PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 2, 1979

HOMOTOPY WITH M-FUNCTIONS

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1. Introduction. M-functions were introduced by G. Darbo [1] and R. Jerrard [5] as a generalization of continuous functions between topological spaces. They are weighted, finitely-valued functions with a property corresponding to that of usual continuity. In [1] and [5] it was shown that ordinary singular homology groups for compact polyhedra are actually m-homotopy type invariants. In [6] it was shown that m-homotopy type is a stronger invariance than homotopy type in the sense that two spaces may have different homotopy types but the same m-homotopy type. R. Schultz [8] has noted that m-homology differs from singular homology on some compact metric spaces. It has also been brought to our attention that in a 1975 letter, G. Bredon indicated a method of proving that m-homotopy classes of PL m-functions on finite complexes are in $1-1$ correspondence with chain homotopy classes of chain maps. His approach is quite different from the one used in this paper. Here we define m-homotopy groups (actually R-modules) and give some of their properties. We show that for a compact polyhedron, the nth singular homology group and the nth m-homotopy group are actually isomorphic.

We show, for example, that the nth m-homotopy group has a natural definition as $m\pi_n(Y) = \hom(S^n, Y)$ in a certain category of m-functions, which is an R-module under the addition of m-functions defined below. This addition turns out to be the extension to m-functions of the usual product in homotopy groups. Since $\hom(X, Y)$ is always an R-module in this category, we see that m-homotopy groups (and hence singular homology groups) are special cases of the R-module $\hom(X, Y)$, which is a joint m-homotopy (and topological) invariant of X and Y.

Next we show that m-homotopy theory is a homology theory by proving it satisfies the Eilenberg-Steenrod axioms [4]. The excision axiom is of special interest since it completely fails to hold for usual homotopy. It is proven to hold in m-homotopy theory by introducing several combinatorial lemmas (§4).

There is a connection between the results here and the Dold-Thom theorem [2]. They showed that $H_m(Y) \cong \pi_m(AG(Y))$ where $AG(Y)$ is the topological free abelian group on the pointed polyhedron Y. There is a natural relationship between m-functions from X to Y and functions from X to $AG(Y)$. However, we show that there are m-functions $X \to Y$ with no corresponding continuous function

$X \to AG(Y)$ and vice versa.

2. *M-functions.* We give below a brief definition of m-functions. For motivation we refer the reader to [5].

Let X and Y be Hausdorff spaces and R a ring with identity and without zero divisors (in most examples $R = \boldsymbol{Z}$ or $\boldsymbol{R}$). Suppose we are given that:

(i) $f: X \to Y$ is a multiple-valued function such that each $f(x)$ is a finite or empty subset of Y,

(ii) $\bar{f}: X \times Y \to R$ is a (standard) function which defines f as a subset of $X \times Y$ by $f = \text{cl}\{(x, y) | \bar{f}(x, y) \neq 0\}$, and

(iii) for any $x \in X$ and any open set $V \subset Y$ such that $\partial V \cap f(X) = \emptyset$ there exists a neighborhood U of x such that for $x' \in U$,

$$\sum_{y \in V} \bar{f}(x, y) = \sum_{y \in V} \bar{f}(x', y) .$$

Then an *m-function* (denoted just by f) is f together with the weighting factor determined by the *defining function* $\bar{f}$. The *multiplicity* of f is $m(f) = \sum_{y \in Y} \bar{f}(x, y)$; it is independent of x if X is connected. The empty m-function, denoted by $\emptyset$ is defined by $\bar{\emptyset}: X \times Y \to 0$. Any continuous function can be regarded as an m-function by assiging it multiplicity one.

The composition of $f: X \to Y$ and $g: Y \to Z$ is defined by $\overline{g \circ f}(x, z) = \sum_{y \in Y} \bar{f}(x, y)\bar{g}(y, z)$, so Hausdorff spaces and m-functions over R form a category R-$T2$, with $T2$ as a subcategory. Any two m-functions may be added: $f + g$ is defined by $\overline{f + g} = \bar{f} + \bar{g}$. Also, if $a \in R$ we define the m-function af by $\overline{af} = a\bar{f}$. Then $\hom(X, Y)$ is an R-module and there are functors $\hom(\_\_, Z)$ and $\hom(Z, \_\_)$: R-$T2 \to (R \text{ modules})$. The restriction of $f: X \to Y$ to a subset $A \subset X$ is defined by $f | A = f \circ i$ when i is the inclusion $i: A \to X$. An m-function $F: X \times I \to Y$ is an *m-homotopy* between $F | X \times \{0\}$ and $F | X \times \{1\}$ (denoted by $\sim_m$). One can form m-homotopy classes of m-functions and these preserve the ring structure, that is, $[f + g] = [f] + [g]$ and $[af] = a[f]$.

We shall work primarily in the category R_0-$phT2$ of pointed pairs of Hausdorff spaces and m-homotopy classes of m-functions over R of multiplicity zero, together with its hom-sets (they are R-modules) and its hom-functors (see [7]). An m-function on pointed pairs $f: (X, A, x_0) \to (Y, B, y_0)$ must satisfy $f | A: A \to B$ and $f | x_0: x_0 \to y_0$.

LEMMA 2.1. *In R_0-$phT2$ the above condition for an m-function to be pointed is equivalent to $f | x_0 = \emptyset$; also, for*

$$f: X \longrightarrow Y , \quad f: (X, A, x_0) \longrightarrow (Y, y_0, y_0)$$

if and only if $f|A = \varnothing$. *In particular the morphisms do not depend upon the choice of base point in the image space.*

Proof. We know from the pointedness condition that $\bar{f}(x_0, y) = 0$ if $y \neq y_0$, and then from the zero multiplicity that $\bar{f}(x_0, y_0) = 0$. Thus x_0 has no image points of nonzero multiplicity and $f|x_0 = \varnothing$. A similar argument gives the second conclusion, and the converses are trivial.

