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A CHARACTERIZATION OF DIMENSION OF TOPOLOGICAL SPACES BY TOTALLY BOUNDED PSEUDOMETRICS

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For a compact metrizable space X, for a metric d on X, and for $\varepsilon>0$, the number $N(\varepsilon, X, d)$ is defined as the minimum number of sets of d-diameter not exceeding ε required to cover X. A classical theorem characterizes the topological dimension of X in terms of the numbers $N(\varepsilon, X, d)$. In this paper, two extensions of this result are given: (i) a direct one, to separable metrizable spaces, involving totally bounded metrics; (ii) a more complicated one, involving the set of continuous totally bounded pseudometrics on the space as well as a special order on this set.

The dimension function involved is the so-called Katětov dimension, i.e., covering dimension with respect to covers by cozero sets. Let d be a metric for the compact metrizable space X. Define

$$k(X, d) = \sup \left\{\inf \left\{-\frac{\log N(\varepsilon, X, d)}{\log \varepsilon} \,\middle|\, \varepsilon < \varepsilon_0\right\} \,\middle|\, \varepsilon_0 > 0\right\} .$$

Then we have the classical

THEOREM A (*L. Pontrjagin and L. Schnirelmann* [4]).

$$\dim X = \inf\{k(X, d) \mid d \textit{ is a metric for } X\} .$$

REMARK. The number $\log N(\varepsilon, X, d)$ is often referred to as the $\varepsilon/2$-entropy of X (with respect to d).

The extension of Theorem A to separable metrizable spaces is given by Theorem 2, while the general case is covered by Theorem 1. The referee has pointed out that Lemma 5 below, needed in the proof of Theorem 1, can be derived from two theorems by Katětov ([3], Theorems 1.9 and 1.16). The author wishes to thank Professor J. Nagata for drawing his attention to Theorem A and to the problem of finding its generalization.

2. Definitions and notations. All spaces considered will be nonempty. A zeroset (cozeroset) in a space X is a set of the form $f^{-1}(\{0\})(f^{-1}((0, 1]))$, where $f\colon X\to[0, 1]$ is continuous. The symbols U, U_i, V, V_i, etc. will denote cozerosets throughout; F, F_i, F_j^k etc. will denote zerosets. If $\mathscr{A} = \{A_\gamma \mid \gamma \in \Gamma\}$ is a collection of subsets of X, the order of $\mathscr{A}$ (ord $\mathscr{A}$) is defined as $\sup\{|\mathscr{A}'| \mid \mathscr{A}' \subset \mathscr{A}$ and

$\cap \mathscr{A}' \neq \emptyset\}$. Dim X will be the Katětov dimension of X, i.e.,

$\dim X \leqq n$ iff every finite cover $\mathscr{U} = \{U_1, \cdots, U_k\}$ has a finite refinement $\mathscr{V} = \{V_1, \cdots, V_l\}$ with $\operatorname{ord} \mathscr{V} \leqq n+1$;

$\dim X = n$ iff $\dim X \leqq n$ but not $\dim X \leqq n-1$;

$\dim X = \infty$ iff not $\dim X \leqq n$ for any n.

Note that in the above definition, U_i and V_j are cozerosets by notation. For normal spaces, Katětov dimension coincides with ordinary covering dimension [1, p. 268].

A continuous pseudometric on a space X is a continuous function $d: X \times X \to [0, \infty)$ which is symmetric, satisfies the triangle inequality and has the property that $d(x, x) = 0$ for all $x \in X$. A pseudometric d is totally bounded iff for every $\varepsilon > 0$ there exists a finite ε-net in X with regard to d. $\mathscr{R}$ will be the set of all totally bounded, continuous pseudometrics on X. For $d \in \mathscr{R}$, $\varepsilon > 0$ and $x \in X$, $U_\varepsilon^d(x)$ is defined as the set $\{y \in X \mid d(x, y) < \varepsilon\}$. This is a cozeroset. On $\mathscr{R}$ we introduce the following relation: $d_1 \succ d_2$ iff for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $U_\delta^{d_1}(x) \subset U_\varepsilon^{d_2}(x)$ for all $x \in X$. For $d \in \mathscr{R}$ and $A \subset X$, the diameter of A with regard to d is the number $d\text{-diam}\, A = \sup\{d(x, y) \mid x, y \in A\}$. We define $|d| = d\text{-diam}\, X$. $|d|$ is always finite. Finally, if $\mathscr{U}$ is a cover of X and $d \in \mathscr{R}$, we say that $\mathscr{U}$ is d-uniform iff there exists $\varepsilon > 0$ such that the cover $\{U_\varepsilon^d(x) \mid x \in X\}$ refines $\mathscr{U}$.

