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SPHERICAL MEAN PERIODIC FUNCTIONS ON SEMI SIMPLE LIE GROUPS

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Let G be a connected semisimple noncompact Lie group with finite center. We define the notion of a smooth spherical mean periodic function (with respect to a fixed maximal compact subgroup K of G) and show that the classical results of L. Schwartz for mean periodic functions on the real line hold in this context.

1. Introduction. The study of mean periodic functions started with Delsarte ([1]) who was interested in solving the convolution equation

$$\mu * f = 0$$

where μ is a measure of compact support on $\boldsymbol{R}$ and f a continuous function on $\boldsymbol{R}$. He was able to show that under certain conditions a general solution f can be written as a linear combination of "exponential monomial" solutions of the above equation. A mean periodic function on $\boldsymbol{R}$ is a continuous function f satisfying the above convolution equation for a nontrivial μ. In his famous paper ([11]) L. Schwartz studied mean periodic functions in detail, introduced the notion of the spectrum of a mean periodic function and showed that a mean periodic function f can be approximated by finite linear combinations of the functions in the spectrum of f. Malgrange in [10] studied the case of mean periodic functions on $\boldsymbol{R}^n$ for $n > 1$ and showed that a weaker version of Schwartz's result holds in this case.

The study of smooth mean periodic functions for the group SL$(2, \boldsymbol{R})$ was taken up by Ehrenpreis and Mautner in [4], [5] and results analogous to those of Schwartz were obtained by them. Since then harmonic analysis of spherical functions on semisimple Lie groups has been studied extensively ([2], [6], [7], [9], [12]).

The purpose of this paper is to use these powerful results along with the original results of Schwartz and Malgrange to study the case of spherical mean periodic functions on a noncompact semisimple Lie group G with finite center.

2. Preliminaries. Throughout §2 and §3 G will denote a noncompact semisimple Lie group with finite center and of real rank 1, K a fixed maximal compact subgroup of G and $\boldsymbol{R}$ the real line. Any unexplained terminology in this section can be found in [8].

Certain function spaces on G and $\boldsymbol{R}$: Let $C^\infty(\boldsymbol{R})$ be the set of all (complex valued) C^∞-functions on $\boldsymbol{R}$ equipped with the topology of uniform convergence along with all derivatives on compacta. By $(C^\infty(\boldsymbol{R}))^e$ we denote the closed subspace of $C^\infty(\boldsymbol{R})$ consisting of functions f which are even (i.e., $f(x)=f(-x)$ for all $x \in \boldsymbol{R}$). Equip $(C^\infty(\boldsymbol{R}))^e$ with the relative topology from $C^\infty(\boldsymbol{R})$. Let $C_c^\infty(\boldsymbol{R})$ $((C_c^\infty(\boldsymbol{R}))^e)$ denote the subspace of $C^\infty(\boldsymbol{R})$ (respectively $(C^\infty(\boldsymbol{R}))^e$) consisting of the compactly supported functions.

We denote by $(C^\infty(\boldsymbol{R}))'$ the dual of $C^\infty(\boldsymbol{R})$ and equip it with the strong topology. Then every $T \in (C^\infty(\boldsymbol{R}))'$ is a distribution of compact support. Let $\mathscr{E}$ be the dual of $(C^\infty(\boldsymbol{R}))^e$ equipped with the strong topology. Then $\mathscr{E}$ can be identified as a topological vector space with the subspace of compactly supported even distributions on $\boldsymbol{R}$. (A distribution T on $\boldsymbol{R}$ is said to be even if for all $f \in C_c^\infty(\boldsymbol{R})$, $T(f)=T(f^e)$ where $f^e(x)=f(x)+f(-x)/2$).

For $f \in C_c^\infty(\boldsymbol{R})$ (resp. $T \in (C^\infty(\boldsymbol{R}))'$) let $\tilde{f}$ (resp. $\tilde{T}$) denote the usual (Euclidean) Fourier transform of f (resp. T).

