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ON THE REDUCTION OF CERTAIN DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF SP (n, C)

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This paper has its origins in the problem of proving irreducibility or reducibility for principal series representations of certain noncompact, complex, semi-simple groups by Fourier-analytic methods; for example, the abelian methods of Gelfand-Naimark for Sl (n, C), and the non commutative (nilpotent) methods of K. Gross for Sp (n, C). As is well-known, principal series representations are induced from unitary characters of a parabolic subgroup, the series being termed "nondegenerate" if the parabolic is minimal (i.e., the Borel subgroup) and otherwise "degenerate". Here we consider degenerate principal series for Sp (n, C) corresponding to maximal parabolic subgroups (more general than the situation studied by Gross) and reduce them with respect to the "opposite" parabolic. Let n_1 denote the complex dimension of the isotropic subspace corresponding to the maximal parabolic, let $0 < n_1 < n$, and $n_0 = n - n_1$. The resulting reduction is described in terms of the natural representation of the complex orthogonal group $O(n_1, C)$ acting on the space $L^2(C^{n_1 \times n_0})$ and the tensor product of n_1 copies of the oscillator representation of Sp (n_0, C). In the terminology introduced by R. Howe, this harmonic analysis reduces to the theory of a "dual reductive pair", and any further resolution of the question of irreducibility by these methods will depend upon the study of the oscillator representations for such a dual reductive pair.

We now describe our work in more detail. As a presentation of the complex symplectic group, take

$$\sum_n = \{g \in C^{2n \times 2n}: gM_n g' = M_n\} ,$$

where $M_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$, I_n is the $n \times n$ identity matrix, and g' denotes the transpose of g. Specify a complete set of conjugacy class representatives of the maximal parabolic subgroups H in Σ_n (c.f., [9], §8) by defining $H = Z'SA$, where the subgroups Z, S, and A are given below. Let the isotropic subspace of C^{2n} corresponding to H have dimension n_1, with $0 < n_1 \leqq n$ and $n_0 = n - n_1$. Then the blocking scheme used in defining Z, S, and A has diagonal blocks of dimensions $n_1 \times n_1$, $n_0 \times n_0$, $n_1 \times n_1$, and $n_0 \times n_0$ from upper left to lower right.

$$Z = \left\{ \begin{Vmatrix} I & 0 & 0 & 0 \\ -y' & I & 0 & 0 \\ \eta & x & I & y \\ x' & 0 & 0 & I \end{Vmatrix} : \eta - \eta' = yx' - xy' \right\}$$

$$S = \left\{ \begin{Vmatrix} I & 0 & 0 & 0 \\ 0 & s_{11} & 0 & s_{12} \\ 0 & 0 & I & 0 \\ 0 & s_{21} & 0 & s_{22} \end{Vmatrix} : s = \begin{Vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{Vmatrix} \in \Sigma_{n_0} \right\}$$

$$A = \left\{ \begin{Vmatrix} \alpha & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha'^{-1} & 0 \\ 0 & 0 & 0 & I \end{Vmatrix} : \alpha \in \mathrm{Gl}(n_1, C) \right\}$$

Clearly, S is isomorphic to $\mathrm{Sp}(n_0, C)$, A is isomorphic to $\mathrm{Gl}(n_1, C)$ and elements of S commute with elements of A. Also, it is easily shown that Z and Z' are normalized by SA and hence, ZSA and $Z'SA$ are semidirect products.

The maximal parabolic subgroup $H = Z'SA$ gives rise to a degenerate principal series of representations T_χ of Σ_n induced from unitary characters χ on H. We shall realize T_χ in the Hilbert space $L^2(Z)$ as follows: Let dz denote Haar measure on the unimodular group Z. Denote by $d_l h$ and $d_r h$, respectively, fixed left and right Haar measures on H, and let δ_H be the modular function defined by $\delta_H(h) = d_l h/d_r h$. By direct calculation (cf., [3], §6), HZ is an open subset of Σ_n whose complement is a set of Haar measure zero. Thus, we can extend the positive character δ_H and any unitary character χ on H to functions defined almost everywhere on Σ_n by defining $\delta_H(hz) = \delta_H(h)$ and $\chi(hz) = \chi(h)$ for any $hz \in HZ$. Also, each right coset of H in Σ_n, except for a set of cosets whose union is a null set, contains a unique element of Z. It follows that the canonical action of Σ_n on the right coset space $H\backslash\Sigma_n$ gives rise to an "action" of Σ_n on Z: for any $g \in \Sigma_n$ and $z \in Z$, let $z\bar{g}$ be the unique element of Z such that $H(z\bar{g}) = Hzg$, provided that such an element exists. To be specific, denote by Z^g the subset of Z such that $z\bar{g}$ exists, then Z^g is an open subset of Z whose complement is a null set. Therefore, if $f \in L^2(Z)$ then the function $z \to f(z\bar{g})$, for fixed $g \in \Sigma_n$, is defined almost everywhere in Z. Now, the formula ([2], §30) defining the (continuous) unitary representations T_χ of Σ_n, which form a (degenerate) principal series is

$$T_\chi(g)f(z) = \delta_H(zg)^{-1/2}\chi(zg)f(z\bar{g}) \quad (g \in \Sigma_n,\ f \in L^2(Z))\ . \tag{1.1}$$