3. An extension of Theorem A. For $d \in \mathscr{R}$ and $\varepsilon > 0$, let $N(\varepsilon, X, d)$ be defined as the minimum number of sets of d-diameter not exceeding ε required to cover X. Put

$$k(X, d) = \sup \left\{ \inf \left\{ -\frac{\log N(\varepsilon, X, d)}{\log \varepsilon} \,\middle|\, \varepsilon < \varepsilon_0 \right\} \,\middle|\, \varepsilon_0 > 0 \right\},$$

just as in the introduction.

Then we have

THEOREM 1. *If $k(X, d)$ is defined as above, then*

$$\dim X = \sup\{\inf\{k(X, d) \mid d \succ d_0,\, d \in \mathscr{R}\} \mid d_0 \in \mathscr{R}\}.$$

Before we give the proof, we will state and prove a few lemmas.

LEMMA 1. *Let $\delta > 0$, and let $\mathscr{U} = \{U_1, \cdots, U_k\}$ be a cover of X. Then there exists $d \in \mathscr{R}$ such that $\mathscr{U}$ is d-uniform and $|d| \leqq \delta$.*

Proof. For the sake of completeness, we include an elementary proof. Let $f_i: X \to [0, 1]$ be continuous, with $f_i^{-1}((0, 1]) =$

$U_i(1 \leqq i \leqq k)$.

Define $f\colon X \to \boldsymbol{R}^k$ by the formula

$$f(x) = \left(\frac{f_1(x)}{\Sigma_{i=1}^k f_i(x)}, \cdots, \frac{f_k(x)}{\Sigma_{i=1}^k f_i(x)}\right).$$

Define $d_1\colon X \times X \to [0, \infty)$ by $d_1(x, y) = \|f(x) - f(y)\|$. It is not difficult to show that $d_1 \in \mathscr{R}$. Now

$$f(X) \subset \Delta = \{(\lambda_1, \cdots, \lambda_k) | \Sigma_{i=1}^k \lambda_i = 1 \quad \text{and} \quad \lambda_i \geqq 0 (1 \leqq i \leqq k)\}.$$

Denoting the set $\{(\lambda_1, \cdots, \lambda_k) \in \Delta | \lambda_j > 0\}$ by V_j, we have $U_j = f^{-1}(V_j)$ $(1 \leqq j \leqq k)$. $\{V_1, \cdots, V_k\}$ is an open cover of the compact set Δ, so there exists $\varepsilon > 0$ such that the cover $\{U_\varepsilon(p) \cap \Delta | p \in \Delta\}$ refines $\{V_1, \cdots, V_k\}$. Let $x \in X$. Then there exists j, $1 \leqq j \leqq k$, such that $U_\varepsilon((f(x)) \subset V_j$. It follows that $U_\varepsilon^{d_1}(x) \subset f^{-1}(V_j) = U_j$. Thus $\mathscr{U}$ is d_1-uniform. Finally putting $d = \delta/|d_1| \cdot d_1$ we get the desired element of $\mathscr{R}$.

