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PERIODIC SOLUTIONS OF HIGHER ORDER SYSTEMS

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Results are given providing sufficient conditions for the existence of periodic solutions higher order nonlinear vector differential equations. The conditions include the possibility of both sublinear and superlinear growth in the nonlinear terms.

1. Introduction. Consider the system of equations

$$u^{(n)} + g(t, u, u', \cdots, u^{(n-1)}) = f(t) , \qquad 0 < t < T , \tag{1.1}$$

where u, g, and f are k-vectors. We will establish the existence of T-periodic solutions to (1.1) for large classes of nonlinearities g and forcing terms f. In particular we include cases where g is bounded, sublinear, or superlinear in u, with arbitrary growth in the other arguments of g in the latter case. We also include cases in which g is mildly singular in t or vanishes at an endpoint of the interval.

The second order scalar version of (1.1) has received attention recently with new and interesting results (see, e.g., [2] and [3]). The second order vector case of (1.1) has also been investigated by several authors recently with interesting results, often based on the sign of $x \cdot g$ (see, e.g., [1], [5], [6]). Recently the second author in [8] obtained results which apply to higher order vector equations which lead to operators nonnegative outside a large ball.

Our results here do not depend on sign conditions in the sense that if the function g in (1.1) satisfies our conditions so does $-g$. Also our results do not depend on the existence of a Nagumo function for g. Roughly speaking we assume that the nonlinearity g is either sublinear at infinity (see Cor. 2.2) or superlinear at the origin (see Cor. 2.3). Although we see our results as being of principle interest for higher order scalar and vector equations we obtain results which appear to be new even in the second order scalar case (see Example 3.3). We wish also to remark that while our results are all stated for periodic boundary conditions, our methods could be applied equally well to some other boundary value problems, e.g., the Neumann problem $x'(0) = x'(T) = 0$ for second order equations.

For the proof of our result we will rely on an abstract result of Mawhin [4] which is an extension of the Leray-Schauder continuation theorem.

Let X and Z be normed vector spaces, $L: D(L) \subseteq X \to Z$ a linear

Fredholm mapping of index zero and $N: X \to Z$ a continuous mapping. It follows that there exist continuous projections $P: X \to X$ and $Q: Z \to Z$ such that $\mathrm{Im}\,(P) = \mathrm{Ker}\,(L)$ and $\mathrm{Im}\,(L) = \mathrm{Ker}\,(Q) = \mathrm{Im}\,(I - Q)$. Moreover the mapping

$$L: D(L) \cap \mathrm{Ker}\,(P) \subsetneqq (I - P)X \to \mathrm{Im}\,L$$

is invertible; denote its inverse by L^{-1}. Let G be an open bounded subset of X. The mapping N is said to be L-compact on $\bar{G}$ if $QN(\bar{G})$ is bounded and $L^{-1}(I - Q)N: \bar{G} \to X$ is compact. Let J be an isomorphism from $\mathrm{Im}\,(Q)$ onto $\mathrm{Ker}\,(L)$; such a J exists since these subspaces have the same finite dimension.

THEOREM (*Mawhin; see* [4], p. 40). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{G}$. Suppose*

(i) *For each $\lambda \in (0, 1)$, every solution u of*

$$Lu = \lambda N u$$

is such that $u \notin \partial G$.

(ii) *$QNu \neq 0$ for each $u \in \mathrm{Ker}\,(L) \cap \partial G$ and $d(JQN, G \cap \mathrm{Ker}\,L, 0) \neq 0$. Then the equation $Lu = Nu$ has at least one solution in $D(L) \cap \bar{G}$. Here $d(\cdot, \cdot, \cdot)$ refers to the Brouwer degree.*

We will use the notation:

$$|a| = \max_{1 \leqq i \leqq k} |a_i| \quad \text{for} \quad a \in R^k \,, \quad L_1(J, R^k)$$

for the integrable functions defined on an interval J and taking values in R^k, with norm denoted by $\|\cdot\|_1$, and $C(J, R^k)$ for the continuous functions, with norm defined by $\|h\| = \sup\{|h(t)|: t \in J\}$.

2. The results. Consider the nth order nonlinear system.

$$u^{(n)} + g(t, u, u', \cdots, u^{(n-1)}) = f(t) \,, \qquad t \in J = (0, T) \,, \tag{2.1}$$

where $g: J \times R^{k \times n} \to R^k$ is continuous and $f \in L_1(J, R^k)$. We will impose general conditions on g and f which will allow us to use the Mawhin theorem stated in §1, and then give special classes of such functions in our corollaries. Section 3 contains some examples.

