PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 1, 1979

HARMONIC ANALYSIS ON COMPACT HYPERGROUPS

RICHARD C. VREM

Let K be a compact hypergroup (convo) as defined by R. Jewett. It is shown that Trig (K) is uniformly dense in $C(K)$ and the Peter-Weyl theorem holds. A generalization of the Weil character formula is obtained and a Fourier transform is defined. Analogues of the Riemann-Lebesgue lemma, Parseval's identity and the Riesz-Fischer theorem are proved in this setting. The space $A(K)$ of functions in $L^1(K)$ with absolutey convergent Fourier series is shown to be the linear span of the positive-definite functions on K and the equality $A(K) = L^2(K) * L^2(K)$ is established.

1. Introduction. There has recently been considerable interest shown by some harmonic analysts in the question of which topological spaces have enough structure so that a convolution on the corresponding space of all finite regular Borel measures can be defined. Dunkl [3], Jewett [5] and Spector [10] have all addressed this question and they have given axioms which are essentially the same. Jewett calls these objects convos while Dunkl and Spector refer to them as hypergroups. We shall use the latter terminology but we adopt Jewett's axioms. For a survey of the subject, the interested reader is referred to Ross [8].

This article will be primarily concerned with compact nonabelian hypergroups. In a subsequent paper we will consider lacunarity on compact hypergroups. Throughout this paper K will denote a hypergroup and $M(K)$ the space of finite regular Borel measures on K. In §2 the representation theory of (locally) compact hypergroups is studied. If $K^\wedge$ denotes the set of equivalence classes of continuous irreducible representations of K then it is shown that $K^\wedge$ separates points of K. If K is compact then the elements of $K^\wedge$ are finite-dimensional and an analogue of the Peter-Weyl theorem is obtained. It is also shown that Trig (K) is uniformly dense in the space $C(K)$ of continuous functions on K. §3 contains basic results regarding the Fourier-Stieltjes transform on $M(K)$. It is also shown that $K^\wedge$ will consist of unitary representations precisely when K is a group. The Fourier-Stieltjes series of a regular Borel measure is defined in §4 and the space $A(K)$ of $L^1(K)$ functions with absolutely convergent Fourier series is considered. It is shown that $A(K)$ is the linear span of the positive-definite functions on K and can be written as $L^2(K) * L^2(K)$ (throughout this paper $*$ will refer to the convolution on $M(K)$). Finally, we prove $A(K)$ is a regular Banach algebra under convolution and provide an example to show $A(K)$ need not be a Banach algebra under pointwise operations.

The notation used is that of Jewett [5] except δ_x denotes the point mass at x, $x \to x^{\vee}$ is the involution on K and I_A the indicator function of A. For each representation U in $K^{\wedge}$, H_U is the corresponding Hilbert space and if U is finite-dimensional d_U is the dimension of U. If K admits a Haar measure it will be written m and if K is compact then m is assumed to be suitably normalized.

Finally, the author wishes to thank K. A. Ross for his many helpful suggestions and criticisms.

2. Representation theory. We first assume K is an arbitrary locally compact hypergroup. Following Jewett [5, 11.3] we define a representation of K as a non norm-increasing *-representation of the Banach *-algebra $M(K)$. The representation will be called continuous if it is weak operator continuous on $M^+(K)$ with the cone topology. For notational convenience, we write U_x for U_{δ_x} where $x \in K$. We now give a fundamental example.

EXAMPLE 2.1. Suppose K is a locally compact hypergroup admitting a Haar measure m and let $H = L^2(m)$. Jewett [5, 6.2] shows that the left regular representation T of K on H is a faithful representation of K. We show that T separates the points of K. If $a, b \in K$ with $a \neq b$ then there exist disjoint relatively compact neighborhoods N_1, N_2 of $a^{\vee}$ and $b^{\vee}$ respectively. By [5, 3.2 D] there exist open neighborhoods W_1, W_2 of e so that $\{a^{\vee}\} * W_1 \subseteq N_1$ and $\{b\} * W_2 \subseteq N$. It is easy to see that $T_a(I_{N_1})$, is identically 1 on $V = W_1 \cap W_2$ and $T_b(I_{N_2})$ is identically 0 on V. Thus T separates points.

The proof of the next theorem is modeled after a proof of Nachbin [7].