LEMMA 2. (a) *Let $d_1, d_2 \in \mathscr{R}$. Then $d_1 + d_2 \in \mathscr{R}$.*
(b) *Let $d_i \in \mathscr{R} (i \in \boldsymbol{N})$ and let $\Sigma_{i=1}^\infty |d_i| < \infty$. Then $\Sigma_{i=1}^\infty d_i \in \mathscr{R}$.*

Proof. (a) It is easy to see that $d_1 + d_2$ is a continuous pseudometric. To prove that is totally bounded, let $\varepsilon > 0$ and $\{x_1, \cdots, x_k\}$ be an $\varepsilon/3$-net for (X, d_1). Let, for $1 \leqq i \leqq k$, $\{y_1^i, \cdots, y_{n_i}^i\}$ be an $\varepsilon/3$-net for $U_{\varepsilon/3}^{d_1}(x_i)$, with regard to d_2 (the restriction of d_2 to any subset of X is again totally bounded, as can be proved in a standard manner). Put $Y = \{y_j^i | 1 \leqq i \leqq k, 1 \leqq j \leqq n_i\}$. It is not difficult to prove that Y is an ε-net for X with respect to $d_1 + d_2$. This proves (a).

(b) $\Sigma_{i=1}^\infty d_i$ is, as a uniform limit of continuous functions, itself continuous. It is easily seen to be a pseudometric. Let $\varepsilon > 0$, and $N \in \boldsymbol{N}$ so, that $\Sigma_{i>N} |d_i| < \varepsilon/2$. Since by (a), $\Sigma_{i=1}^N d_i \in \mathscr{R}$, there exists a finite $\varepsilon/2$-net for X with respect to $\Sigma_{i=1}^N d_i$. The same set is easily proved to be an ε-net for $(X, \Sigma_{i=1}^\infty d_i)$, which proves (b).

LEMMA 3. *Let Y be a dense subset of X, and let $d \in \mathscr{R}$. Then $k(X, d) = k(Y, d | Y \times Y)$.*

Proof. It is easy to see that $N(\varepsilon, X, d) = N(\varepsilon, Y, d | Y \times Y)$ for all $\varepsilon > 0$. From this the result follows by the very definition of $k(X, d)$ and $k(Y, d | Y \times Y)$.

Now we are ready to go on with the proof of Theorem 1. For shortness, denote $\sup\{\inf\{k(X, d) | d \succ d_0, d \in \mathscr{R}\} | d_0 \in \mathscr{R}\}$ by $k(X)$. First we prove: $k(X) \geqq \dim(X)$. This will follow from the following

LEMMA 4. *Let* $n \geqq 0$ *and* $\dim X \geqq n$. *Then there exists* $d_0 \in \mathscr{R}$ *such that, for all* $d \in \mathscr{R}$ *with* $d \succ d_0$, $k(X, d) \geqq n$. *(This formulation also takes care of the case* $\dim X = \infty$.)

Proof of Lemma 4. Let $\mathscr{U} = \{U_1, \cdots, U_k\}$ be a cover such that every refinement $\mathscr{V} = \{V_1, \cdots, V_l\}$ of $\mathscr{U}$ has order $\geqq n+1$. By Lemma 1, there is a $d_0 \in \mathscr{R}$ such that $\mathscr{U}$ is d_0-uniform. Let $d \succ d_0$, $d \in \mathscr{R}$. Then there exists $\delta > 0$ such that the cover $\{U_\delta^d(x) \mid x \in X\}$ refines $\mathscr{U}$.

Consider the equivalence relation $\sim$ on X defined by $x \sim y$ iff $d(x, y) = 0$. Let X' be the set of evuivalence classes, and $\phi: X \to X'$ the natural projection. Define $d': X' \times X' \to [0, \infty)$ by $d'(\phi(x), \phi(y)) = d(x, y)$. This definition turns (X', d') into a totally bounded metric space. Since d is continuous, ϕ is continuous if we equip X' with the metric topology. Furthermore, if $A \subset X$, then d-diam $A = d'$-diam $\phi(A)$; and if $B \subset X'$, then d'-diam $B = d$-diam $\phi^{-1}(B)$. It follows that $N(\varepsilon, X, d) = N(\varepsilon, X', d')$ for all $\varepsilon > 0$, thus $k(X, d) = k(X', d')$. Let (X'', d'') be the metric completion of (X', d'). Since (X', d') is totally bounded, (X'', d'') is compact. From Lemma 3 it follows that $k(X', d') = k(X'', d'')$. From Theorem A we deduce $k(X'', d'') \geqq \dim X''$. Combining the above results, we infer $k(X, d) \geqq \dim X''$.