Let $r = (r_{ip}) \in R^{k \times n}$ be fixed and define

$$\Omega = \{a \in R^{k \times n}: |a_{ip}| < r_{ip}, 1 \leqq i \leqq k, 0 \leqq p \leqq n - 1\} \,.$$

Suppose that the function g satisfies

(G1) there exists a function $C: R \to R^k$ such that for $L \geqq 0$ and $j = 1, \cdots, k$, $|g_j(t, u^0, \cdots, u^{(n-1)})| \leqq L$ for $t \in J$ and $u = (u_i^p) \in \bar{\Omega}$ implies that $|u_j^0| \leqq C_j(L)$, and

(G2) there exists a function $\beta \in L_1(J, R^k)$ such that if $u \in \bar{\Omega}$, then $|g_j(t, u^0, \cdots, u^{(n-1)})| \leqq \beta_j(t)$, $1 \leqq j \leqq k$.

In the above expressions we are using u_i^p to represent the element in the ith row and pth column of a real $k \times n$ matrix.

Define the projection Q on $L_1(J, R^m)$, $m \geqq 1$, by

$$Qh = \frac{1}{T}\int_0^T h(s)ds \ .$$

THEOREM 2.1. *Suppose that* (G1) *and* (G2) *are satisfied and that for* $1 \leqq i \leqq k$,

$$\begin{cases} C_i(|Qf_i|) + T^{n-1}(\|f_i\|_1 + \|\beta_i\|_1) < r_{i0} \ , \\ \quad T^{n-p-1}(\|f_i\|_1 + \|\beta_i\|_1) < r_{ip} \ , \qquad i \leqq p \leqq n-1 \ . \end{cases} \tag{2.2}$$

Further, suppose that

$$\sum_{i=1}^k a_i \operatorname{sgn}(u_i^0) Q[g_i(\cdot, u^0, 0, \cdots, 0) - f_i] > 0 \tag{2.3}$$

for $(u^0, 0, \cdots, 0) \in \partial\Omega$, *where* $a_i \in \{-1, 1\}$, $1 \leqq i \leqq k$. *Then* (2.1) *has a T-periodic solution.*

Proof. We show that the hypotheses of Mawhin's theorem hold with the operators defined below.

Let $X = C^{(n-1)}(\bar{J}, R^k)$, $Z = L_1(J, R^k)$, and

$$G = \{u \in X\colon (u(t), \cdots, u^{(n-1)}(t)) \in \Omega, t \in \bar{J}\} \ .$$

Define $L\colon \operatorname{dom} L \subseteqq X \to Z$ by

$$\operatorname{dom} L = \{u \in X\colon u^{(n-1)} \in AC(J), u^{(p)}(0) = u^{(p)}(T), 0 \leqq p \leqq n-1\} \ ,$$
$$Lu = u^{(n)} \quad \text{for} \quad u \in \operatorname{dom} L \ .$$

Define $N\colon X \to Z$ by

$$Nu(t) = f(t) - g(t, u(t), \cdots, u^{(n-1)}(t)) \quad \text{for} \quad u \in X \ .$$

It is easily verified that $\operatorname{Im} L$ is closed and that $\operatorname{Ker} L = R^k \cong Z/\operatorname{Im} L$ and, therefore, L is a Fredholm map of index zero. Define the projections $P\colon X \to \operatorname{Ker} L$ and $Q\colon Z \to Z$, such that $\operatorname{Im} L = \operatorname{Ker} Q$, by

$$Qh = \frac{1}{T}\int_0^T h(s)ds = Ph \ .$$

The restriction of L to $(I-P)X \cap \operatorname{dom} L$ is one-to-one and so L has a partial inverse,

$L^{-1}\colon \operatorname{Im} L \to (I-P)X \cap \operatorname{dom} L$. Furthermore, since g is continuous and $\operatorname{dom} L$ is compactly imbedded in X, it follows that

$L^{-1}(I - Q)N\colon X \to X$ is completely continuous.

We claim that if $u \in \operatorname{dom} L \cap \partial G$ and $0 < \lambda < 1$, then $Lu \neq \lambda Nu$.