THEOREM 2.2. *If U is a continuous irreducible representation of a compact hypergroup K then U is finite-dimensional.*

Proof. Fix $\zeta, \lambda \in H$ where $H = H_U$. Let $\zeta \in H$ and define $T(\zeta)$ as the unique vector in H such that

$$\langle T\xi, \eta\rangle = \int_K \langle U_x\xi, \zeta\rangle\langle U_x\eta, \lambda\rangle^- dm(x) \qquad \text{for all} \quad \eta \in H .$$

It is easily shown that $T(\zeta, \lambda)$ is a bounded linear operator on H and that $T(\zeta, \lambda)$ commutes with each U_μ, $\mu \in M(K)$. Thus $T(\zeta, \lambda)$ is scalar, say $T(\zeta, \lambda) = a(\zeta, \lambda)I$. By [5, 7.2A] $m = m^{\vee}$ so that $a(\zeta, \lambda)\langle \xi, \eta\rangle =$

$a(\eta, \xi)\langle\zeta, \lambda\rangle$ and hence $a(\zeta, \lambda) = c\langle\zeta, \lambda\rangle^-$ for some constant c. It follows that

$$(1) \qquad \int_K \langle U_x\xi, \zeta\rangle\langle U_x\eta, \lambda\rangle^- dm(x) = c\langle\xi, \eta\rangle\langle\zeta, \lambda\rangle^- .$$

If we let $\xi = \zeta = \eta = \lambda = \beta$ where $||\beta|| = 1$ then

$$\int_K |\langle U_x\beta, \beta\rangle|^2 dm(x) = c .$$

But the continuous function $x \to |\langle U_x\beta, \beta\rangle|^2$ has value 1 at e so c is positive.

Let $\{\zeta_i\}_{i=1}^n$ be an orthonormal set in H. Let $\zeta = \lambda = \zeta_k$, $1 \leqq k \leqq n$ and $\xi = \eta = \alpha$ in equation (1). Using (1) and the fact that $\{\zeta_k\}$ is an orthonormal set we have

$$nc = \sum_{k=1}^n \int_K |\langle U_x\alpha, \zeta_k\rangle|^2 dm(x) \leqq \int_K ||U_x\alpha||^2 dm(x) \leqq \int_K ||U_x||^2 ||\alpha||^2 \leqq 1 .$$

Hence $\dim(H) \leqq c^{-1}$.

We next want to show there are enough continuous irreducible representations of a locally compact hypergroup to separate points. First we require the following lemma.

LEMMA 2.3. *Let K be a locally compact hypergroup admitting a Haar measure. Let T be a continuous irreducible *-representation of $M(K)$ on $B(H)$ with $T|M_a(K) \neq 0$. Then there is a unique continuous irreducible representation U of K such that $U_\nu = T_\nu$ for all $\nu \in M_a(K)$.*

Proof. Let $\bar{T} = T|M_a(K)$. Since $M(K)$ is a Banach *-algebra [5, 6.1G] we have $||T_\mu|| \leqq ||\mu||$ for all $\mu \in M(K)$. Thus $\bar{T}$ is a bounded *-homomorphism. Suppose $\xi \in H$ and $\bar{T}_\nu(\xi) = 0$ for all $\nu \in M_a(K)$ and let $H_\xi = \{T_\mu(\xi): \mu \in M(K)\}^-$. Since H_ξ is a closed T-invariant subspace of H we have $H_\xi = \{0\}$ or $H_\xi = H$. Using the fact that M_a is an ideal of $M(K)$ and our hypothesis that $\bar{T} \not\equiv 0$ one can show $H_\xi \neq H$. The irreducibity of T then forces $\xi = 0$ and [5, 11.5A] gives the existence of a unique representation U of K such that $U_\nu = T_\nu$ for all $\nu \in M_a(K)$. To show U is irreducible, it suffices to show $\bar{T}$ is irreducible. If X is a closed $\bar{T}$-invariant subspace of H then $(\text{span}\,\bar{T}(X))^-$ is T-invariant since $M_a(K)$ is an ideal. If $(\text{span}\,\bar{T}(X))^- = \{0\}$ it follows that $X = 0$. Since $(\text{span}\,\bar{T}(X))^- \subseteqq X$, $(\text{span}\,\bar{T}(X))^- = H$ implies $X = H$.

THEOREM 2.4. *Let K be a locally compact hypergroup. There* *are enough continuous irreducible representations of K to separate points.*

Proof. By Example 2.1 the regular representation $M(K)$ is faithful so there are enough continuous irreducible *-representations of $M(K)$ to separate points. If $a, b \in K$ with $a \neq b$ then as in Example 2.1 there exists a relatively compact neighborhood W of e so that $\nu = \delta_a * L_W$ and $\mu = \delta_b * L_W$ are supported on disjoint sets. So there exists a continuous irreducible *-representation T of $M(K)$ such that $T_\nu \neq T_\mu$. By Lemma 2.3 there exists a continuous irreducible representation U of K so that $U_\nu \neq U_\mu$, i.e., $U_a \neq U_b$.

COROLLARY 2.5. *If K is a compact hypergroup then there are enough finite-dimensional continuous irreducible representations of K to separate points.*

Proof. This follows from Theorems 2.2 and 2.4.

Unless otherwise stated K will from now on be a compact hypergroup. Suppose $U \in K^\wedge$ and $\{\zeta_j\}_{j=1}^{d_U}$ is an orthonormal basis for H_U. We define coordinate functions for U by $u_{jk}(x) = \langle U_x\zeta_k, \zeta_j\rangle$ where $1 \leqq j,\ k \leqq d_U$. If $\mathrm{Trig}_U(K)$ is the linear span of coordinate functions of U then it is easily seen that $\mathrm{Trig}_U(K)$ is independent of the choice of basis for H_U. $\mathrm{Trig}(K)$ will denote the linear span of $\cup\{\mathrm{Trig}_U(K)\colon U \in K^\wedge\}$.