Let us briefly explain the Fourier-analytic reduction of the restriction of T_χ to the (opposite) parabolic subgroup ZSA. Fix n_1 with $1 \leqq n_1 < n$, fix χ, and let $T = T_\chi$. We observe that the restriction $T|_Z$ of T to Z is just the right regular representation of Z. Thus, it is natural to replace T with the unitarily equivalent representation $\hat{T} = \mathscr{P} T \mathscr{P}^{-1}$ where $\mathscr{P}$ is the Plancherel transform of $L^2(Z)$, for $\hat{T}|_Z$ decomposes as a direct integral. The operator $\mathscr{P}$ maps $L^2(Z)$ unitarily onto the Hibert space $L^2(\Lambda, X, dm(\lambda))$, of X-valued, square-integrable functions on Λ. Here Λ is the dual object of Z, $dm(\lambda)$ is the Plancherel measure on Λ, and $X = HS(L^2(C^{n_1 \times n_0}))$ is the Hilbert space of Hilbert-Schmidt operators on $L^2(C^{n_1 \times n_0})$. It is also the case that $\hat{T}|_S$ decomposes as a direct integral, and one can explicitly analyze $\hat{T}(zs\alpha)$ for all $zs\alpha \in ZSA$. The operators of $\hat{T}|_{ZSA}$ involve a representation $\tilde{I}: S \rightarrow \mathscr{U}(L^2(C^{n_1 \times n_0}))$ which is the tensor product of n_1 copies of the oscillator representation of $\mathrm{Sp}(n_0, C)$, as well as a representation $D: A \rightarrow \mathscr{U}(L^2(C^{n_1 \times n_0}))$, in which $\mathrm{Gl}(n_1, C)$ acts on $L^2(C^{n_1 \times n_0})$ by generalized dilations. The above results are contained in §2 of this paper.

Let $\mathscr{A}'(\hat{T}|_{ZSA})$ denote the commuting algebra of $\hat{T}|_{ZSA}$. There are sufficiently many operators of $\hat{T}|_{ZSA}$ which are diagonalizable to force $\mathscr{A}'(\hat{T}|_{ZSA})$ to be decomposable. Moreover, the components of $\mathscr{A}'(\hat{T}|_{ZSA})$ are essentially copies of the intersection, $\mathscr{A}'(\tilde{I}) \cap \mathscr{A}'(D|_{A_1})$ of the commuting algebras of $\tilde{I}$ and $D|_{A_1}$ where $A_1 \cong O(n_1, C)$. That is, there is an isometric isomorphism of von Neumann algebras, which we exhibit, between $\mathscr{A}'(\hat{T}|_{ZSA})$ and $\mathscr{A}'(\tilde{I}) \cap \mathscr{A}'(D|_{A_1})$. This is the content of §3.

It should be noted that there are two special cases in which complete results are known. The case $n_1 = n$ is special since Z is abelian. It is not difficult to show that $\mathscr{A}'(\hat{T}|_{ZSA})$ is one-dimensional and, hence, for all χ, T_χ is already irreducible upon restriction to ZSA. Also, the irreducibility problem has been completely solved in the case $n_1 = 1$, $n_0 = n - 1$, in [3], which may be regarded as the prototype for the general case. There it is proved that T_χ is irreducible unless χ is the trivial character on H, in which case, T_χ splits into the sum of two irreducible representations of Σ_n. The complete results of [3] rest on the fact that the commuting algebra $\mathscr{A}'(\tilde{I}) \cap \mathscr{A}'(D|_{A_1})$ is just 2-dimensional when $n_1 = 1$. In the general case, this algebra is infinite dimensional and the full analysis of $\hat{T}$ on all of Σ_n depends upon its explicit description.[1]

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[1] R. Howe's results show that the joint representation of $\mathrm{Sp}(n_0, C) \times O(n_1, C)$ decomposes continuously. It follows that the commuting algebra is infinite dimensional.

2. The operators $\hat{T}(zs\alpha)$. In order to analyze $\hat{T}|_{ZSA}$ we need to introduce the dual object of Z, the resulting Plancherel transform of $L^2(Z)$ and the oscillator representation of $\mathrm{Sp}(n_0, C)$.

Procedures of Kirillov [5] can be applied to the simply connected, nilpotent lie group Z to yield the dual object — the set of equivalence classes of irreducible, unitary representations of Z. The results are given below.

Denote the elements of Z by (x, y, t), where $t = \eta - yx'$ so that t is symmetric. In this way Z is identified with $V \times V \times \Lambda_0$, where $V = C^{n_1 \times n_0}$ and $\Lambda_0 = \{t \in C^{n_1 \times n_1} : t = t'\}$. Multiplication in Z is now given by

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 - x_1 y_2' - y_2 x_1') .$$

Also, the center of Z is easily seen to be $\{0\} \times \{0\} \times \Lambda_0$ and the Haar measure of the unimodular group Z is real Lebesgue measure $dz = dx\, dy\, dt$ on the Euclidean space $V \times V \times \Lambda_0$.

For $\lambda \in \Lambda_0$ with rank $\lambda = r$, let $C(\lambda) \in O(n_1, C)$ be such that $C(\lambda)\lambda C(\lambda)' = \begin{vmatrix} \lambda_r & 0 \\ 0 & 0 \end{vmatrix}$, where λ_r is a symmetric, invertible $r \times r$ matrix. Also, let $X = \begin{vmatrix} I_r & 0 \\ 0 & 0 \end{vmatrix} C^{n_1 \times n_0}$, thought of as a measure space with real Lebesgue measure. Finally, throughout this paper we shall let $(u|v) = \mathrm{Re\,tr}\, uv'$ for all $u, v \in C^{p \times q}$.

THEOREM 2.1. *Every irreducible, unitary representation of* Z *is unitarily equivalent to* $\Pi_{(\alpha,\beta,\lambda)}$ *for some choice of* $\alpha, \beta \in V$ *and* $\lambda \in \Lambda_0$, *where* $\Pi_{(\alpha,\beta,\lambda)}$ *is defined as follows*:

(1) *If* $\lambda = 0$ *then* $\Pi_{(\alpha,\beta,\lambda)}$ *is* 1-*dimensional and is given by*

$$\Pi_{(\alpha,\beta,0)}(x, y, t) = \exp 2\pi i[(\alpha|x) + (\beta|y)]$$

for $(x, y, t) \in Z$.