Suppose this is not the case, then there exists $\lambda \in (0, 1)$ and a function $u \in \operatorname{dom} L$ with

$$u^{(m)} + \lambda g(t, u, \cdots, u^{(n-1)}) = \lambda f\,, \qquad t \in J\,. \tag{2.4}$$

Furthermore, since $u \in \partial G$, $\|u_i^{(p)}\| \leqq r_{ip}$, $1 \leqq i \leqq k$, $0 \leqq p \leqq n-1$, and for some m and j there is a $t^* \in \bar{J}$ such that $|u_j^{(m)}(t^*)| = r_{jm}$. Also, by the periodicity condition, there are points $t_p \in J$ such that $u_j^{(p)}(t_p) = 0$, $1 \leqq p \leqq n-1$.

Integrating the jth row of (2.4) and using the mean value theorem for integrals, gives another point, $t_0 \in J$, such that

$$|g_j(t_0, u(t_0), \cdots, u^{(n-1)}(t_0))| = |Qf_j|\,,$$

which implies that $|u_j(t_0)| \leqq C_j(|Qf_j|)$, by (G1). Integrating the jth row of (2.4) $n-m$ times between the indicated limits yields

$$u_j^{(m)}(t^*) - u_j^{(m)}(t_m) = \lambda \int_{t_m}^{t^*} \int_{t_{m+1}}^{s_{m-1}} \cdots \int_{t_{n-1}}^{s_{n-1}} [f_j(s_n) - g_j(s_n, u(s_n), \cdots, u^{(n-1)}(s_n))] ds_n \cdots ds_{m+1}\,.$$

Using (G2), it follows that

$$r_{jm} \leqq T^{n-m-1}(\|f_j\|_1 + \|\beta_j\|_1)\,, \qquad \text{if} \quad 1 \leqq m \leqq n-1$$

and

$$r_{j0} \leqq C_j(|Qf_j|) + T^{n-1}(\|f_j\|_1 + \|\beta_j\|_1)\,, \qquad \text{if} \quad m = 0\,.$$

Either case contradicts the hypotheses of the theorem.

Next, because $\operatorname{Ker} L = R^k$, the condition that $QNu \neq 0$ for $u \in \operatorname{Ker} L \cap \partial G$ becomes

$$Q[f - g(\cdot, u^0, 0, \cdots, 0)] \neq 0 \qquad \text{for} \quad (u^0, 0, \cdots, 0) \in \partial\Omega\,.$$

This is guaranteed by (2.3).

Finally, (2.3) implies that QN is homotopic on $\operatorname{Ker} L \cap \partial G$ to the matrix $A = \operatorname{diag}(-a_i)$, implying that the Brouwer degree, $d(QN, G \cap \operatorname{Ker} L, 0)$, is nonzero.

The hypotheses of Mawhin's theorem are satisfied and, hence, there exists a function $u \in \operatorname{dom} L \cap \bar{G}$ such that $Lu = Nu$, that is, u is a T-periodic solution to (2.1) and satisfies $\|u_i^{(p)}\| \leqq r_{ip}$, $1 \leqq i \leqq k$, $0 \leqq p \leqq n-1$.

REMARKS. (a) The solution, u, lies in $C^{n-1}(\bar{J})$ with $u^{(n)}$ only existing almost everywhere. If we assume that f is continuous, then we can conclude that $u^{(n)}$ is continuous on J.

(b) To extend the solution periodically one must, of course, assume that both f and g are T-periodic in t.

(c) The function C in (G1) is not assumed to be continuous. If $g(t, u) = tu$ on $(0, 1) \times R$, for example, and if $|g(t, u)| \leqq L$ for $t \in (0, 1)$, $|u| \leqq r$, then we may take

$$C(L) = \begin{cases} 0 & \text{if } L = 0 \\ r & \text{if } L > 0 \,. \end{cases} \quad \text{(See Corollary 2.3 and Example 3.3)} \,.$$

(d) The purpose of (2.3) is to ensure that the Brouwer degree, $d(QN, G \cap \operatorname{Ker} L, 0)$, is defined and nonzero. A weaker version of (2.3) that is sometimes used is

$$x \cdot [g(t, x, 0, \cdots, 0) - f(t)] > 0 \qquad \text{for} \quad t \in J,\ (x, 0, \cdots, 0) \in \partial\Omega$$

or

$$x \cdot [g(t, x, 0, \cdots, 0) - f(t)] < 0 \qquad \text{on the same set} \,.$$

Our first corollary deals with the case where g is *sublinear* in a certain sense. First some notation:

For all $r \in R^{k \times n}$, $r_{ip} \geqq 0$, $1 \leqq i \leqq k$, $0 \leqq p \leqq n-1$, define $\Omega_{r1} = R^{k \times n}$ and for $j = 2, \cdots, k$.

$$\Omega_{rj} = \{(u^0, \cdots, u^{(n-1)}) \in R^{k \times n} \colon |u_i^p| \leqq r_{ip},\ i < j,\ 0 \leqq p \leqq n-1\} \,.$$

Define $\beta_j(r, t) = \sup \{|g_j(t, u^0, \cdots, u^{(n-1)})| \colon (u^0, \cdots, u^{(n-1)}) \in \Omega_{rj},\ |u_j^0| \leqq r_{j0}\}$, $1 \leqq j \leqq k$.

COROLLARY 2.2. *Suppose that g satisfies*

$$\frac{1}{r_{j0}} \int_0^T \beta_j(r, t)dt \longrightarrow 0 \qquad \textit{and} \tag{2.5}$$

$$a_j \operatorname{sgn}(u_j^0) g_j(t, u^0, \cdots, u^{(n-1)}) \longrightarrow +\infty \qquad \textit{as} \quad |u_j^0| = r_{j0} \longrightarrow \infty \tag{2.6}$$

uniformly for $(t, u^0, \cdots, u^{(n-1)})$ in $J \times \Omega_{rj}$ where $a_j \in \{-1, 1\}$, for $1 \leqq j \leqq k$. Then 2.1 *has a T-periodic solution for all $f \in L_1(J, R^k)$.*

The proof of the corollary will be omitted. Basically the proof consists of showing that Theorem 2.1 applies, first by choosing $r_{1,0}$, then $r_{1,p}$, $1 \leqq p \leqq n-1$, and then proceeding to define $r \in R^{k \times n}$, a row at a time.

The second corollary deals with the case that g is *superlinear*. The proof will be omitted.

Let $R > 0$ be fixed and set $B = \{x \in R^{k \times n} \colon |x| \leqq R\}$. For $r \in R^{k \times n}$, $0 \leqq r_{ip} \leqq R$, $1 \leqq i \leqq k$, $0 \leqq p \leqq n-1$, define $B_{r1} = B$ and for $1 < j \leqq k$, $B_{rj} = \{x \in B \colon |x_{ip}| \leqq r_{ip},\ i < j\}$.

Define $\alpha_j(r, t) = \sup \{|g_j(t, u^0, \cdots, u^{(n-1)})|: (u^0, \cdots, u^{(n-1)}) \in B_{rj}, |u_j^0| \leqq r_{j0}\}$, $1 \leqq j \leqq k$.

COROLLARY 2.3. *Suppose that for* $(t, u^0, \cdots, u^{(n-1)}) \in J \times B$ *either*

$$C(L) \longrightarrow 0 \quad as \quad L \longrightarrow 0 , \tag{2.7}$$

or

$$C(0) = 0 \quad and\ assume \quad Qf = 0 . \tag{2.8}$$

Suppose further that for $1 \leqq j \leqq k$, g_j *satisfies*

$$\frac{1}{r_{j0}} \int_0^T \alpha_j(r, t)dt \longrightarrow 0 \quad as \quad r_{j0} \longrightarrow 0$$

and

$$a_j \operatorname{sgn}(u_j^0)Qg_j(\cdot, u^0, 0, \cdots, 0) > 0$$

for all small $|u_j^0| > 0$, *for* $(u^0, 0, \cdots, 0) \in \partial B_{rj}$ *where* $a_j \in \{-1, 1\}$. *Then* (2.1) *has a T-periodic solution for all* $f \in L_1(J, R^k)$ *with* $\|f\|_1$ *sufficiently small.*

3. Examples. Here we present some examples to which the results of the previous section apply, hopefully illustrating the conditions which we impose upon the nonlinearity.

EXAMPLE 3.1. Consider the systems

$$\begin{cases} x'' \pm \operatorname{sgn}(x)|x|^\alpha + g_1(t, u, u') = f_1(t) \\ y'' \pm \operatorname{sgn}(y)|y|^\beta + g_2(t, x, x') + g_3(t, u, u') = f_2(t) \\ z'' \pm \operatorname{sgn}(z)|z|^\gamma + g_4(t, x, x', y, y') + g_5(t, u, u') = f_3(t) , \end{cases} \tag{3.1}$$

where $0 < \alpha, \beta, \gamma < 1$, $u = (x, y, z)$, g_i is continuous for each i, $1 \leqq i \leqq 5$, and uniformly bounded for $i = 1, 3, 5$. No other conditions are impossed upon g_2 and g_4. It follows from Corollary 2.2 that (3.1) has a solution u for all $f_i \in L_1(0, T)$ and all $T > 0$, satisfying $u(0) = u(T)$, $u'(0) = u'(T)$.