Let $C^\infty(G)$ be the space of C^∞-functions on G. A function $f \in C^\infty(G)$ is said to be K-bi-invariant if $f(kxk')=f(x)$ for all $k, k' \in K$ and $x \in G$. $C^\infty(K\backslash G/K)$ will be the space of K-bi-invariant functions in $C^\infty(G)$. Topologise $C^\infty(G)$ by means of uniform convergence along with all derivatives on compacta. $C^\infty(K\backslash G/K)$ will have the relative topology from $C^\infty(G)$. $C_c^\infty(K\backslash G/K)$ denotes the subspace consisting of compactly supported functions. We recall that $C_c^\infty(K\backslash G/K)$ is closed under convolution and that convolution is commutative in $C_c^\infty(K\backslash G/K)$.

E will denote the dual of $C^\infty(K\backslash G/K)$ and will have the strong topology. Then every $T \in E$ is a K-bi-invariant distribution of compact support. (A distribution T on G is K-bi-invariant if $T(f)= T({}^{k_1}f^{k_2})$ for all $f \in C^\infty(G)$ and $k_1, k_2 \in K$ where ${}^{k_1}f^{k_2}(x)=f(k_1xk_2)$ for all $x \in G$.)

The spaces $(C^\infty(\boldsymbol{R}))^e$ and $C^\infty(K\backslash G/K)$ are Frechet-Montel spaces, hence reflexive. Thus the duals of $\mathscr{E}$ and E can be identified with $(C^\infty(\boldsymbol{R}))^e$ and $C^\infty(K\backslash G/K)$ respectively.

Spherical Fourier transform: (See [8] for details.) Let $G = KAN$ be the Iwasawa decomposition of G. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{a}$ the Lie algebra of A, $\mathfrak{a}^*$ the dual of $\mathfrak{a}$ and $\mathfrak{a}_c^*$ the complexification of $\mathfrak{a}^*$. Let ρ denote the half sum of the positive roots for the adjoint action of $\mathfrak{a}$ on g: Since G is of real rank 1, dimension $\mathfrak{a}_c^*=1$ and thus $s \in \mathfrak{a}_c^*$ can be written uniquely as $s=\lambda\rho$ with $\lambda \in \boldsymbol{C}$. Then for $\lambda \in \boldsymbol{C}$ let ϕ_λ denote the elementary spherical function associated with $\lambda\rho \in \mathfrak{a}_c^*$. Again, since G is of real rank 1 observe that $\phi_\lambda = \phi_{\lambda'}$ iff $\lambda = \lambda'$ or $\lambda = -\lambda'$.

For $f \in C_c^\infty(K\backslash G/K)$ define the spherical Fourier transform $\hat{f}$ on $\boldsymbol{C}$ by

$$\hat{f}(\lambda) = \int_G f(x)\phi_{-\lambda}(x)dx\left(= \int_G f(x)\phi_\lambda(x)dx\right)$$

where dx is a fixed Haar measure on G. More generally, if $T \in E$, define the spherical Fourier transform $\hat{T}$ by

$$\hat{T}(\lambda) = T(\phi_\lambda)\ , \quad \lambda \in \boldsymbol{C}\ .$$

Let X be the space of all entire functions h on $\boldsymbol{C}$ which are even, i.e., $h(z) = h(-z)$ for all $z \in \boldsymbol{C}$ and satisfying the growth condition:

$$|h(z)| \leqq Ke^{r|\mathrm{Im}\, z|}(1 + |z|)^n$$

for some $r, K > 0$ and nonnegative integer n. Then we have the following Paley-Wiener type result from [2, Theorem 3].

THEOREM 2.1. *The spherical Fourier transform gives a linear bijection of E onto X.*

Finally, for $f \in C_c^\infty(K\backslash G/K)$, define a function F_f on $\boldsymbol{R}$ by

$$F_f(t) = e^{\rho \log a}\int_N f(an)dn \text{ where } a = \exp t\rho\ .$$

Then it is well known that $F_f \in (C_c^\infty(\boldsymbol{R}))^e$ and the map $f \to F_f$ is an isomorphism of $C_c^\infty(K\backslash G/K)$ onto $(C_c^\infty(\boldsymbol{R}))^e$ and, further, $\hat{f} = \tilde{F}_f$.