(2) *If rank* $\lambda = r \neq 0$, *then* $\Pi_{(\alpha,\beta,\lambda)}: Z \to \mathscr{U}(L^2(X))$ *is* ∞-*dimensional and is given by*

$$\Pi_{(\alpha,\beta,\lambda)}(x, y, t)f(u) = \exp 2\pi i\left[\left(\alpha \,\middle|\, C(\lambda)' \begin{vmatrix} 0_r & 0 \\ 0 & I \end{vmatrix} C(\lambda)x\right) + (\beta|y) + (\lambda|t - 2C(\lambda)'uy')\right] f\left(u + \begin{vmatrix} I_r & 0 \\ 0 & 0 \end{vmatrix} C(\lambda)x\right)$$

for $f \in L^2(X)$ *and* $(x, y, t) \in Z$. *Moreover* $\Pi_{(\alpha_1,\beta_1,\lambda_1)}$ *and* $\Pi_{(\alpha_2,\beta_2,\lambda_2)}$ *are unitarily equivalent if and only if* $\lambda_1 = \lambda_2$, $\alpha_1 = \alpha_2 + \lambda_1 a$, *and* $\beta_1 = \beta_2 + \lambda_1 b$ *for some* $a, b \in V$.

The Plancherel transform of $L^2(Z)$ does not require the entire dual object of Z, but only the representations corresponding to "maximal orbits". These are the representation $\Pi_{(0,0,\lambda)}$, where λ is invertible (i.e., $r = n_1$). Thus, let $\Lambda = \{\lambda \in \Lambda_0 : \operatorname{rank} \lambda = n_1\}$ and for $\lambda \in \Lambda$ denote $\Pi_{(0,0,\lambda)}$ by $\hat{\lambda}$, then $\hat{\lambda}$ acts in the Hilbert space $L^2(V)$ by the formula

$$\hat{\lambda}(x, y, t)f(u) = \exp[2\pi i(\lambda | t - 2uy')]f(u + x) \tag{2.2}$$

for $f \in L^2(V)$ and $z = (x, y, t) \in Z$.

Let $\mathscr{S} = \mathscr{S}(V) \times \mathscr{S}(V) \times \mathscr{S}(\Lambda_0)$, where $\mathscr{S}(V)$ (respectively $\mathscr{S}(\Lambda_0)$) is the vector space of all infinitely differentiable, rapidly decreasing functions with domain V (respectively Λ_0). Then $\mathscr{S}$ is a dense subspace of $L^2(Z)$ with which we can state and prove the following results concerning the Plancherel transform $\mathscr{P}$ of $L^2(Z)$.

THEOREM 2.3. (1) *The Plancherel measure m on Λ is given by*

$$dm(\lambda) = 2^{2n_0 n_1 + n_1(n_1 - 1)} \cdot |\det \lambda|^{2n_0} d\lambda$$

where $d\lambda$ is the restriction to Λ of the Lebesgue measure on Λ_0. (2) *The mapping $f \to K_f$ defined for $f \in \mathscr{S}$* by

$$K_f(x, y, \lambda) = \int_V \int_{\Lambda_0} f(x - y, v, t) \exp[2\pi i(\lambda | t - 2vy')]dtdv$$

extends uniquely to a linear isometry of $L^2(Z)$ onto $L^2(V \times V \times \Lambda_0, dxdydm(\lambda))$. This is the function-valued Plancherel transform. (3) *The mapping $f \to \hat{f}$ defined weakly for $f \in \mathscr{S}$ by*

$$\hat{f}(\lambda) = \int_Z f(z)\hat{\lambda}(z)dz$$

extends uniquely to an isometry $\mathscr{P}$ of $L^2(Z)$ onto $L^2(\Lambda, HS(L^2(V)), dm(\lambda))$. This is the Plancherel transform of $L^2(Z)$.

Proof. A computation shows that $\hat{f}(\lambda)$ is an integral operator with kernel K_f. The mapping $f \to K_f$ is decomposed as in [3] (1.8) into ordinary, partial Fourier transforms

$$\mathscr{F}_2 f(x, y, \lambda) = \int_V f(x, v, \lambda) \exp[-2\pi i(y | v)]dv$$

$$\mathscr{F}_3 f(x, y, \lambda) = 2^{-n_1(n_1-1)/2} \int_{\Lambda_0} f(x, y, t) \exp[-2\pi i(\lambda | t)]dt$$

($f \in \mathscr{S}$, the factor $c = 2^{-n_1(n_1-1)/2}$ makes $\mathscr{F}_3$ an isometry) and a transformation $\mathscr{R}: L^2(V \times V \times \Lambda_0, dxdyd\lambda) \to L^2(V \times V \times \Lambda_0, dxdydm(\lambda))$ given by

$$\mathscr{R}f(x, y, \lambda) = cf(x - y, 2\lambda y, \lambda) .$$

In fact, $K_f = \mathscr{R}\mathscr{F}_2\mathscr{F}_3^{-1}f$ for all $f \in \mathscr{S}$. Now $dm(\lambda)$ is chosen so that $\mathscr{R}$ is an isometry and hence $f \to K_f$ is an isometry.

To prove (3) and (2), one shows that the mapping $K_f \to \hat{f}$, defined for $f \in \mathscr{S}$, extends to a linear isometry of $L^2(V \times V \times \Lambda_0, dxdydm(\lambda))$ onto $L^2(\Lambda, HS(L^2(V)), dm(\lambda))$.

The tensor product of n_1 copies of the oscillator representation of S occurs naturally in the present setting. Note that S normalizes Z. Specifically, for $s \in S$ and $z = (x, y, t) \in Z$

$$szs^{-1} = (xs'_{22} - ys'_{21}, ys'_{11} - xs'_{12}, t + xs'_{22}s_{12}x' + ys'_{21}s_{11}y' \\ - xs'_{12}s_{21}y' - ys'_{21}s_{12}x') .$$

Let $\lambda \in \Lambda$ and fix $s \in S$. The mapping $z \to \hat{\lambda}(szs^{-1})$ is an irreducible, unitary representation of Z acting in $L^2(V)$ which agrees with $\hat{\lambda}$ on the center of Z. Thus, these two irreducible representations are unitarily equivalent, and so, there is a unitary operator $\tilde{\lambda}(s)$ on $L^2(V)$ such that

$$\hat{\lambda}(szs^{-1}) = \tilde{\lambda}(s)\hat{\lambda}(z)\tilde{\lambda}(s)^{-1} \quad (z \in Z) .$$

For each s, $\tilde{\lambda}(s)$ is unique up to scalar multiples of absolute value 1, and, as we will show, $\tilde{\lambda}(s)$ can be normalized so that $s \to \tilde{\lambda}(s)$ is a unitary representation of S acting in $L^2(V)$. Let us now be more explicit.