Working only with m-functions of zero multiplicity entails almost no loss of generality. To any given m-function of multiplicity $a \in R$ can be added the constant m-function of multiplicity $(-a)$ and image y_0 to get an m-function of multiplicity zero which is the representative of the given m-function in R_0-$phT2$.

3. **M-homotopy groups.** In this section we define m-homotopy groups and the subsidiary concepts of boundary operator and induced homomorphism. We also obtain the surprising result that the usual product $[f][g]$ of two group elements is actually m-homotopic to $[f + g]$, the addition defined in § 2. Thus the group operation is addition and m-homotopy groups turn out to be hom-sets in R_0-$phT2$, which are R-modules.

For any pair $(X, A) = (X, A, \varnothing)$ and integer $n \geqq 1$ we define the nth *m-homotopy group*, $m\pi_n(X, A)$ to have as underlying set, the set of m-homotopy classes of m-functions (of multiplicity zero) $f: (B^n, S^{n-1}, 1) \to (X, A)$. B^n, S^{n-1}, and 1 are subsets of E^n defined by $B^n = \{x \mid |x| \leqq 1\}$, $S^{n-1} = \{x \mid |x| = 1\}$, and $1 = \{(1, 0, 0, \cdots, 0)\}$. In usual homotopy, $A \neq \varnothing$ and $(X, A) = (X, A, x_0)$. But by Lemma 2.1 our definition will include this one.

To define $m\pi_0(X, A)$ we let X_A be the set of path components of X not meeting A. Then $m\pi_0(X, A)$ consists of the m-homotopy classes of m-functions $f: (S^0, 1) \to (X_A)$ of arbitrary multiplicity.

Note that in the definition of $m\pi_n(X, A)$ we can replace B^n, S^{n-1}, and 1 by I^n, $\dot{I}^n$, and 0 respectively, where $I = [0, 1]$, $\dot{I}^n = \dot{(I^n)}$, and 0 denotes $\{(0, 0, \cdots, 0)\}$.

Before defining the group operation, we note the following implications of our above definition and Lemma 2.1:

(i) For $n \geqq 1$, if $[f] \in m\pi_n(X, A)$, then f has multiplicity zero in every path component of X.

(ii) For $n \geqq 2$, if $[f] \in m\pi_n(X, A)$, then $f|S^{n-1}$ has multiplicity zero in every path component of A.

(iii) $m\pi_0(X) \cong R^m$ where X has m path components.

We define, for $n \geqq 1$, the product of f and g in the traditional way by $fg: (B^{n-1} \times [-1, 1])/\sim \to X$ according to:

$$\overline{fg}(b, t, x) = \begin{cases} \bar{f}(b, 2t+1, x) & -1 \leqq t \leqq 0 \\ \bar{g}(b, 2t-1, x) & 0 \leqq t \leqq 1 \,. \end{cases}$$

(For $n = 1$, drop b from the above.)

THEOREM 3.1. *$fg \sim_m f + g$ (where f and g represent elements of $m\pi_n(X, A)$, for $n \geqq 1$).*

Proof. First define the m-functions $f_1, g_1\colon (B^{n-1} \times [-1, 1])/\sim \to X$ by:

$$\text{for} \quad -1 \leqq t \leqq 0\colon \bar{f}_1(b, t, x) = \bar{f}(b, 2t+1, x), \bar{g}_1(b, t, x) = 0$$
$$\text{for} \quad 0 \leqq t \leqq 1\colon \bar{f}_1(b, t, x) = 0, \bar{g}(b, t, x) = \bar{g}(b, 2t-1, x) \,.$$

Then $\overline{fg} = \bar{f}_1 + \bar{g}_1$ and so $fg = f_1 + g_1$. We need only prove that $f \sim_m f_1$ and $g \sim_m g_1$. The proofs are similar; we give the first.

Consider the family of homeomorphisms $d_\tau\colon B^{n-1} \times [-1, \tau] \to B^{n-1} \times [-1, 0]$ defined by $d_\tau(b, t) = (b, (t+1)/(\tau+1) - 1)(\tau \in I)$. The m-homotopy $F\colon ((B^{n-1} \times [-1, 1])/\sim) \times [-1, 1] \to X$ given by $F(b, t, \tau) = f \circ d_\tau$ for $t \leqq \tau$ and $F(b, t, \tau) = \varnothing$ for $t > \tau$ carries $f(\tau = 0)$ to $f_1(\tau = 1)$.

We extend the group operation to dimension zero by using m-function addition as the operation there also.

COROLLARY 3.2. *For $n \geqq 1$ the m-homotopy group $m\pi_n(X, A)$ is the R-module* $\hom[(B^n, S^{n-1}, 1), (X, A)]$. *Letting $A = \varnothing$, $m\pi_n(X) =$* $\hom[(B^n, S^{n-1}, 1), (X)] \cong \hom[(S^n, 1), (X)]$.

The last isomorphism can be easily proven by analogy to usual homotopy.