Mean periodic functions on $\boldsymbol{R}$: A function $f \in C^\infty(\boldsymbol{R})$, is said to be mean periodic if and only if there exists a nonzero distribution T of compact support such that $T*f = 0$ (where $*$ denotes convolution) or, equivalently, there exists $g \in C_c^\infty(\boldsymbol{R})$, $g \neq 0$, such that $g*f = 0$. It is easy to see that f is mean periodic if and only if the closed linear span of $\{{}^x f; x \in \boldsymbol{R}\}$ is a proper subspace of $C^\infty(\boldsymbol{R})$ where ${}^x f(y) = f(y - x)$ for all $y \in \boldsymbol{R}$.

Examples of mean periodic functions are the functions $F_{\lambda,k}$ where

$$F_{\lambda,k}(x) = i^k x^k \exp(i\lambda x)\ , \quad x \in \boldsymbol{R}$$

for $\lambda \in \boldsymbol{C}$ and k a nonnegative integer. (Schwartz in [11] studies in detail mean periodic functions in $C(\boldsymbol{R})$. However, as he himself points out, these results can be formulated and proved in exactly the same way for $C^\infty(\boldsymbol{R})$, the space of distributions, etc.)

Let V be a proper closed subspace of $C^\infty(\boldsymbol{R})$ such that if $f \in V$ then ${}^x f \in V$ for all $x \in \boldsymbol{R}$ — or, equivalently, if $f \in V$ then $W*f \in V$ for all $W \in (C^\infty(\boldsymbol{R}))'$. Such a V will be called a variety in $C^\infty(\boldsymbol{R})$.

The following theorem is due to Schwartz [11].

THEOREM 2.2. *Let V be a variety and let $f \in V$. Then f is the limit in $C^\infty(\boldsymbol{R})$ of finite linear combinations of functions of the type $F_{\lambda,k}$ with $F_{\lambda,k} \in V$. (Note that if $F_{\lambda,k} \in V$, it can be proved that $F_{\lambda,k'} \in V$ for all $k' \leqq k$.)*

Mean periodic even functions on $\boldsymbol{R}$: We now modify the preceding definitions and results slightly in order to apply them later to K-bi-invariant mean periodic functions on semisimple Lie groups of real rank 1.

DEFINITION 2.1. (a) A function $f \in (C^\infty(\boldsymbol{R}))^e$ is said to be an even mean periodic function if there exists $T \in \mathscr{E}$, $T \neq 0$ such that $T*f = 0$.

(b) A proper closed linear subspace V of $(C^\infty(\boldsymbol{R}))^e$ is said to be a variety if $W*f \in V$ for all $W \in \mathscr{E}$ and $f \in V$.

For $\lambda \in \boldsymbol{C}$ and k a nonnegative integer we define $\psi_{\lambda,k} \in (C^\infty(\boldsymbol{R}))^e$ by

$$\begin{aligned}\psi_{\lambda,k}(x) &= \frac{F_{\lambda,k}(x) + F_{\lambda,k}(-x)}{2} \\ &= \frac{i^k x^k \exp i\lambda x + i^k(-x)^k \exp(-i\lambda x)}{2} .\end{aligned}$$

It is easy to see that $\psi_{\lambda,k}$ is even mean periodic. A minor modification of Theorem 2.2 yields:

THEOREM 2.3. *Let V be a variety in $(C^\infty(\boldsymbol{R}))^e$ and let $f \in V$. Then f can be approximated in $(C^\infty(\boldsymbol{R}))^e$ by finite linear combinations of functions of the type $\psi_{\lambda,k}$ where $\psi_{\lambda,k} \in V$.*

K-bi-invariant mean periodic functions:

DEFINITION 2.2. A function $f \in C^\infty(K\backslash G/K)$ is said to be a spherical (or K-bi-invariant) mean periodic function if there exists $T \in E$, $T \neq 0$ such that $T*f = 0$.