Identify S with Σ_{n_0} and define the following subgroups of Σ_{n_0} using the blocking scheme with two diagonal blocks of size $n_0 \times n_0$:

$$M = \left\{ m(b) = \begin{vmatrix} I & b \\ 0 & I \end{vmatrix} : b = b' \right\}$$

$$L = \left\{ l(a) = \begin{vmatrix} a & 0 \\ 0 & a^{\vee} \end{vmatrix} : a \in \mathrm{Gl}(n_0, C),\ a^{\vee} = (a')^{-1} \right\} .$$

Also, let $p = \begin{vmatrix} 0 & -I \\ I & 0 \end{vmatrix}$. The set $L \cup M \cup \{p\}$ generates Σ_{n_0} ([3], p. 404), so to define $\tilde{\lambda}$ on S it is enough to define it on this generating set.

DEFINITION 2.4. Given $\lambda \in \Lambda$, define $\tilde{\lambda}: L \cup M \cup \{p\} \to \mathscr{U}(L^2(V))$ by

$$\tilde{\lambda}(l(a))f(u) = |\det a|^{n_1} f(ua) \quad (l(a) \in L)$$
$$\tilde{\lambda}(m(b))f(u) = \exp[-2\pi i(\lambda u \mid ub)]f(u) \quad (m(b) \in M)$$
$$\tilde{\lambda}(p)f(u) = \gamma(\lambda)|\det 2\lambda|^{n_0} Uf(2\lambda u)$$

where U is the Fourier transform of $L^2(V)$ defined for $f \in L^1(V)$ by

$$Uf(u) = \int_V f(v) \exp [2\pi i(u|v)]dv$$

and $\gamma(\lambda)$ is a complex number with modulus 1, which will be determined in the proof of Theorem 2.5.

THEOREM 2.5. *The mapping* $\tilde{\lambda}$, *defined on* $L \cup M \cup \{p\}$ *above, extends uniquely to be a continuous unitary representation of* Σ_{n_0} *(and hence* S*) acting in* $L^2(V)$ *which satisfies*

$$\hat{\lambda}(szs^{-1}) = \tilde{\lambda}(s)\hat{\lambda}(z)\tilde{\lambda}(s)^{-1}$$

for all $s \in S$ *and* $z \in Z$.

Proof. It is easy to verify that the restrictions of $\tilde{\lambda}$ to L, M, and LM are continuous unitary representations of these groups. Now apply Lemma 1 of [3]. To prove that condition (2) of Lemma 1 is satisfied, we use the following: Observe that $m(I)pm(I) = pm(-I)p$. Let $m = m(I)$, then $m^{-1} = m(-I)$. From the definitions of the operators $\tilde{\lambda}(m)$ and $\tilde{\lambda}(p)$, a computation shows that

$$\begin{aligned}
&\tilde{\lambda}(m)\tilde{\lambda}(p)\tilde{\lambda}(m)\hat{\lambda}(z)\tilde{\lambda}(m)^{-1}\tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)^{-1} \\
&\quad = \hat{\lambda}(mpmzm^{-1}p^{-1}m^{-1}) \\
&\quad = \hat{\lambda}(pm^{-1}pzp^{-1}mp^{-1}) \\
&\quad = \tilde{\lambda}(p)\tilde{\lambda}(m)^{-1}\tilde{\lambda}(p)\hat{\lambda}(z)\tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)\tilde{\lambda}(p)^{-1}
\end{aligned}$$

and hence $Y_1(\lambda) = \tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)\tilde{\lambda}(p)^{-1}\tilde{\lambda}(m)\tilde{\lambda}(p)\tilde{\lambda}(m) \in \mathscr{A}'(\hat{\lambda})$, the commuting algebra of $\hat{\lambda}$. This is true regardless of the value of $\gamma(\lambda)$ with $|\gamma(\lambda)| = 1$. Thus, letting

$$Y_2(\lambda) = [\gamma(\lambda)^{-1}\tilde{\lambda}(p)]^{-1}\tilde{\lambda}(m)[\gamma(\lambda)^{-1}\tilde{\lambda}(p)]^{-1}\tilde{\lambda}(m)[\gamma(\lambda)^{-1}\tilde{\lambda}(p)]\tilde{\lambda}(m) ,$$

we have $Y_2(\lambda) = \gamma(\lambda)Y_1(\lambda) \in \mathscr{A}'(\hat{\lambda})$. But $\hat{\lambda}$ is irreducible so $\mathscr{A}'(\hat{\lambda})$ is 1-dimensional and hence $Y_2(\lambda) = c(\lambda)I$ for some unique $c(\lambda) \in C$ with $|c(\lambda)| = 1$. *Define* $\gamma(\lambda) = c(\lambda)$ then $Y_1(\lambda) = I$ and it follows that

$$\tilde{\lambda}(m)\tilde{\lambda}(p)\tilde{\lambda}(m) = \tilde{\lambda}(p)\tilde{\lambda}(m^{-1})\tilde{\lambda}(p)$$

which is condition (2) of Lemma 1 of [3].

