(T^2) \cong \pi_1(T^2) \cong Z^2$ (since $\pi_1(T^2)$ is abelian). Thus one can identify $H^1(T^2)$ with Z^2. Now if one has a homomorphism $f: T^2 \to T^2$ represented by the matrix A, then that homomorphism induces a homomorphism on $H^1(T^2) = Z^2 \to Z^2 = H^1(T^2)$. The induced homomorphism we can take to be represented by the transpose A' of the matrix A if we make the following choice of basis for $H^1(T^2)$. Let $\{e_1, e_2\}$ be the basis for $\pi_1(T^2)$ corresponding to the basis in Z^2 by which the matrix A was identified with f. Let $\{e_1', e_2'\}$ be the basis in $H_1(T^2)$ under the

natural transformation between $\pi_1(T^2)$ and $H_1(T^2)$. Then let $\{e_1'', e_2''\} \subset$ $\text{Hom}(H_1(T^2), Z)$ be defined by $e_i''(e_j') = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Then $\{e_1'', e_2''\}$ is a basis for $\text{Hom}(H_1(T^2), Z)$. The image of this basis in $H^1(T^2)$ will be the required basis and with that basis $f^*: H^1(T^2) \to H^1(T^2)$ will be associated with the integral matrix A'. These facts together with the continuity of the Čech cohomology groups implies that $H^1(F)$ is the direct limit of the direct sequence of groups:

$$(1) \qquad Z^2 \xrightarrow{\;q_1\;} Z^2 \xrightarrow{\;q_2\;} Z^2 \xrightarrow{\;q_3\;} \cdots$$

where all the q_i's are given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Clearly the direct limit of this direct sequence of groups is isomorphic to $Z \oplus Q^{(2)}$ where $Q^{(2)} = \{m/2^n \mid m, n \in Z, n \geq 0\}$. Thus all that remains to be proved is that $H^1(F')$ is not isomorphic to $Z \oplus Q^{(2)}$. Now the matrix representing gf is $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. Taking this fact together with the definition of F' into account, it will suffice to prove the following lemma.

LEMMA 3. *Let* $1 \leq n_1 < n_2 < n_3 < \cdots$ *be a sequence of integers and let*

$$(3) \qquad G_0 \xrightarrow{\;q_1\;} G_1 \xrightarrow{\;q_2\;} G_2 \xrightarrow{\;q_3\;} \cdots$$

be a direct sequence of groups such that $G_i = Z^2$ *for all* i *such that the first* n_1 *bonding homomorphisms are given by matrix*

$$(4) \qquad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},$$

the next n_2 *bonding homomorphisms are given by*

$$(5) \qquad F = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

the next n_3 *bonding homomorphisms are given by* H, *etc. Then the direct limit is a group* G *which is not isomorphic to* $Z \oplus Q^{(2)}$.

In §6 we prove Lemma 3 and thus complete the proof of Theorem 1.

6. **Algebraic lemmas.** We shall prove Lemma 3. That is, we shall show that the G in Lemma 3 cannot be isomorphic to $Z \oplus Q^{(2)}$. This will be shown by showing that G has a certain property α, which is invariant under group isomorphism, which is not possessed

by $Z \oplus Q^{(2)}$.

We say that an abelian group A has *property* α provided there exists a monomorphism $\mu: A \to R^2$ and a sequence of elements a_i in A such that $\lim \mu(a_i) = 0$ and such that any two of the elements $\mu(a_i)$ are linearly independent in R^2. Clearly, if A has property α and $A \cong A'$, then A' also has property α. Thus Lemma 3 will follow from the next two lemmas.