What is left to prove, is that $\dim X'' \geqq n$. So suppose $\dim X'' \leqq n-1$. Then there is an open cover $\mathscr{W} = \{W_1, \cdots, W_s\}$ (consisting of cozerosets) such that ord $\mathscr{W} \leqq n$ and d''-diam $W_i < \delta$ for $1 \leqq i \leqq s$. Then $\{\phi^{-1}(W_i) | 1 \leqq i \leqq s\}$ is a refinement of $\mathscr{U}$, consisting of cozerosets, with order $\leqq n$. This is a contradiction. Thus $k(X, d) \geqq \dim X'' \geqq n$, which completes the proof of Lemma 4.

Next we will prove: $k(X) \leqq \dim X$. If $\dim X = \infty$, we have nothing to prove. So suppose $\dim X = n < \infty$.

Then the result will follow from

LEMMA 5. *Let* $d_0 \in \mathscr{R}$, *and* $\varepsilon_0 > 0$. *Then there exists* $d \in \mathscr{R}$, $d \succ d_0$, *such that* $k(X, d) \leqq n + \varepsilon_0$.

Proof. First we prove the following

Claim. There exist $d^* \in \mathscr{R}$, $d^* \succ d_0$, and $\mathscr{F}_k = \{F_1^k, \cdots, F_{m_k}^k\}$ $(k \geqq 0)$ such that

(i) $\mathscr{F}_k$ is a cover and ord $\mathscr{F}_k \leqq n+1$ $(k \geqq 0)$

(ii) d^*-diam $F_i^k \leqq 1/k$ $(k \in \boldsymbol{N}, 1 \leqq i \leqq m_k)$

(iii) For every $\mathscr{F}' \subset \mathscr{F}_k$ with $\cap \mathscr{F}' = \emptyset$, the cover $\{X \backslash F \mid F \in \mathscr{F}'\}$ is d^*-uniform $(k \in \boldsymbol{N})$.

Proof of Claim. We will construct inductively sequences $(d_k)_{k=0}^{\infty}$ of elements of $\mathscr{R}$ and $(\mathscr{F}_k)_{k=0}^{\infty}$ of cozero covers of X in the following way: d_0 is given, put $\mathscr{F}_0 = \{X\}$; let $k \in N$, and suppose $d_0, \cdots, d_{k-1}$ and $\mathscr{F}_0, \cdots, \mathscr{F}_{k-1}$ have been defined in such a way that

(a) $\mathscr{F}_l = \{F_1^l, \cdots, F_{m_l}^l\}$ is a cover and $\text{ord}\,\mathscr{F}_l \leqq n + 1\,(0 \leqq l < k)$

(b) $(d_0 + \cdots + d_{k-1})\text{-diam}\,F_i^l < 1/l \quad (0 < l < k,\, 0 \leqq i \leqq m_l)$

(c) For every $\mathscr{F}' \subset \mathscr{F}$ such that $\cap \mathscr{F}' = \varnothing$, the cover $\{X \backslash F \mid F \in \mathscr{F}'\}$ is d_l-uniform $(0 < l < k)$

(d) $|d_l| \leqq 2^{-l}$ $(0 < l < k)$.

Since $d_0 + \cdots + d_{k-1} \in \mathscr{R}$, by Lemma 2, and since $\dim X = n$, there exists a cover $\mathscr{F}_k = \{F_1^k, \cdots, F_{m_k}^k\}$ of X such that $\text{ord}\,\mathscr{F}_k \leqq n + 1$ and $(d_0 + \cdots + d_{k-1})\text{-diam}\,F_i^k < 1/k$ $(1 \leqq i \leqq m_k)$: simply take $\mathscr{F}_k$ to be a suitable shrinking of a finite cover $\mathscr{U} = \{U_1, \cdots, U_s\}$ with $\text{ord}\,\mathscr{U} \leqq n + 1$ and $(d_0 + \cdots + d_{k-1})\text{-diam}\,U_i < 1/k$ (compare e.g., [1, p. 267]).