Just as the representation $\tilde{\lambda}$ of S arises from interwining operators $\tilde{\lambda}(s)$ between $\hat{\lambda}$ and $z \to \hat{\lambda}(szs^{-1})$, a representation D of A arises from intertwining operators $D(\alpha)$ between $(\alpha\lambda\alpha')^\wedge$ and $z \to \hat{\lambda}(\alpha^{-1}z\alpha)$. For $\alpha \in A$ and $z = (x, y, t) \in Z$, we have

$$\alpha^{-1}z\alpha = (\alpha'x, \alpha'y, \alpha't\alpha) ,$$

and from formula (2.2), the representations $(\alpha\lambda\alpha')^\wedge$ and $z \to \hat{\lambda}(\alpha^{-1}z\alpha)$

are easily seen to agree on the center of Z. Hence, these two irreducible representations of Z are unitarily equivalent and, in fact, the operator $D(\alpha)$ given by

$$D(\alpha)f(u) = |\det \alpha|^{n_0} f(\alpha' u) \quad (f \in L^2(V))$$

intertwines them. Also, the fact that

$$\begin{aligned}\tilde{\lambda}(s)D(\alpha)^{-1}(\alpha\lambda\alpha')^{\wedge}(z)D(\alpha)\tilde{\lambda}(s)^{-1} &= \hat{\lambda}(s\alpha^{-1}z\alpha s^{-1}) = \hat{\lambda}(\alpha^{-1}szs^{-1}\alpha)\\ &= D(\alpha)^{-1}(\alpha\lambda\alpha')^{\sim}(s)(\alpha\lambda\alpha')^{\wedge}(z)(\alpha\lambda\alpha')^{\sim}(s)^{-1}D(\alpha)\end{aligned}$$

suggests that $D(\alpha)$ may also intertwine $\tilde{\lambda}$ and $(\alpha\lambda\alpha')^{\sim}$ and, indeed, this is the case. We summarize these facts in the next theorem.

THEOREM 2.6. *The mapping $D: A \to \mathscr{U}(L^2(V))$ is a continuous unitary representation of A which satisfies*
(1) $D(\alpha)^{-1}(\alpha\lambda\alpha')^{\wedge}(z)D(\alpha) = \hat{\lambda}(\alpha^{-1}z\alpha)$
(2) $D(\alpha)\tilde{\lambda}(s)D(\alpha)^{-1} = (\alpha\lambda\alpha')^{\sim}(s)$
for all $\lambda \in \Lambda$, $\alpha \in A$, $s \in S$, and $z \in Z$.

We omit the proof of the above theorem since it is fairly straightforward (cf., [3], Theorem 2), however, we make the following observation related to the proof. For each invertible symmetric matrix λ there exists $\beta \in \mathrm{Gl}(n_1, C)$ such that $\lambda = \beta\beta'$. Consequently, the action of A on Λ defined by $\alpha \cdot \lambda = \alpha\lambda\alpha'$ is transitive, and from this follow two important facts: First, the function γ defined on Λ in the proof of Theorem 2.5 is *constant*. Secondly, we have the

COROLLARY. *Let $\lambda \in \Lambda$ and let $I \in \Lambda$ denote the $n_1 \times n_1$ identity matrix, then $\tilde{\lambda}$ is unitarily equivalent to $\tilde{I}$.*

View $L^2(V)$ as the tensor product $\bigotimes^{n_1} L^2(C^{n_0})$ by defining

$$f_1 \otimes f_2 \otimes \cdots \otimes f_{n_1}(u) = f_1(u_1)f_2(u_2)\cdots f_{n_1}(u_{n_1}),$$

where u_i is the ith row of u and $f_i \in L^2(C^{n_0})$. By inspection of $\tilde{I}$ and comparison with Theorem 2 of [3], one sees that $\tilde{I}$ is a tensor product of n_1 copies of the oscillator representation $\tilde{I}$ of [3].

We may now compute the operators $\hat{T}(z_0 s\alpha)$ for $z_0 s\alpha \in ZSA$. Recall that the formula (1.1) for $T(g)$ involves the action of g on Z given by $H(z\bar{g}) = Hz\bar{g}$. If $g = z_0 s\alpha$ then the action becomes $z\overline{z_0 s\alpha} = \alpha^{-1}s^{-1}zz_0 s\alpha$. Also, $\delta_H(zg) = \delta_H(zz_0 s\alpha) = \delta_H((s\alpha)(\alpha^{-1}s^{-1}zz_0 s\alpha)) = \delta_H(s\alpha)$, since $\delta_H(hz)$ is defined to be $\delta_H(h)$ for any $hz \in HZ$. Furthermore, $\delta_H(s\alpha) = \delta_H(\alpha)$ because S has no nontrivial characters. Similarly, $\chi(zz_0 s\alpha) = \chi(\alpha)$ for any unitary character χ on H. Thus, (1.1) becomes

$$T(z_0 s\alpha)f(z) = \delta_H(\alpha)^{-1/2}\chi(\alpha)f(\alpha^{-1}s^{-1}zz_0 s\alpha) \tag{2.7}$$

for $f \in L^2(Z)$. Now let $f \in L^1(Z) \cap L^2(Z)$ and $\hat{f} = \mathscr{P}f$, then $\hat{f} \in \mathscr{H} = L^2(\Lambda, HS(L^2(V)), dm(\lambda))$ and $\mathscr{P}(L^1(Z) \cap L^2(Z))$ forms a dense subspace of $\mathscr{H}$. The next theorem determines the transformed representation $\hat{T}|_{ZSA}$.

THEOREM 2.8. *For $f \in L^1(Z) \cap L^2(Z)$ and $zs\alpha \in ZSA$,*

$$\hat{T}(zs\alpha)\hat{f}(\lambda) = \delta_H(\alpha)^{1/2}\chi(\alpha)\tilde{\lambda}(s)D(\alpha)\hat{f}(\alpha^{-1}\cdot\lambda)D(\alpha)^{-1}\tilde{\lambda}(s)^{-1}\hat{\lambda}(z)^{-1}$$

for almost every $\lambda \in \Lambda$.