If $f\colon (X, A) \to (Y, B)$ and $n \geqq 1$ then $f_*\colon m\pi_n(X, A) \to m\pi_n(Y, B)$ is defined by $f_*[g] = [f \circ g]$. For $n = 0$, $f_*[g] = [(f \,|\, f^{-1}(Y_B)) \circ g]$. (Alternatively we could adjust the definition of $m\pi_0$ instead of that of f_*.) The remarks above on hom-functors imply that f_* and the boundary operator ∂_* defined below are well-defined on m-homotopy classes of m-functions. Let $\delta_n\colon B^n \to S^n$ be the natural continuous map implied by $B^n/S^{n-1} \approx S^n$, for $n \geqq 1$ (δ_n collapses S^{n-1} to 1, so that $\delta_n\colon (B^n, S^{n-1}, 1) \to (S^n, 1, 1)$). When we use $(I^n, \dot{I}^n, 0)$, δ_n becomes the map $\delta_n\colon I^n \to \dot{I}^{n+1}$ implied by $I^n/\dot{I}^n \approx \dot{I}^{n+1}$. Let $\delta_0\colon S^0 \to S^0$ (or $\dot{I} \to \dot{I}$) be the identity map. We sometimes drop the subscript n as superfluous. Let $\partial_*\colon m\pi_{n+1}(X, A) \to m\pi_n(A)$, called the boundary operator, be defined by $\partial_*[f] = [f \circ \delta_n]$. Then for injections $i\colon (A) \to (X)$ and $j\colon (X) \to (X, A)$ we have the *m-homotopy sequence*:

$$\cdots \longrightarrow m\pi_{n+1}(X, A) \xrightarrow{\partial_*} m\pi_n(A) \xrightarrow{i_*} m\pi_n(X) \xrightarrow{j_*} m\pi_n(X, A) \longrightarrow \cdots$$
$$\cdots \longrightarrow m\pi_0(A) \xrightarrow{i_*} m\pi_0(X) \xrightarrow{j_*} m\pi_0(X, A) \longrightarrow 0 .$$

The functorial axioms for m-homotopy follow from the fact that hom$[(B^n, S^{n-1}, 1), (_, _)]$ is a functor. Also ∂_* is a natural map since $f_* \circ \partial_*[g] = [f \circ g \circ \delta] = \partial_* \circ f_*[g]$.

THEOREM 3.3. *The m-homotopy sequence is exact.*

Proof. We use the definition of m-homotopy groups which considers m-functions from $(I^n, {}^{\cdot}I^n, 0)$. The proof is divided into four cases with only case d considering $n = 0$.

(a) (Exactness at $m\pi_n(A)$.) Suppose $[f] \in m\pi_{n+1}(X, A)$, so that $f: (I^{n+1}, {}^{\cdot}I^{n+1}, 0) \to (X, A)$. Define $\Gamma: I^{n+1} \times I \to I^{n+1}$ by $\Gamma_t(x) = tx$. Let $(\delta_n \times 1): I^n \times I \to {}^{\cdot}I^{n+1} \times I$ be the function $(\delta_n \times 1)(x, t) = (\delta_n(x), t)$. Now define $G: (I^n, {}^{\cdot}I^n, 0) \times I \to (X)$ by $G = f \circ \Gamma \circ (\delta_n \times 1)$. Since $G_1 = f \circ \delta_n$ and $G_0 = \varnothing$ (a constant m-function of zero multiplicity must be empty), G shows that $i_* \partial_*[f] = 0$, and so im $\partial_* \subset$ ker i_*.

Now suppose that $[f] \in m\pi_n(A)$ (so that $f: (I^n, {}^{\cdot}I^n, 0) \to (A)$) and that $i_*[f] = 0$. Then there exists $F: (I^n, {}^{\cdot}I^n, 0) \times I \to (X)$ with $F_0 = f$ and $F_1 = \varnothing$. There exists a continuous family of continuous functions $D_t: (I^n, {}^{\cdot}I^n) \to ({}^{\cdot}I^{n+1}, {}^{\cdot}I^n \times I \cup I^n \times 1)$ for $t \in I$, with $D_1 = \delta_n$ and $D_0: I^n \to I^n \times 0$ the natural injection. Then $F \circ D_t: (I^n, I^n, 0) \to (A)$, since $F | {}^{\cdot}I^n \times I \cup I^n \times 1 = \varnothing$. So in $m\pi_n(A)$, $[f] = [F_0] = [F \circ D_0] = [F \circ D_1] = [F \circ \delta] = \partial_*[F]$. Hence ker $i_* \subset$ im ∂_*.

(b) (Exactness at $m\pi_n(X)$.) Suppose $[f] \in m\pi_n(A)$, so that $f: (I^n, {}^{\cdot}I^n, 0) \to (A)$. Now for Γ as in (a), $f \circ \Gamma: (I^n, {}^{\cdot}I^n, 0) \times I \to (X, A)$, $f \circ \Gamma_1 = f$ and $f \circ \Gamma_0 = \varnothing$, so $i_* j_*[f] = 0$ and im $i_* \subset$ ker j_*.

Now suppose that $[f] \in m\pi_n(X)$ (so that $f: (I^n, {}^{\cdot}I^n, 0) \to (X)$) and $j_*[f] = 0$. Then there exists $F: (I^n, {}^{\cdot}I^n, 0) \times I \to (X, A)$ with $F_0 = f$ and $F_1 = 0$. There is a continuous function $H: (I^n, {}^{\cdot}I^n) \times I \to (I^{n+1}, {}^{\cdot}I^n \times 0)$ such that $H | I^n \times 1$ is a homeomorphism onto $I^n \times 1 \cup {}^{\cdot}I^n \times I$ and $H | I^n \times 0$ is the identity (H "pulls" the top of I^{n+1} over the sides, while "collapsing" the sides). Then $F \circ H: (I^n, {}^{\cdot}I^n, 0) \times I \to (X)$ and $F \circ H_1: (I^n, {}^{\cdot}I^n, 0) \to (A)$. Hence in $m\pi_n(X)$, $i_*[F \circ H_1] = [F \circ H_1] = [F \circ H_0] = [f]$. Thus ker $j_* \subset$ im i_*.