We next establish orthogonality relations for these coordinate functions.

THEOREM 2.6. *If $U, V \in K^\wedge$ then there exists a constant k_U with $k_U \geqq d_U$ such that*

$$\int_K u_{jk}(v_{rs})^- dm = \begin{cases} k_U^{-1} & \text{if } U = V,\ j = r,\ k = s \\ 0 & \text{otherwise}\ . \end{cases}$$

Moreover, if K is a group then $k_U = d_U$.

Proof. Suppose $U = V$ and $\{\zeta_j\}_{j=1}^{d_U}$ is a fixed orthogonal basis for H_U. Using equation (1) of Theorem 2.2 and the fact that the basis is orthonormal we conclude

$$\int_K u_{jk}\bar{v}_{rs} dm = \begin{cases} c & \text{if } r = j \text{ and } k = s \\ 0 & \text{otherwise}\ . \end{cases}$$

Let $k_U = c^{-1}$. Then $d_U \leqq k_U$ from the last line of the proof of 2.2 and equality occurs when K is a group.

The case where U and V are not equivalent is handled by a standard argument.

COROLLARY 2.7. *The dimension of each* $\mathrm{Trig}_U(K)$ *is* d_U^2. *If fixed coordinate functions are selected for each* $U \in K^\frown$ *then* $\{k_U^{1/2} u_{ij}: U \in K^\frown, 1 \leqq i, j \leqq d_U\}$ *is an orthonormal set in* $L^2(K)$. *Also,* $\mathrm{Trig}(K) = \bigoplus\{\mathrm{Trig}_U(K): U \in K^\frown\}$.

LEMMA 2.8. $M(K)$ *has a nonnegative approximate unit in* $L^2(K)$.

Proof. Use normalized indicator functions of neighborhoods of e and [5, 5.1C].

THEOREM 2.9. $\mathrm{Trig}(K)$ *is dense in* $L^2(K)$.

Proof. Let T denote the regular representation of K on $L^2(K)$. By [5, 7.2C] $L^2(K)$ is the direct sum of its minimal closed ideals and each of these minimal closed ideals is finite-dimensional. Let μ be in $M(K)$ and let $\{t_\alpha\}$ be a bounded nonnegative approximate unit as in Lemma 2.8. If I is a minimal closed ideal of $L^2(K)$ with $f \in I$, then $\mu * f \in L^2(K)$ and hence $t_\alpha * (\mu * f) \to \mu * f$ in $L^2(K)$. Since I is closed, we have $\mu * f \in I$, i.e., I is T-invariant. Hence $T|I$ is a finite-dimensional representation of K which can be written as a direct sum of continuous irreducible subrepresentations, say $L^2(K) = \bigoplus\{H_\beta: \beta \in A\}$. Write $T|H_\beta = T^\beta$ and $d(\beta)$ for the dimensional basis for H_β. Suppose $f \in L^2(K)$ and $\langle f, g\rangle = 0$ for all $g \in \mathrm{Trig}(K)$. Since $T^\beta \in K^\frown$ for each $\beta \in A$ we have

$$0 = \int_K \langle T_x^\beta g_j^\beta, g_i^\beta\rangle \bar{f}(x)dm(x) = \langle T_{\bar{f}} g_j^\beta, g_i^\beta\rangle = \langle \bar{f} * g_j^\beta, g_i^\beta\rangle .$$

Since $\{g_i^\beta: \beta \in A, 1 \leqq i \leqq d(\beta)\}$ is a basis for $L^2(K)$ we have $\bar{f} * h = 0$ for all $h \in L^2(K)$. In particular, $\bar{f} * t_\alpha = 0$ for all α and hence $f = 0$.

The following generalization of the Peter-Weyl theorem for compact groups was known to Spector [10, II. 1.3] (compare with [4, 27.40]).

COROLLARY 2.10. *For* $f \in L^2(K)$ *we have*

$$f = \sum_{U \in K^\frown} \sum_{j,k=1}^{d_U} k_U \langle f, u_{jk}\rangle \cdot u_{jk}$$

where the series is in $L^2(K)$. *Furthermore, if* $\{a_{jk}(U): U \in K^\frown, 1 \leqq j, k \leqq d_U\}$ *is any set of complex numbers such that*

$$\sum_{U \in K^\wedge} \sum_{j,k=1}^{d_U} k_U |a_{jk}(U)|^2 < \infty$$

then there is a unique $g \in L^2(K)$ *such that* $\langle g, u_{jk} \rangle = a_{jk}(U)$ *for all* $U \in K^\wedge$, $1 \leqq j$, $k \leqq d_U$ *and for which* $g = \sum_{U \in K^\wedge} \sum_{j,k=1}^{d_U} k_U a_{jk}(U) u_{jk}$.