LEMMA 4. *The group G described in Lemma 3 has property α.*

LEMMA 5. *The group $Z \oplus Q^{(2)}$ fails to have property α.*

Proof of Lemma 4. We shall first define by induction a sequence of monomorphisms $\mu_i: Z^2 \to R^2$ such that μ_0 is the inclusion $G_0 = Z^2 \subset R^2$ and such that the following diagram commutes:

$$(1) \qquad \begin{array}{ccccccc} G_0 & \xrightarrow{q_1} & G_1 & \xleftarrow{b_2} & G_2 & \xrightarrow{q_3} & \cdots \\ \mu_0 \downarrow & & \mu_1 \downarrow & & \mu_2 \downarrow & & \\ R^2 & \xrightarrow[\text{id}]{} & R^2 & \xrightarrow[\text{id}]{} & R^2 & \longrightarrow & \cdots \end{array}$$

Let us assume that μ_{i-1} has already been defined. Since $\det F = \det H = 2 \neq 0$, q_i is a monomorphism. Thus the subgroup $q_i(G_{i-1})$ of G_i is a free abelian group of rank 2. Thus there is a basis $\{a_1, a_2\}$ of $Z^2 = G_i$ and integers $n_1, n_2 \geq 1$ such that $\{n_1 a_1, n_2 a_2\}$ is a basis for $q_i(G_{i-1})$. Since $q_i: G_{i-1} \to q_i(G_{i-1})$ is an isomorphism, there exists also a basis $\{e_1, e_2\}$ of G_{i-1} such that

$$(2) \qquad q_i(e_1) = n_1 a_1 \quad \text{and} \quad q_i(e_2) = n_2 a_2 .$$

Since R^2 is a divisible group one can find elements b_1, b_2 in R^2 such that

$$(3) \qquad n_1 b_1 = \mu_{i-1}(e_1) \quad \text{and} \quad n_2 b_2 = \mu_{i-1}(e_2) .$$

Now define $\mu_i: G_i = Z^2 \to R^2$ by putting

$$(4) \qquad \mu_i(a_1) = b_1 \quad \text{and} \quad \mu_i(a_2) = b_2 .$$

Clearly, $\mu_i q_i(e_1) = \mu_i(n_1 a_1) = n_1 \mu_i(a_1) = n_1 b_1 = \mu_{i-1}(e_1)$. Similarly, $\mu_i q_i(e_2) = \mu_{i-1}(e_2)$, so that induced $\mu_i q_i = \mu_{i-1}$. Now we show that μ_i is a monomorphism. If k_1, k_2 are in Z and $\mu_i(k_1 a_1 + k_2 a_2) = 0$, then it follows that $k_1 b_1 + k_2 b_2 = 0$ and therefore $\mu_{i-1}(n_2 k_1 e_1 + n_1 k_2 e_2) = n_1 n_2(k_1 b_1 + k_2 b_2) = 0$. Since μ_{i-1} is a monomorphism, we conclude that $n_2 k_1 e_1 + n_1 k_2 e_2 = 0$. However, $\{e_1, e_2\}$ is a basis for Z^2 so that $n_2 k_1 = n_1 k_2 = 0$. Thus $k_1 = k_2 = 0$.

It can be readily seen that the monomorphisms μ_i induce an isomorphism μ of the direct limit G of (G_i, q_i) onto the subgroup $B = \bigcup_{i=0}^{\infty} \mu_i(G_i) \subset R^2$.

We now consider the points a_i in $G_{n_1+\cdots+n_i} = Z^2$ defined as follows.

$$(5) \qquad a_i = \begin{pmatrix} (1, 1) & i \text{ ood} \\ (0, 1) & i \text{ even} \end{pmatrix}.$$

Let

$$(6) \qquad b_i = \mu_{n_1+\cdots+n_i}(a_i), \quad b_0 = (0, 1).$$

We shall show that

$$(7) \qquad b_{i+1} = 2^{-n_{i+1}}(b_i + (-1)^i(1, 0)).$$

First assume that i is even. Then the homomorphism $q_{n_1+\cdots+n_{i+1}} \circ \cdots \circ q_{n_1+\cdots+n_i+1}$ is given by the matrix $H^{n_{i+1}}$. Since

$$(8) \qquad H^{n_{i+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2^{n_{i+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

(1) implies that

$$(9) \qquad \begin{aligned} 2^{n_{i+1}} b_{i+1} &= \mu_{n_1+\cdots+n_{i+1}} 2^{n_{i+1}}(1, 1) \\ &= \mu_{n_1+\cdots+n_i}(1, 1). \end{aligned}$$

Since

$$(10) \qquad F \begin{pmatrix} 1 \\ 0 \end{pmatrix} = H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

it follows that

$$(11) \qquad \begin{aligned} \mu_{n_1+\cdots+n_i}(1, 1) &= (1, 0) + \mu_{n_1+\cdots+n_i}(0, 1) \\ &= (1, 0) + b_i. \end{aligned}$$

Now (9) and (11) yield (7) for even i.