Let $0 < \delta < \min\{2^{-k}, \min\{1/l - (d_0 + \cdots + d_{k-1})\text{-diam}\,F \mid 0 < l \leqq k, F \in \mathscr{F}_l\}$.

Let $\{\mathscr{U}_1, \cdots, \mathscr{U}_t\}$ be the set of all covers of the form $\{X \backslash F \mid F \in \mathscr{F}'\}$, where $\mathscr{F}' \subset \mathscr{F}_k$ and $\cap \mathscr{F}' = \varnothing$. By Lemma 1, there exist $d^i \in \mathscr{R}$ such that $|d^i| \leqq \delta/t$ and $\mathscr{U}_i$ is d^i-uniform $(1 \leqq i \leqq t)$. Put $d_k = d^1 + \cdots + d^t$. It is not difficult to prove that for these choices of $\mathscr{F}_k$ and d_k the conditions (a)-(d) are satisfied for k instead of $k - 1$. This completes the inductive construction.

Now put $d^* = \Sigma_{i=0}^{\infty} d_i$. By Lemma 2, $d^* \in \mathscr{R}$. It is easy to see that $d^* \succ d_0$. The conditions (i)-(iii) are readily verified. This proves our claim.

Now, let as before $\sim$ be the equivalence relation on X defined by $x \sim y$ iff $d^*(x, y) = 0$. Let X' be the set of equivalence classes and $\phi\colon X \to X'$ be projection. Let $d'\colon X' \times X' \to [0, \infty)$ be defined by $d'(\phi(x), \phi(y)) = d^*(x, y)$. Again ϕ is continuous. Let (X'', d'') be the (compact) completion of (X', d'). We will prove: $\dim X'' \leqq n$. It will suffice to show that, for every $k \in N$, there exists a closed cover of X'' with order$\leqq n + 1$ and such that its elements have d''-diameter not exceeding $1/k$. So, let $k \in N$. Define $G_i = \text{Cl}(\phi(F_i^k))$ $(1 \leqq i \leqq m_k)$, where the closure is taken in X'', and put $\mathscr{G} = \{G_1, \cdots, G_{m_k}\}$. Then $\mathscr{G}$ is a closed cover of X'', and $d''\text{-diam}\,G_i = d''\text{-diam}\ \phi(F_i^k) = d'\text{-diam}\,\phi(F_i^k) = d^*\text{-diam}\,F_i^k \leqq 1/k$.

It is left to prove that $\text{ord}\,\mathscr{G} \leqq n + 1$. Let $\mathscr{G}' \subset \mathscr{G}$, $|\mathscr{G}'| = n + 2$. For convenience we assume that $\mathscr{G}' = \{G_1, \cdots, G_{n+2}\}$. Let $\mathscr{F}' = \{F_1^k, \cdots, F_{n+2}^k\}$. Since $\text{ord}\,\mathscr{F}_k \leqq n + 1$, $\cap \mathscr{F}' = \varnothing$. Thus the cover $\{X \backslash F_i^k \mid 1 \leqq i \leqq n + 2\}$ is d^*-uniform and there exists $\delta > 0$ such that for all $x \in X$ $U_\delta^{d^*}(x) \subset X \backslash F_i^k$ for some i with $1 \leqq i \leqq n + 2$.

Suppose $\cap \mathscr{G}' \neq \varnothing$, say $z \in \cap \mathscr{G}'$. Since $G_i = \text{Cl}(\phi(F_i^k))$, there exists $x_i \in F_i^k$ such that $d''(\phi(x_i), z) < \delta/2$ $(1 \leqq i \leqq n + 2)$. Thus

$d^*(x_i, x_j) = d'(\phi(x_i), \phi(x_j)) < \delta$ for $1 \leqq i, j \leqq n + 2$. It follows that $U_\delta^{d^*}(x_1) \cap F_i^k \neq \emptyset$ $(1 \leqq i \leqq n + 2)$, which is a contradiction. So $\bigcap \mathscr{G}' = \emptyset$, and $\operatorname{ord} \mathscr{G} \leqq n + 1$. This proves $\dim X'' \leqq n$.