Since Trig (K) is not an algebra of functions ([10, II. 1.3]) we cannot apply Stone-Weierstrass. In order to prove Trig (K) is uniformly dense in $C(K)$ we require the following lemmas.

LEMMA 2.11. *Let* $\{h_\alpha\} \subseteqq L^1(K)^+$ *with* $\|h_\alpha\|_1 = 1$ *for all* α. *Then* $\{h_\alpha\}$ *is a left approximate unit in* $L^1(K)$ *if* $\lim_\alpha \|h_\alpha I_{K-W}\|_1 = 0$ *for all neighborhoods* W *of* e. *Moreover,* $h_\alpha \to \delta_e$ *in the weak-* topology.*

Proof. From [5, 5.4H] we have $\lim_{y \to e} \|f_{y^\vee} - f\|_1 = 0$ and [5, 3.3B] shows that $\|f_{y^\vee}\|_1 \leqq \|f\|_1$. The proof that $\{h_\alpha\}$ is a left approximate unit now follows as in [4, 28.52]. A standard argument shows that $h_\alpha \to \delta_e$ in the weak-* topology.

LEMMA 2.12. *There is a bounded left approximate unit* $\{h_\alpha\}$ *in* $L^1(K)$ *such that for all* α:

(i) $h_\alpha \in \mathrm{Trig}^+(K)$

(ii) $\|h_\alpha\|_1 = 1$

(iii) h_α *is a finite sum of functions of the form* $g * g^*$ *where* $g \in \mathrm{Trig}(K)$.

Proof. Let $\{k_W\}$ be the approximate unit described in 2.8. Let $\psi_W = k_W * k_W$. The proof now proceeds as in the group case [4, 28.53] (note that this proof does require Corollary 2.10).

The next theorem answers a question of Dunkl [3, 3.7].

THEOREM 2.13. Trig (K) *is uniformly dense in* $C(K)$.

Proof. Suppose $f \in C(K)$. By Lemma 2.12 there exists a left approximate unit $\{t_\alpha\} \subseteqq \mathrm{Trig}^+(K)$ for $L^1(K)$. If $h \in L^1(K)$ and $f \in C(K)$ it is easy to see that $\|h * f\|_u \leqq \|f\|_u \|h\|_1$ Thus $C(K)$ is a left $L^1(K)$-module. By Lemma 2.11 $t_\alpha \to \delta_e$ in the weak-* topology and hence $t_\alpha * f \to \delta_e * f = f$ uniformly [5, 4.2F]. Thus $L^1(K) * C(K)$ is dense in $C(K)$. By the Cohen Factorization theorem [4, 32.22] there exist $h \in L^1(K)$, $g \in C(K)$ so that $f = h * g$. Now

$$\|t_\alpha * f - f\|_u \leqq \|t_\alpha * h - h\|_1 \|g\|_u \longrightarrow 0$$

and $t_\alpha * f \in \mathrm{Trig}(K)$ so Trig (K) is uniformly dense in $C(K)$.

REMARKS 2.14. (a) For $U \in K^\wedge$ we define $\chi_U(x) = \mathrm{tr}(U_x)$. Then two finite-dimensional representations U and V of K are equivalent if and only if their characters are the same, i.e., $\chi_U = \chi_V$.

(b) If $h \in C(K)$, then $x \to h(x * y * x^\vee * z)$ is continuous on K for each $y, z \in K$; see [5, 3.1B, 3.1G].

We now generalize Weil's character formula for compact groups [4, 27.54].

THEOREM 2.15. *A nonzero continuous function h on K satisfies*

$$(1) \qquad h(y)h(z) = \int_K h(x * y * x^\vee * z)dm(x)$$

if and only if $h(x) = k_U^{-1}\chi_U(x)$ for some $U \in K^\wedge$.

Proof. We first show that $h = k_U^{-1}\chi_U$ satisfies equation (1). Let $U \in K^\wedge$, $d = d_U$ and $\{\zeta_j\}_{j=1}^d$ an orthonormal basis for H_U. By equation (1) of Theorem 2.2

$$\sum_{j=1}^d k_U^{-1}\langle U_y\zeta_i, \zeta_j\rangle\langle U_z\zeta_k, \zeta_k\rangle = \int_K \sum_{j=1}^d \langle U_xU_y\zeta_j, \zeta_k\rangle\langle U_{x^\vee} U_z\zeta_k, \zeta_j\rangle dm(x)$$
$$= \int_K \langle U_xU_yU_{x^\vee} U_z\zeta_k, \zeta_k\rangle dm(x) .$$

Thus

$$k_U^{-1}\chi_U(y)\chi_U(z) = \sum_{k=1}^d \sum_{j=1}^d k_U^{-1}\langle U_y\zeta_j, \zeta_j\rangle\langle U_z\zeta_k, \zeta_k\rangle$$
$$= \int_K \operatorname{tr}(U_xU_yU_{x^\vee} U_z)dm(x)$$

and a straightforward calculation shows that $\operatorname{tr}(U_xU_yU_{x^\vee} U_z) = \chi_U(x * y * x^\vee * z)$ which implies (1) as desired.