Note that if f is mean periodic then the closure of the subspace $\{W*f;\ W \in E\}$ which will be denoted by V_f in the sequel, is a proper subspace of $C^\infty(K\backslash G/K)$. (The converse of this assertion is also true — as will follow easily from Lemma 3.3 and the corresponding fact for $(C^\infty(\boldsymbol{R}))^e$).

EXAMPLES. (1) ϕ_λ, $\lambda \in \boldsymbol{C}$ is mean periodic in the above sense.

For, let f be any nonzero function in $C_c^\infty(K\backslash G/K)$ such that $\hat{f}(\lambda)=0$ (such functions certainly exist). Now

$$f*\phi_\lambda(x) = \int_G f(y)\phi_\lambda(y^{-1}x)dy .$$

However, using the well known functional equation

$$\int_K \phi_\lambda(x_1kx_2)dk = \phi(x_1)\phi(x_2)$$

the above integral is merely equal to $\phi_\lambda(x)\hat{f}(\lambda)$.

(2) Let $0 \neq f \in C^\infty(G) \cap L^1(K\backslash G/K)$. Then we show that f cannot be mean periodic. As is well known the spherical Fourier transform is defined on a horizontal band containing the real axis. If now $T*f=0$ for a nonzero $T \in E$, then we have $\hat{T}\cdot\hat{f}=0$. Since $\hat{T}$ is entire its zeros on $\boldsymbol{R}$ are isolated. So $\hat{f}=0$ on $\boldsymbol{R}$ and hence $f=0$.

(3) A similar argument shows that if f belongs to any of the Harish-Chandra Schwartz spaces $\mathscr{C}^p(K\backslash G/K)$ (see [12] for definition of these spaces) then f cannot be mean periodic in the sense of Definition 2.2.

DEFINITION 2.3. Let $\lambda \in \boldsymbol{C}$ and k a positive integer. Let

$$\phi_{\lambda,k}(x) = \frac{d^k}{d\lambda^k}\phi_\lambda(x) , \quad x \in G$$

and

$$\phi_{\lambda,0}(x) = \phi_\lambda(x) , \quad x \in G .$$

Following Schwartz we now introduce the concept of spectrum of a mean periodic function.

DEFINITION 2.4. Let $f \in C^\infty(K\backslash G/K)$ be mean periodic. Let V_f denote the closure in $C^\infty(K\backslash G/K)$ of the subspace $\{W*f: W \in E\}$. By spectrum f we mean the collection $\{\phi_{\lambda,k};\ \phi_{\lambda,k} \in V_f\}$.

It will follow from Lemma 3.3 and the corresponding fact about the spectrum of a variety in $(C^\infty(\boldsymbol{R}))^e$ that if $\phi_{\lambda,k} \varepsilon$ spectrum f then $\phi_{\lambda,k'} \varepsilon$spectrum f for all $k' \leqq k$.

3. The main result for groups of real rank 1. As in §2 G will stand for a semisimple Lie group of real rank one. We begin with a proposition which is implicit in the work of Inoue, Okamoto and Tanaka [9].

PROPOSITION 3.1. *There exists a linear topological isomorphism*

S from E onto $\mathscr{E}$ such that for any $f \in C_c^\infty(K\backslash G/K)$ (considered as a K-bi-invariant compactly supported distribution)

$$S(f) = F_f \ .$$

Also, if $\Lambda_1, \Lambda_2 \in E$, then $S(\Lambda_1 * \Lambda_2) = S(\Lambda_1) * S(\Lambda_2)$.