Proof. For every $\lambda \in \Lambda$, $\hat{T}(z_0 s\alpha)\hat{f}(\lambda) = \mathscr{P}T(z_0 s\alpha)\mathscr{P}^{-1}\mathscr{P}f(\lambda) = \mathscr{P}T(z_0 s\alpha)f(\lambda) = \int_Z T(z_0 s\alpha)f(z)\hat{\lambda}(z)dz = \int_Z \delta_H(\alpha)^{-1/2}\chi(\alpha)f(\alpha^{-1}s^{-1}zz_0 s\alpha)\hat{\lambda}(z)dz$

$$
\begin{aligned}
(1)\quad &= \delta_H(\alpha)^{-1/2}\chi(\alpha)\int_Z f(z)\hat{\lambda}(s\alpha z\alpha^{-1}s^{-1}z_0^{-1})d(s\alpha z\alpha^{-1}s^{-1}z_0^{-1})\\
(2)\quad &= \delta_H(\alpha)^{-1/2}\chi(\alpha)\int_Z f(z)\hat{\lambda}(s\alpha z\alpha^{-1}s^{-1}z_0^{-1})\delta_H(\alpha)dz\\
(3)\quad &= \delta_H(\alpha)^{1/2}\chi(\alpha)\left[\int_Z f(z)\tilde{\lambda}(s)\hat{\lambda}(\alpha z\alpha^{-1})\tilde{\lambda}(s)^{-1}dz\right]\hat{\lambda}(z_0)^{-1}\\
(4)\quad &= \delta_H(\alpha)^{1/2}\chi(\alpha)\tilde{\lambda}(s)\left[\int_Z f(z)D(\alpha)(\alpha^{-1}\cdot\lambda)\hat{\ }(z)D(\alpha)^{-1}dz\right]\tilde{\lambda}(s)^{-1}\hat{\lambda}(z_0)^{-1}\\
&= \delta_H(\alpha)^{1/2}\chi(\alpha)\tilde{\lambda}(s)D(\alpha)\left[\int_Z f(z)(\alpha^{-1}\cdot\lambda)\hat{\ }(z)dz\right]D(\alpha)^{-1}\tilde{\lambda}(s)^{-1}\hat{\lambda}(z_0)^{-1},
\end{aligned}
$$

which gives the theorem. Equation (1) is a change of variables, (2) is the fact $d(s\alpha z\alpha^{-1}s^{-1}) = \delta_H(\alpha)dz$ (cf., [6], II. 7), (3) is an application of Theorem 2.5, and (4) is from Theorem 2.6 (1). The formula in the Theorem is said to hold "almost everywhere" since there may be a null set in Λ where the right-hand-side is not in $HS(L^2(V))$.

3. The commuting algebra of $\hat{T}|_{ZSA}$. We seek necessary and sufficient conditions for $B \in \mathscr{L}(\mathscr{H})$, the bounded linear operators on the Hilbert space $\mathscr{H} = L^2(\Lambda, HS(L^2(V)), dm(\lambda))$, to be in the commuting algebra $\mathscr{A}'(\hat{T}|_{ZSA})$. Suppose $B \in \mathscr{A}'(\hat{T}|_{ZSA})$. Then, in particular, B commutes with $\hat{T}(z)$ and $\hat{T}(s)$ for all $z \in Z$ and $s \in S$. We will first see what conditions on B these facts impose. Then we will obtain additional conditions from the fact that B commutes with $\hat{T}(\alpha)$ for $\alpha \in A$.

Realize $HS(L^2(V))$ as $L^2(V) \mathbin{\overline{\otimes}} L^2(V)$. From the observation that $\mathscr{H} = L^2(\Lambda, L^2(V) \mathbin{\overline{\otimes}} L^2(V), dm(\lambda)) = \int_\Lambda L^2(V) \mathbin{\overline{\otimes}} L^2(V)dm(\lambda)$ is a direct integral of Hilbert spaces, we have the notions of decomposable and diagonalizable operators ([7], I. 3). From Theorem 2.8, it is clear that $\hat{T}(z)$ and $\hat{T}(s)$ are decomposable operators for $z \in Z$ and $s \in S$ and can be denoted:

$$\hat{T}(z) = \int_{\Lambda} I \,\overline{\otimes}\, \hat{\lambda}(z) dm(\lambda)$$

$$\hat{T}(s) = \int_{\Lambda} \tilde{\lambda}(s) \,\overline{\otimes}\, \tilde{\lambda}(s) dm(\lambda) .$$

To show from this that B is decomposable requires the following technical lemma of Stone-Weierstrass type.

Let X be a locally compact, σ-compact, Hausdorff space with positive Borel measure μ, which is finite on compact sets. Let H be a separable Hilbert space. For $a \in L^\infty(X, \mu)$, let $M(a)$ denote the diagonalizable operator on $L^2(X, H, \mu)$ given by $M(a)f(x) = a(x)f(x)$.

LEMMA 3.1. *Let $\mathscr{A}$ be a subalgebra of $C(X) \cap L^\infty(X, \mu)$ over C such that*

(a) $1 \in \mathscr{A}$,

(b) *$a \in \mathscr{A}$ implies $\bar{a} \in \mathscr{A}$,*

(c) *$\mathscr{A}$ separates points of X.*

If $B \in \mathscr{L}(L^2(X, H, \mu))$ and $BM(a) = M(a)B$ for all $a \in \mathscr{A}$, then $BM(a) = M(a)B$ for all $a \in L^\infty(X, \mu)$.