Conversely, suppose h satisfies (1). If $U_h = 0$ for all $U \in K^\wedge$ then $0 = \langle h, u_{jk}\rangle$ for all coordinate functions u_{jk} and Corollary 2.10 implies $h = 0$ contrary to hypothesis. Suppose U in $K^\wedge$ satisfies $\int_K U_xh(x)dm(x) \neq 0$. Let $z \in K$ and $g = h^z$. Then

$$h(z)U_h = \int_K \int_K g^x(x^\vee * y)U_ydm(y)dm^\vee(x) .$$

So if $t \in K$ and $\zeta, \eta \in H_U$ then using [5, 5.1D] and Fubini's theorem repeatedly we have

$$\langle U_th(z)U_h\xi, \eta\rangle = \int_K \int_K g^x(y)\langle U_tU_xU_y\xi, \eta\rangle dm(y)dm(x)$$
$$= \int_K \int_K g_y(t^\vee * x)\langle U_xU_y\xi, \eta\rangle dm(x)dm(y)$$
$$= \int_K \int_K g^x(x^\vee * y)\langle U_yU_t\xi, \eta\rangle dm(y)dm(x) = \langle h(z)U_hU_t\xi, \eta\rangle .$$

Since U is irreducible we have $h(z)U_h$ is scalar for all $z \in K$. Since $h \neq 0$, U_h is scalar, say $U_h = \alpha I$. Using equation (1) and [5, 5.1D].

$$h(z)U_h = \int_K \int_K h(y) U_{x^\vee} U_y U_{z^\vee} U_x dm(y)dm(x) = U_h \int_K U_{x^\vee} U_{z^\vee} U_x dm(x)$$

so, in particular, $h(e)I_U = \int_K U_{x^\vee} U_x dm(x)$. If $\xi \in H_U$ with $||\xi|| = 1$ then as in the proof of Theorem 2.2 $d_U k_U^{-1} = \int_K \langle U_x\xi, U_x\xi\rangle dm(x)$ and hence $d_U k_U^{-1} = h(e)$. Now

$$h(z)d_U = \operatorname{tr} \int_K U_{x^\vee} U_{z^\vee} U_x dm(x) = \operatorname{tr} \left(U_{z^\vee} \int_K U_{x^\vee} U_x dm(x)\right) = \chi_U(z^\vee) k_U^{-1} d_U \,,$$

which implies $h(z) = k_{\bar{U}}^{-1} \chi_{\bar{U}}(z)$ where $\bar{U}$ is the conjugate representation of U. Since $\bar{U} \in K^\wedge$ the proof is complete.

3. Fourier transform. The development and notation in this section follows closely that found in Chapter 28 of [4]. We continue to assume K is a compact hypergroup. The *-algebra $\prod_{U \in K^\wedge} B(H_U)$ will be denoted by $\mathscr{E}(K^\wedge)$; scalar multiplication, addition, multiplication and the adjoint of an element are defined coordinatewise. Let $E = (E_U)$ be an element of $\mathscr{E}(K^\wedge)$. For $1 \leqq p < \infty$ we define

$$||E||_p = \left(\sum_{U \in K^\wedge} k_U ||E_U||_{\varphi_p}^p\right)^{1/p} \quad \text{and} \quad ||E||_\infty = \sup \{||E_U||_{\varphi_\infty}\} \,.$$

The norms $||\cdot||_{\varphi_p}$ are the operator norms of [4, D. 37, D. 36(e)] and the notation $\mathscr{E}_p(K^\wedge)$, $\mathscr{E}_{00}(K^\wedge)$ and $\mathscr{E}_0(K^\wedge)$ is as in [4, 28.24].

DEFINITION 3.1. For $\mu \in M(K)$ let $\mu^\wedge(U) = \bar{U}_\mu$ for each $U \in K^\wedge$. Then $\mu^\wedge \in \mathscr{E}(K^\wedge)$ and is called a Fourier-Stieltjes transform of μ. If $f \in L^1(K)$ then $f^\wedge(U) = \bar{U}_f$ and we call $f^\wedge$ a Fourier transform of f.

THEOREM 3.2. *For each $\mu \in M(K)$ the mapping $\mu \to \mu^\wedge$ is a non norm-increasing *-isomorphism of the algebra $M(K)$ into the algebra $\mathscr{E}_\infty(K^\wedge)$.*

Proof. Since $\bar{U} \in K^\wedge$ it is immediate that the map is a *-homomorphism and that $||\mu^\wedge||_\infty \leqq ||\mu||$. If $\bar{U}_\mu = 0$ for all $U \in K^\wedge$ then $\int_K u_{jk} d\mu = 0$ for all coordinate functions u_{jk}. Thus the continuity of the map $f \to \int_K f d\mu$ and 2.13 imply $\int_K f d\mu = 0$ for all $f \in C(K)$ so that $\mu = 0$.

THEOREM 3.3. *The map $f \to f^\wedge$ is a non norm-increasing *-isomorphism of $L^1(K)$ onto a dense subalgebra of $\mathscr{E}_0(K^\wedge)$.*

Proof. Imitate the proof in [4, 28.40].