Proof. Given $w \in E$, let $\hat{w}$ denote its spherical Fourier transform. Then $\hat{w} \in X$ (see § 2) and there exists a unique $u \in \mathscr{E}$ such that $\tilde{u} = \hat{w}$ (by the Paley-Wiener theorem for $\boldsymbol{R}$). Define $S(w) = u$. It is easy to see from Theorem 2.1 and the Paley-Wiener theorem for $\boldsymbol{R}$ that S is one-to-one and onto. On X we impose the topology defined by Ehrenpreis (see [3, p. 414]). This makes X and $\mathscr{E}$ topologically isomorphic. The important observation made in [9, Prop. 1] is that for X equipped with this topology, the spherical Fourier transform is an isomorphism of E onto X. Thus it follows that the map S defined above is a topological isomorphism from E onto $\mathscr{E}$. Finally, the fact that $S(f) = F_f$ if $f \in C_c^\infty(K\backslash G/K)$ follows from the equality $\hat{f} = \tilde{F}_f$ (see § 2). The last statement is a consequence of the relations $(w_1 * w_2)^\wedge = \hat{w}_1 \cdot \hat{w}_2$ and $(S(w_1) * S(w_2))^\sim = S(w_1)^\sim S(w_2)^\sim$.

PROPOSITION 3.2. *There exists a linear topological isomorphism T from $C^\infty(K\backslash G/K)$ onto $(C^\infty(\boldsymbol{R}))^e$ such that*

$$S(w)(T(f)) = w(f) \text{ for all } w \in E \text{ and } f \in C^\infty(K\backslash G/K) \ .$$

Further, under this isomorphism

$$T(\psi_{\lambda,k}) = \phi_{\lambda,k}$$

for all $\lambda \in \boldsymbol{C}$ and k nonnegative integer.

Proof. Define T as above. Since $C^\infty(K\backslash G/K)$ and $(C^\infty(\boldsymbol{R}))^e$ are Frechet-Montel spaces, they are reflexive. Hence the duals of E and $\mathscr{E}$ (equipped with the respective strong topologies) are $C^\infty(K\backslash G/K)$ and $(C^\infty(\boldsymbol{R}))^e$ respectively. Since by the previous proposition S is a linear topological isomorphism, T is also a linear topological isomorphism.

For the second assertion, first observe that $T(\phi_\lambda) = \psi_\lambda$. Let $w \in E$. Then $S(w)T(\phi_\lambda) = w(\phi_\lambda)$ by definition of T. However $w(\phi_\lambda) = \hat{w}(\lambda) = \widetilde{S(w)}(\lambda) = S(w)(\psi_\lambda)$.

So, $S(w)(T(\phi_\lambda)) = S(w)(\psi_\lambda)$ for all $w \in E$. Since S is an isomorphism this implies

$$\Lambda(T(\phi_\lambda)) = \Lambda(\psi_\lambda) \text{ for all } \Lambda \in \mathscr{E} \ .$$

Thus $T(\phi_\lambda) = \psi_\lambda$. Next, observe that the function

$$\frac{\psi_{\lambda+h} - \psi_\lambda}{h} \longrightarrow \frac{d}{d\lambda}\psi_\lambda$$

in the topology of $(C^\infty(\boldsymbol{R}))^e$ as $h \to 0$. Since T is a topological isomorphism, it follows that $(\phi_{\lambda+h} - \phi_\lambda)/h$ converges in $C^\infty(K\backslash G/K)$ but clearly it has to converge to $(d/d\lambda)\phi_\lambda$. Hence

$$T\left(\frac{d}{d\lambda}\phi_\lambda\right) = \frac{d}{d\lambda}(\psi_\lambda) .$$

Iteration of the above gives that

$$T(\phi_{\lambda,k}) = \psi_{\lambda,k}$$

for all $\lambda \in \boldsymbol{C}$ and k nonnegative integer.

LEMMA 3.3. *Let $f \in C^\infty(K\backslash G/K)$ and $w, w' \in E$. Then*

$$S(w')*T(w*f) = T(w'*w*f) .$$

Proof. Let $\Lambda \in E$. Then

$$\begin{aligned} S(\Lambda)(Sw'*T(w*f)) &= S(\Lambda)*(S(w')*T(w*f))^\vee(0) \\ &= S(\Lambda)*(T(w*f))^\vee*(S(w'))^\vee(0) \\ &= S(\Lambda)*(S(w'))^\vee*(T(w*f))^\vee(0) \\ &= (\Lambda*w')(w*f) . \end{aligned}$$

Similarly we can show

$$S(\Lambda)(T(w'*w*f)) = (\Lambda*w')(w*f) .$$

Note. (1) However in checking the above one needs to use the fact that if $w \in E$ and $f \in C^\infty(K\backslash G/K)$ then $w*f = f*w$, where the right hand side should be viewed as the convolution of two distributions.