THEOREM 3.2. *Suppose $B \in \mathscr{A}'(\hat{T}|_{ZSA})$. Then*

(a) *B is decomposable.*

(b) *There exists a mapping $\lambda \to B(\lambda)$ of Λ into $\mathscr{L}(L^2(V))$, defined* a.e. $[m]$, *such that $B = \int_{\Lambda} B(\lambda) \,\overline{\otimes}\, I dm(\lambda)$.*

(c) *$B(\lambda) \in \mathscr{A}'(\tilde{\lambda})$ for almost every $\lambda \in \Lambda$.*

Proof. (a) Let $\mathscr{A} = \{\lambda \to \exp[2\pi i(\lambda | t)]: t \in \Lambda_0\}$. $\mathscr{A}$ is a subalgebra over C of $C(\Lambda) \cap L^\infty(\Lambda, m)$, which satisfies the conditions of Lemma 3.1. Furthermore, if $a \in \mathscr{A}$ then the associated diagonalizable operator $M(a) \in \mathscr{L}(\mathscr{H})$, given by $M(a)\hat{f}(\lambda) = a(\lambda)\hat{f}(\lambda)$, is an operator of the representation $\hat{T}|_Z$. In fact, if $a(\lambda) = \exp[2\pi i(\lambda | t)]$ then $M(a) = \hat{T}(z)$, where $z = (0, 0, t)$. Therefore $BM(a) = M(a)B$ for every $a \in \mathscr{A}$ and the lemma implies that this holds for every $a \in L^\infty(\Lambda, m)$. Since $\{M(a): a \in L^\infty(\Lambda, m)\}$ is exactly the set of diagonalizable operators on $\mathscr{H}$, B must be decomposable ([7], I. 3.2).

(b) Since B is decomposable, there exists an essentially bounded mapping $\lambda \to B_\lambda$, defined a.e. $[m]$ with values in $\mathscr{L}(L^2(V) \,\overline{\otimes}\, L^2(V))$, such that $B = \int_{\Lambda} B_\lambda dm(\lambda)$.

Since B commutes with $\hat{T}(z) = \int_{\Lambda} I \,\overline{\otimes}\, \hat{\lambda}(z) dm(\lambda)$ for every $z \in Z$, B_λ commutes with $I \,\overline{\otimes}\, \hat{\lambda}(z)$ except for λ in an m-null set N_z. Z is separable; let $\{z_i: i \in \mathscr{I}\}$ be a countable dense subset of Z and let $N = \bigcup_{i \in \mathscr{I}} N_{z_i}$. Then $m(N) = 0$ and for $\lambda \in N^c$, B_λ commutes with $I \,\overline{\otimes}\, \hat{\lambda}(z_i)$ for all $i \in \mathscr{I}$. Thus we have two continuous maps, $z \to$

$B_\lambda(I \mathbin{\overline{\otimes}} \hat{\lambda}(z))$ and $z \to (I \mathbin{\overline{\otimes}} \hat{\lambda}(z))B_\lambda$, which agree on a dense subset of Z. It follows that $B_\lambda \in \mathscr{A}'(I \mathbin{\overline{\otimes}} \hat{\lambda})$ for all $\lambda \in N^c$. Since $\hat{\lambda}$ is irreducible, for each $\lambda \in N^c$ there exists $B(\lambda) \in \mathscr{L}(L^2(V))$ such that $B_\lambda = B(\lambda) \mathbin{\overline{\otimes}} I$ ([1], VI. 3.14). Therefore, $B = \int_\Lambda B(\lambda) \mathbin{\overline{\otimes}} I dm(\lambda)$.

(c) Since B commutes with $\hat{T}(s) = \int_\Lambda \tilde{\lambda}(s) \mathbin{\overline{\otimes}} \tilde{\lambda}(s) dm(\lambda)$ for every $s \in S$, $B(\lambda) \mathbin{\overline{\otimes}} I$ commutes with $\tilde{\lambda}(s) \mathbin{\overline{\otimes}} \tilde{\lambda}(s)$ a.e. $[m]$. Just as in the proof of (b), since S is separable there exists an m-null set N such that for every $\lambda \in N^c$ and every $s \in S$,

$$(B(\lambda) \mathbin{\overline{\otimes}} I)(\tilde{\lambda}(s) \mathbin{\overline{\otimes}} \tilde{\lambda}(s)) = (\tilde{\lambda}(s) \mathbin{\overline{\otimes}} \tilde{\lambda}(s))(B(\lambda) \mathbin{\overline{\otimes}} I) .$$

It follows that $B(\lambda)\tilde{\lambda}(s) \mathbin{\overline{\otimes}} \tilde{\lambda}(s) = \tilde{\lambda}(s)B(\lambda) \mathbin{\overline{\otimes}} \tilde{\lambda}(s)$ and hence $B(\lambda)\tilde{\lambda}(s) = \tilde{\lambda}(s)B(\lambda)$. Thus, $B(\lambda) \in \mathscr{A}'(\tilde{\lambda})$ for almost every λ.

Continue to suppose that $B \in \mathscr{A}'(\hat{T}|_{ZSA})$ so that B satisfies (a), (b), and (c) of Theorem 3.2. We will now make use of the condition that $B\hat{T}(\alpha) = \hat{T}(\alpha)B$ for all $\alpha \in A$. Recall that α denotes both an element of $\mathrm{Gl}(n_1, C)$ and the corresponding element $\begin{vmatrix} \alpha & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha'^{-1} & 0 \\ 0 & 0 & 0 & I \end{vmatrix}$ of A.

Concerning the transitive action of A on Λ given by $\alpha \cdot \lambda = \alpha\lambda\alpha'$, let A_1 be the stability subgroup of A at $I \in \Lambda$. That is, $A_1 = \{\alpha \in A : \alpha\alpha' = I\}$, which can be identified with $O(n_1, C)$. Let $p: A \to \Lambda$ be the projection $p(\alpha) = \alpha \cdot I = \alpha\alpha'$. We will need to know that a measurable set N is an m-null set in Λ if and only if $p^{-1}(N)$ is a null set in A with respect to Haar measure. This result can be obtained by first showing that $d\eta(\lambda) = |\det \lambda|^{-(n_1+1)} d\lambda$ is an A-invariant measure on Λ. It follows (as in [1], V. 3) that N is an η-null set in Λ if and only if $p^{-1}(N)$ is a null set in A. Since m and η are clearly equivalent we have the needed result. We are now able to prove

THEOREM 3.3. *If $B \in \mathscr{A}'(\hat{T}|_{ZSA})$ then there exists a mapping $\lambda \to B(\lambda)$ of Λ into $\mathscr{L}(L^2(V))$ which is weakly continuous and satisfies:*

(a) $B = \int_\Lambda B(\lambda) \mathbin{\overline{\otimes}} I dm(\lambda)$;

(b) $B(\alpha\alpha') = D(\alpha)B(I)D(\alpha)^{-1}$ *for all* $\alpha \in A$;

(c) $B(I) \in \mathscr{A}'(D|_{A_1})$;

(d) $B(\lambda) \in \mathscr{A}'(\tilde{\lambda})$ *for all* $\lambda \in \Lambda$.