THEOREM 3.4. *The map $f \to f^\wedge$ is an inner product preserving map of $L^2(K)$ onto $\mathscr{E}_2(K^\wedge)$. In particular, $\|f^\wedge\|_2 = \|f\|_2$. For $f \in L^2(K)$ we have*

$$f = \sum_{U \in K^\wedge} k_U \sum_{j,k=1}^{d_U} \langle f^\wedge(U)\zeta_k^U, \zeta_j^U \rangle u_{jk}$$

where the series converges in the L^2-norm.

Proof. Use Corollary 2.10.

The next theorem and its corollaries show that the notation of unitary representation is appropriate for a compact hypergroup precisely when the hypergroup is in fact a group. Also, these results generalize [3, 2.2] and [8, 3.1].

THEOREM 3.5. *Let K be a compact hypergroup, $U \in K^\wedge$ and T a weak operator closed subgroup of the unitary operators on H_U. Then $S = \{x \in K: U_x \in T\}$ is a closed subhypergroup of K.*

Proof. Clearly $e \in S$ and $S^\vee = S$. We need only show $S * S \subseteq S$. Let $x, y \in S$ and $\xi \in H_U$. Consider

$$\langle \xi, \xi \rangle = \langle U_x U_y \xi, U_x U_y \xi \rangle = \int_K \langle U_z \xi, U_x U_y \xi \rangle d\delta_x * \delta_y(z)$$

and note $|\langle U_z \xi, U_x U_y \xi \rangle| \leq \langle \xi, \xi \rangle$. Also, the map $z \to \langle U_z \xi, U_x U_y \xi \rangle$ is continuous and the support of $\delta_x * \delta_y$ is compact so a straightforward argument shows that $\langle \xi, \xi \rangle = \langle U_z \xi, U_t \xi \rangle$ for all $\xi \in H_U$, $z, t \in$ support $\delta_x * \delta_y$. In particular, choosing $z = t$ it follows that U_Z is unitary for all z in the support of $\delta_x * \delta_y$. Now if $z, t \in$ support $\delta_x * \delta_y$ then $\langle \xi, \xi \rangle = \langle U_{t^\vee} U_z \xi, \xi \rangle$ which implies U is constant on the support of $\delta_x * \delta_y$, i.e., if $z \in$ support $\delta_x * \delta_y$

$$U_z = \int_K U_t d\delta_x * \delta_y(t) = U_x U_y \in T .$$

Thus $S * S \subseteq S$.

COROLLARY 3.6. *If K and U are as in 3.5 then $S = \{x \in K: U_x = I\}$ is a closed subhypergroup of K.*

COROLLARY 3.7. *Let $N = \bigcap_{U \in K^\wedge} \{x \in K: U_x$ is unitary$\}$. Then N is the maximal subgroup of K.*

Proof. Let M denote the maximal subgroup of K. If $x \in M$ then $U_x U_x^* = U_e = I$ so that $M \subseteqq N$. Notice that N is a closed subhypergroup of K by Theorem 3.5. If $x \in N$ and $U \in K^\wedge$, we have $U_{\delta_{x^*x}{}^\vee} = U_x U_{x^\vee} = I = U_e$ and hence $(\delta_x * \delta_x) = \delta_e$. Theorem 3.2 implies $\delta_x * \delta_{x^\vee} = \delta_e$ so that $x \in M$.

4. Functions with absolutely convergent Fourier series. In this section we define the Fourier-Stieltjes series of a measure and study in some detail the set $A(K)$ of those $L^1(K)$ functions with absolutely convergent Fourier series.

DEFINITIONS 4.1. Let $\mu \in M(K)$ and $U \in K^\wedge$. Set $A_U = \mu^\wedge(\bar{U})^*$ and write A for the element (A_U) of $\mathscr{E}(K^\wedge)$. The A_U are called the coefficient operators of μ and the formal expression $\sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U U)$ is called the Fourier-Stieltjes series of μ. If $\mu = f dm$ for some $f \in L^1(K)$ we call $\sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U U)$ the Fourier series of f. If $f \in L^1(K)$, $f \approx \sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U U)$ with $\sum_{U \in K^\wedge} k_U \|A_U\|_{\varphi_1} < \infty$ we say f has an absolutely convergent Fourier series. For $f \in A(K)$ we define $\|f\|_{\varphi_1} = \|f^\wedge\|_1$.

PROPOSITION 4.2. *Let $f \in A(K)$, $f \approx \sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U U)$ Then f is equal* a.e., *to the continuous function $\sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U U_x)$ and so can be regarded as an element of $C(K)$. Also, $\|f\|_U \leqq \|f\|_{\varphi_1}$. Furthermore, the mapping $f \to f^\wedge$ is a norm-preserving linear isomorphism of $A(K)$ onto $\mathscr{E}_1(K^\wedge)$ and so $A(K)$ is a Banach space.*

Proof. The proof here is similar to the group case [4, 34.5, 34.6, 34.7].