Thus $\phi: X \to X''$ is a continuous map into the compact metric space X'', which satisfies $\dim X'' \leqq n$. By Theorem A, there exists a metric $d^\cdot$ on X'' with $k(X'', d^\cdot) \leqq n + \varepsilon_0$. Put $d(x, y) = d^\cdot(\phi(x), \phi(y))$ for $x, y \in X$. From the compactness of X'' and the continuity of ϕ it follows that $d \in \mathscr{R}$. Also $d^\cdot \succ d''$ on X'', again since X'' is compact. From the formulas $d^*(x, y) = d''(\phi(x), \phi(y))$ and $d(x, y) = d^\cdot(\phi(x), \phi(y))$ it follows then that $d \succ d^*$. Since $d^* \succ d_0$, we also have $d \succ d_0$. Furthermore, just as before, $k(X, d) = k(X'', d^\cdot) \leqq n + \varepsilon_0$. This completes the proof of Lemma 5.

Combining Lemma 4 and Lemma 5, finally, we get the proof of Theorem 1.

REMARK. If X is a compact, nonempty, metrizable space, then

(a) all (pseudo) metrics on X are totally bounded

(b) for every two metrics d_1 and d_2, we have $d_1 \succ d_2$

(c) for every metric d and every pseudometric d', $d' \succ d$ implies that d' is a metric, compatible with the topology.

(N. B. all these (pseudo) metrics are supposed to be continuous.)

We did prove:

$$\dim X = \sup\{\inf\{k(X, d) \mid d \succ d_0, d \in \mathscr{R} \mid d_0 \in \mathscr{R}\} .$$

It follows, that for fixed $d_1 \in \mathscr{R}$

$$\dim X = \sup\{\inf\{k(X, d) \mid d \succ d_0, d \in \mathscr{R}\} \mid d_0 \succ d_1, d_0 \in \mathscr{R}\} .$$

(Here the fact that the pseudo-order $\succ$ is directed (cf. Lemma 1) is needed.) Now, if we take d_1 to be a fixed metric for X, we infer from (a)-(c):

$$\begin{aligned}\dim X &= \sup\{\inf\{k(X, d) \mid d \succ d_0, d \in \mathscr{R}\} \mid d_0 \succ d_1, d_0 \in \mathscr{R}\} \\ &= \inf\{k(X, d) \mid d \text{ is a metric for } X\}\end{aligned}$$

which is Theorem A. Thus our result includes Theorem A as a special case.

4. The separable metrizable case. In the case of a separable metrizable space X another, more direct generalization of Theorem A is available. Namely, we have

THEOREM 2. *Let X be a nonempty, separable metrizable space. Then* $\dim X = \inf\{k(X, d) \mid d$ *is a totally bounded metric for* $X\}$.

Proof. Denote $k(X) = \inf\{k(X, d) \mid d$ is a totally bounded metric for $X\}$. First we prove: $k(X) \leqq \dim X$. If $\dim X = \infty$, we have nothing to prove. So suppose $\dim X = n \geqq 0$. Let $\tilde{X}$ be a metrizable compactification of X with $\dim \tilde{X} = n$[2, p. 65]. Let $\varepsilon > 0$ and d_0 be a metric for $\tilde{X}$ such that $k(\tilde{X}, d_0) \leqq n + \varepsilon$ (Theorem A). The restriction of d_0 to X is totally bounded, and by Lemma 3, $k(X, d_0 \mid X \times X) = k(\tilde{X}, d_0) \leqq n + \varepsilon$. Thus $k(X) \leqq n = \dim X$.

Next we prove: $k(X) \geqq \dim X$. Let d be any totally bounded metric for X. The completion $(\tilde{X}, \tilde{d})$ of (X, d) is then compact, so $k(\tilde{X}, d) \geqq \dim X$, again by Theorem A. By Lemma 3, $k(X, d) = k(\tilde{X}, \tilde{d})$. This completes the proof of Theorem 2.

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