Proof. From Theorem 3.2, we have a weakly measurable mapping $\lambda \to B_1(\lambda)$ such that $B = \int_\Lambda B_1(\lambda) \mathbin{\overline{\otimes}} I dm(\lambda)$. The major part of this proof is to show that there is an equivalent mapping which is weakly continuous. By Theorem 2.8, if $\alpha \in A$ and $\hat{f} \in \mathscr{P}(L^1(Z) \cap L^2(Z))$ then

$$\hat{T}(\alpha)\hat{f}(\lambda) = \delta_H(\alpha)^{1/2}\chi(\alpha)D(\alpha)\hat{f}(\alpha^{-1} \cdot \lambda)D(\alpha)^{-1}$$

for almost every $\lambda \in \Lambda$, where $\alpha^{-1} \cdot \lambda = \alpha^{-1}\alpha'^{-1}$. The condition that $B\hat{T}(\alpha) = \hat{T}(\alpha)B$ for all $\alpha \in A$ is seen to be equivalent to

(1) For all $\alpha \in A$, $B_1(\lambda) = D(\alpha)^{-1}B_1(\alpha\lambda\alpha')D(\alpha)$ a.e. $[m(\lambda)]$.

Consider the weakly measurable mapping $\alpha \to D(\alpha)^{-1}B_1(\alpha\alpha')D(\alpha)$ of A into $\mathscr{L}(L^2(V))$. For fixed $\alpha \in A$, (1) implies

(2) $D(\alpha\beta)^{-1}B_1(\alpha\beta\beta'\alpha')D(\alpha\beta) = D(\beta)^{-1}B_1(\beta\beta')D(\beta)$ a.e. $[d\beta]$,

since a null set in Λ pulls back under p^{-1} to a null set in A. Fix $\phi, \psi \in L(V)$ and define the measurable, essentially bounded function $w: A \to C$ by

$$w(\beta) = (D(\beta)^{-1}B_1(\beta\beta')D(\beta)\phi, \psi),$$

where $(,)$ is the inner product of $L(V)$. Then, by (2), $w(\beta)d\beta$ defines a left invariant Borel measure on A. By uniqueness of Haar measure, w must be almost everywhere constant. Moreover, if this number is denoted $w_{\phi,\psi}$ then, by application of the Riesz representation theorem to the bilinear form $(\phi, \psi) \to w_{\phi,\psi}$, there exists a unique $L \in \mathscr{L}(L^2(V))$ such that

(3) $D(\beta)^{-1}B_1(\beta\beta')D(\beta) = L$ a.e. $[d\beta]$.

Consider the weakly measurable maps $\beta \to B_1(\beta\beta')$ and $\beta \to D(\beta)LD(\beta)^{-1}$ of A into $\mathscr{L}(L^2(V))$. Since $A_1 = \{\alpha \in A: \alpha\alpha' = I\}$, the first map is constant on left cosets of A_1. The second map is continuous (with respect to either the strong or the weak operator topology of $\mathscr{L}(L^2(V))$). Also, (3) implies that the two maps coincide almost everywhere. We can conclude that $\beta \to D(\beta)LD(\beta)^{-1}$ is both continuous and constant on left cosets of A_1. Because of this fact, the mapping $\beta\beta' \to D(\beta)LD(\beta)'$ is well-defined on Λ. It is also continuous since $\beta \to D(\beta)LD(\beta)^{-1}$ is continuous and p is open. Define $B(\beta\beta') = D(\beta)LD(\beta)'$, then $B(\beta\beta')$ and $B_1(\beta\beta')$ differ only on a "strip" set of measure zero in A, which projects to a null set in Λ. Thus, $\lambda \to B(\lambda)$ is a continuous mapping such that (a) holds.

Parts (b) and (c) follow immediately from the definition of $B(\lambda)$. To prove (d), recall from Theorem 3.2 (c) that $B(\lambda) \in \mathscr{A}'(\tilde{\lambda})$ for almost every $\lambda \in \Lambda$. In particular, $B(\lambda_0) \in \mathscr{A}'(\tilde{\lambda}_0)$ for some $\lambda_0 = \beta\beta' \in \Lambda$. Let $\lambda \in \Lambda$, then $\lambda = \alpha\lambda_0\alpha' = \alpha\beta\beta'\alpha'$ for some $\alpha \in A$. Now apply the definition of $B(\lambda)$ along with Theorem 2.6 (2).

We now have the main theorem.

THEOREM 3.4. *The mapping*

$$B(I) \longrightarrow B = \int_{\Lambda} D(\beta)B(I)D(\beta)^{-1} \,\overline{\otimes}\, Idm(\beta\beta')$$

is an isomorphism of von Neumann algebras from $\mathscr{A}'(D|_{A_1}) \cap \mathscr{A}'(\tilde{I})$ *onto* $\mathscr{A}'(\hat{T}|_{ZSA})$.

Proof. The mapping in the theorem makes sense because the condition $B(I) \in \mathscr{A}'(D|_{A_1})$ guarantees that $\beta\beta' \to D(\beta)B(I)D(\beta)^{-1} \,\overline{\otimes}\, I$ is well-defined. It is straightforward to verify separately that $B \in \mathscr{A}'(\hat{T}|_Z)$, $B \in \mathscr{A}'(\hat{T}|_S)$, and $B \in \mathscr{A}'(\hat{T}|_A)$. Also, using properties of decomposable operators, it is easy to show that $B(I) \to B$ is an isometric, *-algebra isomorphism. The fact that it is surjective is proved in Theorem 3.3.

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