(2) For any function f on a group G by $f^\vee$ we mean the function $f^\vee(g) = f(g^{-1})$ and if T is a distribution, $T^\vee$ is defined by $T^\vee(f) = T(f^\vee)$.

PROPOSITION 3.4. *Let f be mean periodic in $C^\infty(K\backslash G/K)$. Then $T(f)$ is mean periodic in $(C^\infty(\boldsymbol{R}))^e$ and spectrum f (=spectrum V_f)= spectrum TV_f.*

Proof. If f is mean periodic then $V_f \neq C^\infty(K\backslash G/K)$ and hence $TV_f \neq (C^\infty(\boldsymbol{R}))^e$. From Lemma 3.3 we conclude that TV_f is a variety in $(C^\infty(\boldsymbol{R}))^e$. Thus $T(f)$ is mean periodic. The second assertion follows from the definition of spectrum and Proposition 3.2.

We are now in a position to state our main theorem.

THEOREM 3.5. *Let $f \in C^\infty(K\backslash G/K)$ be mean periodic. Then f is in the closed linear span of spectrum of f, that is, f can be approximated in the topology of $C^\infty(K\backslash G/K)$ by finite linear combinations of functions of the type $\phi_{\lambda,k}$ where $\phi_{\lambda,k} \varepsilon$ spectrum f.*

Proof. Immediate from Proposition 3.4 and Theorem 2.3.

Note. (i) One could have studied mean periodic distributions instead of mean periodic functions and one would obtain results analogous to Theorem 3.5.

(ii) As in Schwartz [11] and Ehrenpreis-Mautner [4, p. 52] one can, by means of grouping of terms and Abel convergence factors, represent a given mean periodic function $f \in C^\infty(K\backslash G/K)$ by an infinite series $f \sim \sum d_{\lambda,k}\phi_{\lambda,k}$ where $d_{\lambda,k}$ are constants and $\phi_{\lambda,k} \varepsilon$ spectrum f.

4. **The case of arbitrary rank.** In this section we will drop the assumption on the rank of G. Hence G stands for an arbitrary semisimple Lie group with finite center. As before, K will be a fixed maximal compact subgroup and KAN, $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{a}^*$, $\mathfrak{a}_c^*$, ρ and finally $C^\infty(K\backslash G/K)$ and E will have the same meaning as in § 2 and § 3. (Note that now $\dim \mathfrak{a} = n =$ real rank of $G \geqq 1$.) Let W be the Weyl group of the pair (G, K) (see [8]). Let $e_1, \cdots, e_n$ be an orthonormal basis of $\mathfrak{a}$ with respect to the Killing form B restricted to $\mathfrak{a}$ and $e_1^*, \cdots, e_n^*$ be the dual basis of $\mathfrak{a}_c^*$. Then any $\lambda \in a^*$ can be written uniquely as

$$\lambda = z_1e_1^* + \cdots + z_ne_n^* , \quad z_i \in \boldsymbol{C} , \quad i = 1, \cdots, n .$$

We denote by ϕ_λ the elementary spherical function associated with $\lambda \in \mathfrak{a}_c^*$. Let α be a multi-index, i.e., $\alpha = (\alpha_1, \cdots, \alpha_n)$ where $\alpha_1, \cdots, \alpha_n$ are nonnegative integers. Define $\phi_{\lambda,\alpha}$ to be the function $\partial^{|\alpha|}\phi_\lambda/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$ (where $|\alpha| = \sum_{i=1}^n \alpha_i$).