(c) (Exactness at $m\pi_n(X, A)$.) Suppose $[f] \in m\pi_n(X)$, so that $f: (I^n, {}^{\cdot}I^n, 0) \to (X)$. Since $\delta_{n-1}: I^{n-1} \to {}^{\cdot}I^n$, $f \circ \delta_{n-1} = \varnothing$. So in $m\pi_n(X)$, $\partial_* j_*[f] = [f \circ \delta_{n-1}] = 0$, and im $j_* \subset$ ker ∂_*.

Now suppose that $[f] \in m\pi_n(X, A)$ (so that $f: (I^n, {}^{\cdot}I^n, 0) \to (X, A)$) and $\partial_*[f] = 0$. Then there exists $F: (I^{n-1}, {}^{\cdot}I^{n-1}, 0) \times I \to (A)$ with $F_0 = f \circ \delta$ and $F_1 = \varnothing$. There are continuous functions for $0 \leqq \tau \leqq 1$:

$$\alpha_\tau\colon I^{n-1} \times [0, \tau/2] \longrightarrow I^{n-1} \times I$$

with

$$\alpha_\tau(x, t) = (x, \tau - 2t)$$

and

$$\Delta_\tau\colon I^{n-1} \times \left[\frac{\tau}{2}, 1\right] \longrightarrow I^{n-1} \times I$$

$$\text{such that } \begin{cases} \Delta_\tau \mid I^{n-1} \times \left\{\frac{\tau}{2}\right\} = \delta \\ \Delta_\tau\left(\left(\cdot I^{n-1} \times \left[\frac{\tau}{2}, 1\right]\right) \cup (I^{n-1} \times 1\right) = 0 \\ \Delta_\tau \mid \operatorname{int}\left(I^{n-1} \times \left[\frac{\tau}{2}, 1\right]\right) \text{ is a homeomorphism onto} \\ \quad \operatorname{int}(I^{n-1} \times I)\,. \end{cases}$$

We define $H\colon [(I^{n-1}, {}^{\cdot}I^{n-1}, 0) \times I] \times I \to (X, A)$ by

$$H(x, t, \tau) = \begin{cases} F \circ \alpha_\tau & 0 < t \leqq \tau/2 \\ f \circ \Delta_\tau & \tau/2 \leqq t \leqq 1\,. \end{cases}$$

Then $H|(\tau = 1)\colon (I^n, {}^{\cdot}I^n, 0) \to (X)$, so $[H|(\tau = 1)] \in m\pi_n(X)$, and $H|(\tau = 0) = f \circ \Delta_0 \in [f]$, since $\Delta_0 \sim 1_{I^n}$. Thus $j_*[H|(\tau = 1)] = [f]$ and $\ker \partial_* \subset \operatorname{im} j_*$.

(d) (Exactness for $n = 0$.) At $m\pi_0(A)$, $\operatorname{im} \partial_* \subset \ker i_*$ follows from the first part of (a). Now suppose $[f] \in m\pi_0(A)$ (so $f\colon (\cdot I, 0) \to (A)$) and that $i_*[f] = 0$. Then there exists $F\colon (\cdot I, 0) \times I \to (X)$ with $F_0 = f$ and $F_1 = \varnothing$. Define $g\colon (I, {}^{\cdot}I, 0) \to (X, A)$ by $g(t) = F_t(1)$. Then $\partial_*[g] = [g \circ \delta_0] = [(g|0) + (g|1)] = [f|1] = [f]$. So $\ker i_* \subset \operatorname{im} \partial_*$.

At $m\pi_0(X)$ we have $\operatorname{im} i_* \subset \ker j_*$ because if $[f] \in m\pi_0(A)$ then $f\colon ({}^{\cdot}I, 0) \to (A)$ and hence $f|X_A = \varnothing$. Note that $j_*[g] = [(j|X_A) \circ g]$. Now suppose $[f] \in m\pi_0[X]$ and $j_*[f] = 0$. Then $f|X_A \sim_m g$ where $g\colon ({}^{\cdot}I, 0) \to (A)$. It follows that $[g] \in m\pi_0(A)$ and $i_*[g] = [f]$.

At $m\pi_0(X, A)$ we take $[f]_A \in m\pi_0(X, A)$. Then $f\colon ({}^{\cdot}I, 0) \to (X)$, $[f] \in m\pi_0(X)$, and $j_*[f] = [f]_A$. Thus j_* is onto.

4. Three lemmas about boxes. In order to prove the excision axiom, we introduce several definitions and lemmas. By an n-dimensional box we mean the Cartesian product of n (orthogonal) compact line segments. By a k-face of a box, we mean any sub-box which is formed by taking the product of k of the original segments and replacing the remaining $n - k$ segments by (in each case) either endpoint. If T is a collection of boxes, we let $O(T)(E(T))$ be the subcollection of those boxes of odd (even) dimension. $|\cdot|$ represents the cardinality of a set.

LEMMA 4.1. *Let t be a proper k-face of an n-dimensional box, V, and let T be the set of faces of V containing t. Then* $|O(T)| = |E(T)|$.

Proof. We may assume that $V = I^n$. Note that a face of V is then determined by an ordered n-tuple where the entries are chosen from among I, 0, and 1.