We call a complex-valued function f on K positive-definite (p.d.) if f is continuous and $0 \leqq \sum_{i=1}^n \sum_{j=1}^n a_i \bar{a}_j f(x_i * x_j^\vee)$ for each choice of complex numbers a_i and elements $x_i \in K$. We denote the set of p.d. functions by $P(K)$.

LEMMA 4.3. *If $f \in P(K)$ then $\langle f^\wedge(U)\xi, \xi\rangle \geqq 0$ for all $U \in K^\wedge$ and $\xi \in H_U$. In particular,* $\operatorname{tr}(f^\wedge(U)) \geqq 0$ *for all $f \in P(K)$.*

Proof. Clearly, we may assume $\|\xi\| = 1$. Now extend ξ to an orthonormal basis $\{\zeta_j\}$ for H_U where $\xi = \zeta_1$. It follows that $\langle f^\wedge(U)\xi, \xi\rangle = \int_K u_{11} f dm$. However, $u_{11} = k_U^{1/2} u_{11} * k_U^{1/2} u_{11}^*$ which implies

$$\langle f^\wedge(U)\xi, \xi\rangle = \int_K f d(k_U^{1/2} u_{11} m * k_U^{1/2} u_{11}^* m) \geqq 0$$

where the last inequality follows from [5, 11.1A, 11.1B].

The next theorem is instrumental in our characterization of $A(K)$. The proof given here applies Mercer's theorem following a method of Krein [6].

THEOREM 4.4. *If* $f \in P(K)$ *then* $f(e) = \|f\|_u = \sum_{U \in K^\wedge} k_U \operatorname{tr}(f^\wedge(U))$ *where the series converges absolutely.*

Proof. [5, 11.1E] gives $f(e) = \|f\|_u$. Define $J(x, y) = f(y^\vee * x)$ which is continuous by [5, 3.1A]. Now define the operator T_J: $L^2(K) \to L^2(K)$ by $T_J(g)(x) = \int_K J(x, y)g(y)dm(y) = g * f(x)$ for all $g \in L^2(K)$. Since T_J is just right convolution by f, T_J is a bounded linear operator on $L^2(K)$ which is also compact [2, VI. 9.56]. Clearly $J(x, y) = \overline{J(y, x)}$ and $\langle T_J g, g\rangle \geqq 0$ since $f \in P(K)$. Thus T_J satisfies the conditions of Mercer's theorem [2, XI. 8.57, XI. 8.58]. Therefore we may write $J(x, y) = \sum_{i=1}^\infty \lambda_i \Phi_i(x)\overline{\Phi_i(y)}$ where $\{\Phi_i\}_{i=1}^\infty$ is an orthonormal set of eigenfunctions for T_J with corresponding eigenvalue λ_i and the series converges absolutely and uniformly on $K \times K$. We have $\langle \Phi_i, f\rangle = \Phi_i * f(e) = \lambda_i \Phi_i(e)$ and $J(x, y) = f(y^\vee * x)$ so by setting $y = e$ we obtain

$$f(x) = \sum_{i=1}^{\infty} \langle f, \Phi_i\rangle \Phi_i(x)$$

with the series converging absolutely and uniformly on K. For $V \in K^\wedge$ the uniform convergence implies

$$\langle f, v_{rs}\rangle = \sum_{i=1}^{\infty} \langle f, \Phi_i\rangle\langle \Phi_i, v_{rs}\rangle . \tag{1}$$

Since $f, \Phi_i \in C(K)$ we have $f^\wedge, \Phi_i^\wedge \in \mathscr{E}_2(K^\wedge)$ (Theorem 3.4) so that $f^\wedge \Phi_i^\wedge = \lambda_i \Phi_i^\wedge \in \mathscr{E}_1(K^\wedge)$ Proposition 4.2 implies

$$\Phi_i(x) = \sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U(\Phi_i)U_x)$$

with the series converging absolutely and uniformly. Thus

$$f(x) = \sum_{j=1}^{\infty} \langle f, \Phi_j\rangle \sum_{U \in K^\wedge} k_U \operatorname{tr}(A_U(\Phi_j)U_x)$$

and so by equation (1)

$$\begin{aligned} f(e) &= \sum_{j=1}^{\infty} \langle f, \Phi_j\rangle \sum_{U \in K^\wedge} k_U \operatorname{tr}(\Phi_j^\wedge(U)) = \sum_{U \in K^\wedge} \sum_{j=1}^{\infty} \sum_{k=1}^{d_U} \langle f, \Phi_j\rangle k_U \langle \Phi_j, u_{kk}\rangle \\ &= \sum_{U \in K^\wedge} k_U \operatorname{tr}(f^\wedge(U)) . \end{aligned}$$

Finally, Lemma 4.3 shows that the series $\sum_{U \in K^\wedge} k_U \operatorname{tr}(f^\wedge(U))$ converges absolutely.