On the other hand a can be identified with $\boldsymbol{R}^n$ by means of the orthonormal basis $e_1, \cdots, e_n$. Since W acts on a this identification will induce a natural action of W on $\boldsymbol{R}^n$. Let $(C^\infty(\boldsymbol{R}^n))^W$ denote the space of C^∞-functions which are invariant under the action of the Weyl group W. Topologise $(C^\infty(\boldsymbol{R}^n))^W$ as a closed subspace of $C^\infty(\boldsymbol{R}^n)$ with the usual topology. Let $\mathscr{E}$ stand for the strong dual of $(C^\infty(\boldsymbol{R}^n))^W$. (Then $\mathscr{E}$ is really the space of compactly supported distributions on $\boldsymbol{R}^n$ which are invariant under W.) Now using the work in [9] just as in § 3 we can establish the isomorphisms

$$S: E \longrightarrow \mathscr{E}$$

and

$$T: C^\infty(K\backslash G/K) \longrightarrow (C^\infty(\boldsymbol{R}^n))^W$$

such that $S(\Lambda)(T(g)) = \Lambda(g)$, $\Lambda \in E$, $g \in C^\infty(K\backslash G/K)$. Lemma 3.3 would be valid in this set up and an easy application of this lemma together with an approximate identity argument would yield:

$$(*) \qquad S(\Lambda)*T(f) = 0 \text{ if and only if } \Lambda*f = 0 .$$

For a function $f \in C^\infty(\boldsymbol{R}^n)$ define $f^W \in (C^\infty(\boldsymbol{R}^n))^W$ by

$$f^W(x) = \frac{1}{|W|}\sum_{s \in W} f(sx) .$$

For $\lambda \in \boldsymbol{C}^n$ and α a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ let

$$\psi_{\lambda,\alpha} = (F_{\lambda,\alpha})^W$$

where

$$F_{\lambda,\alpha}(x) = \frac{\partial^{|\alpha|}(e^{i\lambda\cdot x})}{\partial\lambda_1^{\alpha_1}\cdots\partial\lambda_n^{\alpha_n}}, \quad x \in \boldsymbol{R}^n .$$

With the identification of α used above it can be shown exactly as in § 3 that $T(\phi_{\lambda,\alpha}) = \psi_{\lambda,\alpha}$ for all $\lambda \in \boldsymbol{C}^n$ and multi-index α. The following theorem is due to Malgrange [10].

THEOREM 4.1. *Let f be a nonzero function in $C^\infty(\boldsymbol{R}^n)$ and T a nonzero distribution in $(C^\infty(\boldsymbol{R}^n))'$ such that $T*f = 0$. Then f can be approximated in $C^\infty(\boldsymbol{R}^n)$ by finite linear combinations of functions of the type $F_{\lambda,\alpha}$ where the $F_{\lambda,\alpha}$ satisfy the convolution equation*

$$T*F_{\lambda,\alpha} = 0 .$$

Just as with Schwartz's main result, we adapt the above to yield the following

THEOREM 4.2. *Let $0 \neq f \in (C^\infty(\boldsymbol{R}^n))^W$ and $0 \neq T \in \mathscr{E}$ such that $T*f = 0$. Then f can be approximated in $(C^\infty(\boldsymbol{R}^n))^W$ by finite linear combinations of functions of the type $\psi_{\lambda,\alpha}$ where the $\psi_{\lambda,\alpha}$ satisfy the convolution equation $T*\psi_{\lambda,\alpha} = 0$.*

In view of the isomorphisms S and T and of $(*)$ Theorem 4.2 translates into the following result (which is a weaker version of Theorem 3.5 when $n = 1$).

THEOREM 4.3. *Let* $0 \neq f \in C^\infty(K\backslash G/K)$ *and* $0 \neq T \in E$ *such that* $T*f = 0$. *Then* f *can be approximated in the topology of* $C^\infty(K\backslash G/K)$ *by finite linear combinations of functions of the type* $\phi_{\lambda,\alpha}$ *where the* $\phi_{\lambda,\alpha}$ *satisfy* $T*\phi_{\lambda,\alpha} = 0$.

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