Let t correspond to an n-tuple consisting of k entries of I and $(n-k)$ entries of a single point, 0 or 1. Choosing an m-face containing t is equivalent to choosing $(m-k)$ of the $(n-k)$ positions consisting of a single point, to be replaced by I. So we must show that

$$\sum_{m \text{ odd}} \binom{n-k}{m-k} = \sum_{m \text{ even}} \binom{n-k}{m-k} \quad \text{where} \quad k \leqq m \leqq n .$$

This is equivalent to showing that $\sum_{s \text{ odd}} \binom{r}{s} = \sum_{s \text{ even}} \binom{r}{s}$, for $r = n-k$, $0 \leqq s \leqq r$. One sees that this is true by considering

$$(x-1)^r = \sum_{s \text{ even}} \binom{r}{s} x^{r-s} - \sum_{s \text{ odd}} \binom{r}{s} x^{r-s}, \quad \text{for} \quad x = 1 .$$

Now suppose I^n is subdivided into finitely many boxes by subdividing each I in the product $I^n = I \times I \times \cdots \times I$ into segments. Let T be the collection of all these n-dimensional boxes and all those faces which meet the interior of I^n. For $t \in T$, we identify the box t with the function $t: I^n \to I^n$ for which $t(x)$ is the point of t closest to x (if $x \in t$, $t(x) = x$). For $t \in T$ and $f: I^n \to X$ an m-function, we let $f_t = f \circ t$.

LEMMA 4.2. *For f and T as above,* $f = (\sum_{s \in E(T)} f_s - \sum_{r \in O(T)} f_r)(-1)^n$.

Proof. $\sum_{s \in E(T)} f_s - \sum_{r \in O(T)} f_r = \sum_{s \in E(T)} f \circ s - \sum_{r \in O(T)} f \circ r = f \circ (\sum_{s \in E(T)} s - \sum_{r \in O(T)} r)$. So it suffices to show that

$$g = \Big(\sum_{s \in E(T)} s - \sum_{r \in O(T)} r\Big)(-1)^n$$

is the identity m-function on I^n. (The functions are added here by considering them as m-functions.) Let $\{v_i\}_{i=1}^m$ be the n-dimensional elements of T. Fix $a \in \operatorname{int} v_k$, for some k. To each v_i, associate v_i^*, the collection of elements, t, of T, such that $t(a) = v_i(a)$. We next show that $\{v_i^*\}$ partitions T and that each v_i^* consists of those faces of v_i containing a special face t_i.

In general, for u a set formed by Cartesian product of subsets of each coordinate axis, let u_k be the projection of u onto the kth coordinate axis. We then write $u = u_1 \times u_2 \times \cdots \times u_n$. Given $t \in T$, $t(a)_k$ is the point of t closest to a_k. ($t(a)_k$ is either a_k, one of the two endpoints of t_k, or, if t_k is a point, t_k itself. Only one of these can occur.) So v_i^* and v_j^* are disjoint for $i \neq j$.

Let $t \in T$ be fixed. Suppose we replace some of the t_k which are points by the interval in our subdivision of the kth coordinate axis with endpoint t_k between a_k and the other endpoint. Then the new Cartesian product gives us a box $s \in T$ with $s(a) = t(a)$ and t a face of s. Making all possible such replacements, we get $t \in v_i^*$ for some i, with t a face of v_i. Hence $\{v_i^*\}$ partitions T.

The elements of v_i^* can be constructed from v_i by considering each $(v_i)_k$. If $a_k \notin (v_i)_k$ we replace $(v_i)_k$ by its endpoint nearest a_k. The new Cartesian product will give us an element of v_i^*, and any element of v_i^* is of this form. By making all such replacements we get t_i, the element of v_i^* which we require.

Now, for $a \in \operatorname{int} v_k$, we have

$$g = v_k + (-1)^n \sum_{i \neq k} \Big(\sum_{s \in E(v_i^*)} s - \sum_{r \in O(v_i^*)} r \Big) .$$

But for $i \neq k$, $(\sum_{s \in E(v_i^*)} s - \sum_{r \in O(v_i^*)} r)$ maps a to $v_i(a)$ with multiplicity $|E(v_i^*)| - |O(v_i^*)| = 0$. So the only image point of a under g with nonzero multiplicity is a itself, which has multiplicity one. But this is true for any a in $\bigcup_{k=1}^m \operatorname{int} v_k$, a dense open subset of I^n. Hence g is the identity m-function on I^n.

Note that the image of f_t is the image of $f|t$. This lemma allows us to "break up" m-functions in a manner which corresponds to the subdivision operator on simplices used in simplicial homology.

LEMMA 4.3. *Given an m-function, $f: I^n \to Y$, and an open cover of Y, $\{U_\alpha\}$, there exist m-functions f_α such that $f = \Sigma f_\alpha$ and* $\operatorname{im}(f_\alpha) \subset U_\alpha$. *Further, if we choose Z, a face of I^n such that $f|Z = \varnothing$ (assuming such a face exists), then we may choose $\{f_\alpha\}$ such that $f_\alpha|Z = \varnothing$ for all α.*

Proof. For $x \in I^n$, let $f(x) = \{y_i\}_{i=1}^m$ with r_i the weight at (x, y_i). By the definition of m-functions, there exists $W_x \subset I^n$, a neighborhood of x, and $\{V_i\}_{i=1}^m$, disjoint open subsets of elements of $\{U_\alpha\}$, such that $f|W_x$ is an m-function with image in $\bigcup_{i=1}^m V_i$. Clearly each component of $f|W_x$ has its image in some V_i. Partition I^n into cubes of mesh less than the Lebesgue number of the cover $\{W_x\}_{x\in I^n}$. Let T be the collection of these cubes together with those faces (of any dimension) which meet $\operatorname{int} I^n$. Then $f = (\sum_{s\in E(T)} f_s - \sum_{r\in O(T)} f_r)(-1)^n$ by Lemma 4.2. For each $t \in T$, f_t equals the sum of its component m-functions, each of which has its image in some U_α. We partition the component m-functions of f_t and add so that $f_t = \sum_\alpha f_{t\alpha}$ and the image of $f_{t\alpha}$ lies in U_α. Letting $f_\alpha = \sum_t f_{t\alpha}$, the first part of the lemma is proved.