EXAMPLE 3.2. Let g be continuous and uniformly bounded on $[0, T] \times R^n$ and let $0 \leqq \alpha < 1$, $0 < \beta < 1$, then

$$u^{(n)} + t^{-\alpha} \operatorname{sgn}(u)|u|^\beta = g(t, u, \cdots, u^{(n-1)}) \tag{3.2}$$

has a solution u such that $u^{(i)}(0) = u^{(i)}(T)$, $0 \leqq i \leqq n - 1$. This also follows directly from Corollary 2.2.

EXAMPLE 3.3. Consider the equation

$$(3.3)\qquad \begin{cases} u'' + kt^{\alpha}\sin u = f(t)\ , & 0 < t < 1\ , \\ u(0) = u(1)\ , \quad u'(0) = u'(1)\ , \end{cases}$$

where $\alpha > -1$ and $\int_0^1 f(t)dt = 0$. Theorem 2.1 implies that (3.3) has a solution, u provided $|k|/(\alpha + 1) + ||f||_1 < \pi$. Furthermore, $|u(t)| < \pi$ and $|u'(t)| \leqq |k|/(\alpha + 1) + ||f||_1$.

EXAMPLE 3.4. Consider the fourth order system

$$(3.4)\qquad \begin{cases} x^{iv} + x^3 g_1(t, u, u', u'', u''') = f_1(t) \\ y^{iv} + y^3 g_2(t, u, u', u'', u''') = f_2(t)\ , & 0 < t < T\ , \end{cases}$$

where $u = (x, y)$ and g_i is continuous and uniformly bounded away from zero, $i = 1, 2$. Corollary 2.3 implies that (3.4) has a solution u with $u^{(i)}(0) = u^{(i)}(T)$, $0 \leqq i \leqq 3$, for all $f_1, f_2 \in L_1(0, T)$ with sufficiently small norm.

Concluding remarks. The results of this paper are related to some of the results of Mawhin ([7], Theorems 6.1 and 7.1) and Gaines and Mawhin ([4], Theorem IX. 3). Those results rely on one of Mawhin's coincidence degree theorems, as do our results here. The results of [7] and Theorem IX. 3 of [4] both include vector equations of the form considered here, but require the nonlinearity to have sublinear growth. For example, Theorem 6.1 of [7] applied to our equation (1.1) requires that for all $\varepsilon > 0$ there exists $\gamma > 0$ such that $|g(t, u^0, \cdots, u^{(n-1)})| \leqq \varepsilon(|u^0| + \cdots + u^{(n-1)})| + \gamma$ for all $(t, u^0, \cdots, u^{(n-1)}) \in R^{n+1}$. Moreover, those results all have hypotheses which exclude periodic nonlinearities, such as $\sin(u)$, which our results allow (Example 3.3). In comparing our results with Theorem IX. 3 of [4], one sees that conditions (IX. 14) and (ii) are similar to, but stronger than, conditions (2.5) and (2.6) of our Corollary 2.2. In addition, we allow nonlinearities in the derivative terms and mild singularities in t. Theorems 6.1, 7.1 of [7] and IX. 3 of [4] do, however, allow for a more general linear part of the equations considered than do our results here, and Theorem 6.1 of [6] includes functional differential equations.

In our Corollary 2.3 the nonlinear part is superlinear near the origin, and our methods require that the forcing function f be small. If the superlinearity is assumed to hold at infinity, e.g., if $g(x)/x \to +\infty$ as $|x| \to \infty$, then our results do not apply. Here the sign of $g(x)$ assumes critical importance (at least in affecting the difficulty of the problem). If the equation considered is, e.g., $(-1)^n x^{(2n)} + g(x) = f$, with $g(x)x \geqq 0$ then a number of results are available in case $n = 1$ (concluding vector equations); see, e.g., [1] and many results in [5], and many results in [4], and in case $n \geqq 1$ see [8]. For equations such as $(-1)^n x^{(2n)} - g(x) = f$, with g as before, much less seems to be known, but Fucik and Lovicar have shown in [2] that if $n = 1$ there is a periodic solution for any periodic f. The results of Gaines [3] also apply in some cases of this type.

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