Using the fact that

$$(12) \qquad F^{n_{i+1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2^{n_{i+1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

one can verify (7) for odd i in a similar way.

Let $b_i = (x_i, y_i)$ and let $s_k = \sum_{j=1}^{k} n_j$. From the recursive formula (7) it is easy to derive the formulas.

$$(13) \qquad x_i = 2^{-s_i} \left(1 + \sum_{j=1}^{i-1} (-1)^j 2^{s_j} \right)$$

(14) $$y_i = 2^{-s_i} .$$

Now we shall show that b_n and b_m are linearly independent in R^2 for $n < m$. Indeed, assume that

(15) $$\lambda b_n + \mu b_m = 0 \quad n < m .$$

Then (14) yields

(16) $$\lambda 2^{-s_n} + \mu 2^{-s_m} = 0 .$$

The analogous relation for the first coordinates simplifies using (16) to the following.

(17) $$\lambda 2^{-s_n} \sum_{j=1}^{n-1} (-1)^j 2^{s_j} + \mu 2^{-s_m} \sum_{j=1}^{m-1} (-1)^j 2^{s_j} = 0 .$$

Now (16) and (17) yields:

(18) $$\mu 2^{-s_m} \sum_{j=n}^{m-1} (-1)^j 2^{s_j} = 0 .$$

The number

$$\sum_{j=n}^{m-1} (-1)^j 2^{s_j} = 2^{s_n}\left((-1)^n + \sum_{j=n+1}^{m-1} (-1)^j 2^{s_j - s_n}\right)$$

is $\neq 0$ because the expression in the parentheses is an odd number. Consequently, (18) yields $\mu = 0$. Then (16) yields $\lambda = 0$.

Finally, to show that $\lim b_i = 0$ it suffices to show that

(19) $$|b_i| \leqq 2 \cdot 2^{-n_i} .$$

This holds for $i = 1$ because $b_1 = 2^{-n_1}(1, 1)$.

If we assume (19) for i, then by (7)

$$|b_{i+1}| \leqq 2^{-n_{i+1}}(|b_i| + 1) \leqq 2 \cdot 2^{-n_{i+1}} .$$

This completes the proof of Lemma 4.

Proof of Lemma 5. Let $u: Z \oplus Q^{(2)} \to R^2$ be any monomorphism. One readily sees that u extends to a linear map $v: R^2 \to R^2$. Therefore $v(0 \times R)$ is a straight line $L \subset R^2$ through 0. Also, $v(n \times R)$, $n \in Z$, all coincide with L or are straight lines parallel to L such that the distance between any two consecutive lines is the same number $r > 0$. The first case is trivial. In the second case let U denote the open disc around 0 of radius r, then $U \cap u(Z \oplus Q^{(2)}) \subset L$. Consequently, all pairs of points of $u(Z \oplus Q^{(2)})$ which are close enough to 0 must be linearly dependent. Hence the group $Z \oplus Q^{(2)}$ fails to have property α.

This completes the proof of Lemma 5 and thus completes the proof of Theorem 1.

Added in proof. K. R. Goodearl and T. B. Rushing have recently shown that there are uncountably many fibers of different shape type in the example of the paper. Details are in "Direct limit groups and the Keesling-Mardesic shape fibration" (to appear in Pacific J. Math.).

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UNIVERSITY OF FLORIDA
GAINESVILLE, FL 32611
AND
UNIVERSITY OF ZAGREB
ZAGREB, YUGOSLAVIA