Suppose we choose Z a face of I^n such that $f|Z = \varnothing$. For $t \in T$, let $\bar{t}$ be the set of points of t closest to Z. Let $[t] = \{r \in T | \bar{t} = \bar{r}\}$.

We may assume that each projection Z_k is either I or 0. If $Z_k = 0$, then $(\bar{t})_k$ is a single point. Also, $[t]$ consists precisely of those $r \in T$ such that $r_k = (\bar{t})_k$ if $Z_k \neq 0$ and $r_k = [(\bar{t})_k, b]$ if $Z_k = 0$ (and b may equal $(\bar{t})_k$). It follows that $[t]$ contains an element, t', of maximal dimension (namely, $\dim t' = (\dim \bar{t}) + (n - \dim Z)$) and that $[t]$ consists of the faces of t' containing $\bar{t}$.

Note that if $z \in Z$ and $\bar{r} = \bar{s}$(i.e., $r, s \in [t]$ for some t) then $r(z) = s(z)$. If Z is p-dimensional, let Z' be the open dense subset of Z minus the $(p-1)$-dimensional boxes in the subdivision of I^n. But now, by the same argument as in the proof of Lemma 4.2, for $z \in Z'$, $z \notin t'$, $(-1)^n(\sum_{r\in E([t])} r - \sum_{s\in O([t])} s)$ has multiplicity zero at each image point of z. Choose an equivalence class $[t]$. Then t' and all its faces lie in some single W_x. So we can write $f\circ(-1)^n(\sum_{r\in E([t])} r - \sum_{s\in O([t])} s) = \sum_\alpha f^*_{t'\alpha}$ (allowing some $f^*_{t'\alpha}$'s to be empty) so that the image of $f^*_{t'\alpha}$ lies in U_α (just by partitioning the components and summing as before). Note that for $z \in Z' - t'$, $f^*_{t'\alpha}$ has multiplicity zero at each image point of z. Let $f^*_\alpha = \Sigma f^*_{t'\alpha}$ where we let t take one value in each equivalence class.

Now $f = \Sigma f^*_\alpha$ so it remains to show that $f^*_\alpha | Z = \varnothing$ for each α. Fix $a \in Z'$ and let v be the single (n-dimensional) box in T containing a. Then $v' = v$ and $f^*_\alpha = f^*_{v\alpha} + \sum_{t'\neq v} f^*_{t'\alpha}$. But $f^*_{v\alpha}$ is just the sum of certain components of $\pm f_v$ and $f_v(a) = f(a)$. Since $f|Z = \varnothing$, f^*_α has multiplicity zero at each image point of a. Hence $f^*_\alpha|Z' = \varnothing$. It follows that $f^*_\alpha | Z = \varnothing$ and we are done.

5. M-homotopy theory is a homology theory.

In §2 we described m-homotopy theory, $m\pi_n$. We wish to show this is a homology theory. In §3 we proved the exact sequence axiom and noted that the functorial axioms are satisfied. We also noted that the dimension axiom is satisfied, i.e., that $m\pi_0(Z) = R^m$ where m is the number of path components of Z.

THEOREM 5.1 (*The excision axiom*). *If $\bar{U} \subset \operatorname{int} A$, then the inclusion map $i: (X - U, A - U) \to (X, A)$ induces an isomorphism $i_*: m\pi_n(X - U, A - U) \to m\pi_n(X, A)$ for all n.*

Proof. Suppose $[f] \in m\pi_n(X, A)$, so $f: (I^n, \cdot I^n, 0) \to (X, A)$. By Lemma 4.3, there exists $g, h: (I^n, 0) \to (X)$ such that $f = g + h$ and $\operatorname{im}(g) \subset X - \bar{U}$ and $\operatorname{im}(h) \subset \operatorname{int} A$. But then in $m\pi_n(X, A)$, $[h] = 0$ (just consider $h \circ \Gamma_t$ where $\Gamma_t: I^n \to I^n$, $\Gamma_t(x) = tx$). Since f and h represent elements of $m\pi_n(X, A)$, so does $g = f - h$; in fact $[g] = [f]$. But $\operatorname{im}(g) \subset X - U$, and $\operatorname{im}(g|\cdot I^n) \subset A$, so $\operatorname{im}(g|\cdot I^n) \subset A - U$. Hence g represents an element of $m\pi_n(X - U, A - U)$ and i_* maps the m-homotopy class of g to the m-homotopy class of f and so is onto.

In the present paragraph, $[\cdot]$ will represent an m-homotopy class in $m\pi_n(X - U, A - U)$. Suppose $[f] \in m\pi_n(X - U, A - U)$ and there exists $F: (I^n, \cdot I^n, 0) \times I \to (X, A)$ with $F_0 = f$, $F_1 = \emptyset$ (i.e., f is null-m-homotopic in $m\pi_n(X, A)$). Then $F: (I^{n+1}, 0 \times I) \to (X)$, so by Lemma 4.3, we write $F = G + H$ with $G: (I^{n+1}, 0 \times 1) \to (X - \bar{U})$ and $H: (I^{n+1}, 0 \times I) \to (\operatorname{int} A)$. Since $f = G_0 + H_0$, $\operatorname{im}(H_0) \subset \operatorname{im}(f) \cup \operatorname{im}(G_0) \subset X - U$. So $\operatorname{im}(H_0) \subset A - U$, and $[H_0] = 0$ (consider $H_0 \circ \Gamma_t$). Hence, as before, $[f] = [G_0]$. Now $F_1 = \emptyset = G_1 + H_1$, so $\operatorname{im}(G_1) = \operatorname{im}(H_1) \subset A - U$. But then $[G_1] = [H_1] = 0$. Since $\operatorname{im}(G) \subset X - U$, $G = F - H$, $\operatorname{im}(F|\cdot I^n \times I) \subset A$, and $\operatorname{im}(H) \subset A$, we can conclude that $\operatorname{im}(G|\cdot I^n \times I) \subset A - U$. So $G: (I^n, \cdot I^n, 0) \times I \to (X - U, A - U)$ and $G: [G_0] = [G_1]$. Hence $[f] = [G_0] = [G_1] = 0$, and i_* is one-to-one.

We have neglected the case where $n = 0$. In this case, $m\pi_0(Y, B)$ is essentially the possible finite assignments of multiplicities to components of Y_B. Let X' be a component of X. If X' is disjoint from A, then X' is a component of $X - U$. If X' meets A, let X'' be a component of $X' - U$. Suppose $u \in \bar{X}'' \cap \bar{U}$. Then, since $\bar{U} \subset \operatorname{int} A$, some neighborhood of u lies in A and also meets X''. Hence X'' meets A. On the other hand, if $\bar{X}'' \cap \bar{U} = \emptyset$, then $\bar{X}''$ is a component of X and so $X' \cap U = \emptyset$ and $X'' = X'$. In either case, X'' meets $A - U$. So $X_A = (X - U)_{A-U}$. It is now easy to check that i_* is an isomorphism for $n = 0$ also.

Hence m-homotopy theory is a homology theory. By uniqueness we can conclude that $m\pi_n(X, A) = H_n(X, A)$ where (X, A) is any compact polyhedral pair and H_n is singular homology.

6. Examples; the Dold-Thom theorem. In this final section we consider the connections between m-homotopy groups, singular homology groups, and the Dold-Thom expression of homology groups as homotopy groups.

PROPOSITION 6.1. *The m-function image of a compact set is compact.*

Proof. Suppose $f: X \to Y$ is an m-function, $A \subset X$ is compact, and $\{V_\alpha\}$ is an open cover of $f(A)$. For $a \in A$, if $f(a) = \{y_1, \cdots, y_n\}$, choose V_{α_i} such that $y_i \in V_{\alpha_i}$. Then by the definition of m-function, there exist neighborhoods $V_i^*(y_i) \subset V_{\alpha_i}$ and $U(a)$ such that for $y \notin \bigcup_{i=1}^n V_i^*$ and $x \in U$, $y \notin f(x)$. Now let $U_1, \cdots, U_m$ cover A, with $V_1^*, \cdots, V_k^*$ the collection of all corresponding neighborhoods in Y. These cover $f'(A)$, and for each V_j^* some $V_{\alpha j}$ contains V_j^*. So $V_{\alpha 1}, \cdots, V_{\alpha k}$ is an open subcover of $f'(A)$.

EXAMPLE 6.2. Although m-functions are pointwise finite, they need not be globally or even locally finite. And $\bar{f}$ (say with $R = \boldsymbol{Z}$ or $\boldsymbol{R}$) need not attain a maximum, even on a compact set. In Figure 1 we sketch the graph of an m-function $f: I \to I$, such that as $x \nearrow 1$, $|f(x)| \to \infty$ and $\max_y \bar{f}(x, y) \to \infty$.

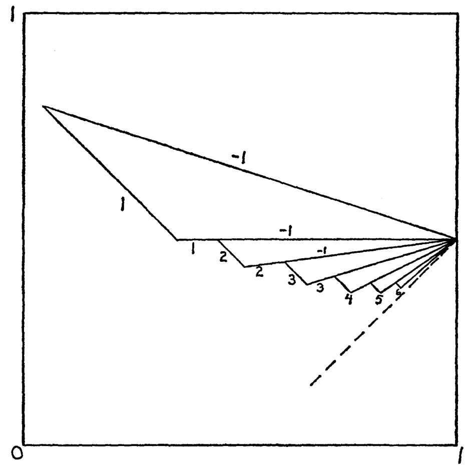

FIGURE 1

For an m-function, f, an ordinary point (as opposed to a tangent point) is a point (x, y) where f is locally single-valued (in [5] it is shown that in a neighborhood of an ordinary point f is a continuous function). Call a point $x \in X$ ordinary if $\{(x, y) \mid y \in f(x)\}$ consists only of ordinary points. One can show that if X is Baire and Y is metric or 2nd countable then the ordinary points form an open dense subset of X. There are examples with X not Baire and $Y = I$ such that X has no ordinary points.

COROLLARY 6.3. *M-homotopy theory each and m-homology theory* (*see* [5]; *we denote this latter theory by* mH_n) *both satisfy the axiom* *of compact supports (see* [9]).

Proof. If $[f] \in m\pi_n(X, A)$, then $f: (I^n, \cdot I^n, 0) \to (X, A)$ and we let $X' = \text{im}(f)$, $A' = \text{im}(f | \cdot I^n)$. Then $[f]^* \in m\pi_n(X', A)$ (where $[\cdot]^*$ represents an equivalence class in $m\pi_n(X', A)$) and the map i_* induced by the injection $i: (X', A') \to (X, A)$ maps $[f]^*$ to $[f]$. In the case where $[f] \in mH_n(X, A)$, $f: \Delta^n \to X$ and $X' = f'(\Delta^n)$ is compact. Let $A' = X' \cap A$. Then $[f]^\wedge \in mH_n(X', A')([\cdot]^\wedge$ represents an equivalence class in $mH_n(X', A'))$, and i_* maps $[f]^\wedge$ to $[f]$.

We conclude (see [4]) from this corollary that if (X, A) is any polyhedral pair, $H_n(X, A) \approx mH_n(X, A) \approx m\pi_n(X, A)$. For example, suppose for some specific pair, (X, A), we know that $z \in H_n(X, A)$ is nonzero. Then $mH_n(X, A)$ is nontrivial, but what is the m-function corresponding to z? This question is answered by describing the two isomorphisms above. For $[z] \in H_n(X, A)$, $z = \sum_{i=1}^n r_i\sigma_i$, a formal sum where each $\sigma_i: \Delta^n \to X$ is a singular simplex. Let $f = \sum_{i=1}^n r_i\sigma_i$ be an m-function sum. Then the map $z \to f$ induces a homomorphism from $H(X, A)$ to $mH(X, A)$ which is the unique isomorphism between them. Similarly, if $[f] \in m\pi_n(X, A)$, $f: (\Delta^n, \cdot\Delta^n, 0) \to (X, A)$. In particular, $f: \Delta^n \to X$ and so f determines an element $[f]^\wedge$ in $mH_n(X, A)$. This map ($f \to f$, but with the second f we ignore the last two elements of the triple) induces a homomorphism from $m\pi(X, A)$ to $mH(X, A)$ which must be the unique isomorphism.

There is a relationship between the results here and the Dold-Thom theorem [2]: $H_m(Y) \cong \pi_m(AG(Y))$ where Y is a pointed polyhedron and $AG(Y)$ is the topological free abelian group on Y. We next define $AG(Y)$ and describe this relationship.

Regarding $Y \vee Y$ as a subset of $Y \times Y$ we use the notation $y = (y, *)$, $-y = (*, y)$, $* = (*, *)$. Starting with an element of $Y' = \sum_{q=1}^\infty \prod_{i=1}^q (Y \vee Y)$ we remove any simultaneous occurrences of y and $-y$, remove all occurrences of $*$, and identify two resulting k-tuples if one is a permutation of the other. (The summation above is free union.) This equivalence relation, R, gives us a quotient map, $\pi: Y' \to Y'/R = AG(Y)$. Addition in $AG(Y)$ is by juxtaposition of representatives of elements followed by π.

Given spaces X and Y there is a natural mapping from m-functions $f: X \to Y$ to (standard) functions $f^*: X \to AG(Y)$. Namely, if $f(x) = \{y_1, \cdots, y_n\}$ then let $f^*(x) = \sum_{i=1}^n \bar{f}(x, y_i)y_i$. Although this correspondence seems to identify a class of "nice" m-functions to a class of "nice" continuous functions from X to $AG(Y)$, we show below that there are degenerate examples on each side.

EXAMPLE 6.4. f may be m-function, but f^* not continuous. The graph in Figure 1 indicates how we might define an m-function f such that $f^*: I \to AG(I)$ fails to be continuous (at 1). It is convenient to take $* = 0$. Thus we can identify $I \vee I$ to $[-1, 1]$ in a natural way. Let $U \subset AG(I)$ consist of those points $\langle w_1, w_2, \cdots, w_n \rangle$ for which $\sum_{i=1}^n w_i < 1/10$ (the addition is in $\boldsymbol{R}$ and is independent of representation of the point). It is easy to verify that $\pi^{-1}(U)$ and hence U is open. But we can define f so that $f^{*-1}(U) \cap [1/2, 1] = 1$. Along any vertical line the distances between adjacent lines, L_1 and L_2, L_2 and L_3, etc., can be taken to be proportional to 1, 1/4, 1/9, 1/16, $\cdots$, $1/n^2$, etc. Given a vertical line, if we choose n large, the sum of the vertical distances from L_n to $L_i (i < n)$ can be made arbitrarily large. Define f using this information.

EXAMPLE 6.5. f^* may be continuous, but f not an m-function. In Figure 2, we indicate an example of a weighted multiple-valued function f for which we can define f^* as above. Then, although f fails to be an m-function ($f(0)$ is infinite) f^* is continuous. The only point where the continuity of f^* is questionable is at 0. There $f^*(0) = *$. If U is a neighborhood of $*$ in $AG(I)$, then $\pi^{-1}(*) \subset \pi^{-1}(U)$ and $\pi^{-1}(*) \cap ((I \vee I) \times (I \vee I)) \subset \pi^{-1}(U) \cap ((I \vee I) \times (I \vee I))$ which is open in $(I \vee I) \times (I \vee I)$. Since I is compact there exists $\varepsilon > 0$ such that if $|x - y| < \varepsilon$ then $\langle x, -y \rangle \in U$ (the brackets represent unordered pair, so $\langle x, -y \rangle = x + (-y)$ where addition is in $AG(I)$ not $\boldsymbol{R}$). So, from the figure we see that there exists $\delta > 0$ such that if $x < \delta$, $f^*(x) \in U$.

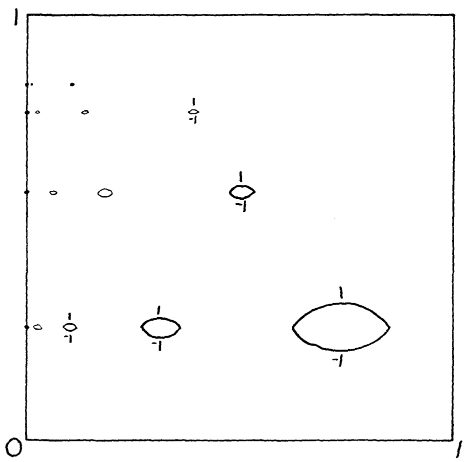

FIGURE 2

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Received August 1, 1978.

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