LEMMA 4.5. *Let K be any locally compact hypergroup. If $f, g \in P(K)$ then $\bar{f} \in P(K)$ and $\alpha f + \beta g \in P(K)$ for all $\alpha, \beta \geqq 0$. Also, the pointwise limit of p.d. functions is p.d.*

Proof. The only statement requiring proof here is the last one. Suppose $f_n \to f$ pointwise with $f_n \in P(K)$. By Theorem 4.4, $\|f_n\|_u = f_n(e)$. A standard argument shows that $\sup \{\|f_n\|_u: n = 1, 2, \cdots\} < \infty$. Since support $(\delta_x * \delta_{y^\vee})$ is compact, the lemma follows easily by an application of Lebesgue's Dominated Convergence theorem.

THEOREM 4.6. *$f \in P(K)$ if and only if $f \in A(K)$ and each A_U is p.d. The condition each A_U is p.d. is equivalent to each operator $f^\wedge(U)$ being p.d.*

Proof. Sufficiency follows from Lemma 4.4 and an argument found in [4, 34.10]. We assume $f \in P(K)$. Lemma 4.3 shows that $f^\wedge(U)$ is p.d. for each $U \in K^\wedge$. Moreover, $\mathrm{tr}\,(f^\wedge(U)) = \|f^\wedge(U)\|_{\varphi_1}$ ([4, D.46]). By Theorem 4.4

$$\sum_{U \in K^\wedge} k_U \|f^\wedge(U)\|_{\varphi_1} = \|f\|_u = f(e) < \infty$$

and hence $f \in A(K)$.

THEOREM 4.7. *$A(K)$ is precisely the linear span of $P(K)$. In fact, every $f \in P(K)$ has the form $f = f_1 - f_2 + i(f_3 - f_4)$ where $f_i \in P(K)$.*

Proof. This follows directly from Theorem 4.6 and [4, D.47].

THEOREM 4.8. *If $f, g \in L^2(K)$ then $f * g \in A(K)$ and $\|f * g\|_{\varphi_1} \leqq \|f\|_2 \|g\|_2$.*

Proof. Use Theorems 3.2, 3.4 and Hölder's inequality.

THEOREM 4.9. *$A(K) = L^2(K) * L^2(K)$.*

Proof. Apply 4.8 and mimic the argument in [4, 34.15].

The next theorem establishes regularity for $A(K)$; compare with [1, 2.9] and [4, 34.21].

THEOREM 4.10. *Let X, Y be disjoint, nonvoid, closed subsets of K. There is a function $f \in A(K)$ such that $f(X) = \{1\}$, $f(Y) = \{0\}$ and $f(K) \subseteqq [0, 1]$.*

Proof. Select a symmetric neighborhood W of e so that $W*W*X \subseteq K - Y$. Let $f = m(W)^{-1}I_W * I_{W*X}$. Clearly f is in $A(K)$ and it is not hard to show f has the desired properties.

REMARKS 4.11. Since $\mathscr{C}_1(K^\wedge)$ is Banach algebra [4, 28.32(v)] it follows that $A(K)$ is a regular Banach algebra with convolution as multiplication and $||\cdot||_{\varphi_1}$ as norm. However, in contrast to the group case [4, 34.18], $A(K)$ may not form a Banach algebra under pointwise operations. In fact, we give an example of a finite abelian hypergroup where $A(K)$ fails to be a pointwise Banach algebra.

EXAMPLE 4.12. Let $K = \{e, a, b\}$ and $K^\wedge = \{1, \chi, \psi\}$ be as in [5, 9.1C]. Since $\chi \in P(K)$ we have $||\chi||_{\varphi_1} = \chi(e) = 1$ but $||\chi^2||_{\varphi_1} = (666/612) > 1$, i.e., $||\chi^2||_{\varphi_1} > ||\chi||_{\varphi_1}||\chi||_{\varphi_1}$. The difficulty here is that the product of p.d. functions need not be p.d.

REFERENCES

1. A. Chilana and K. A. Ross, *Spectral synthesis in hypergroups*, to appear.
2. N. Dunford and J. T. Schwartz, *Linear operators I, II*, Interscience Publishers Inc., 1958 and 1963.
3. C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc., **179** (1973), 331–348.
4. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* II, Springer-Verlag, 1970.
5. R. I. Jewett, *Spaces with an abstract convolution of measures*, Advances in Math., **18** (1975), 1–101.
6. M. G. Krein, *Hermitian-positive kernels on homogeneous spaces, I and II*, Ukrain. Mat. Z., **1** (1949), 64–98. English translation: Amer. Math. Soc. Translation, Ser. 2, Vol. **34** (1963), 69–164.
7. L. Nachbin, *On the finite dimensionality of every irreducible representation of a compact group*, Proc. Amer. Math. Soc., **12** (1961), 11–12.
8. K. A. Ross, *Hypergroups and centers of measure algebras*, Ist. Naz. Alta. Mat. (Symposia Math.), volume **XXXII**, to appear.
9. ———, *Centers of hypergroups*, Trans. Amer. Math. Soc., to appear.
10. R. Spector, *Apercu de la theorie des hypergroupes*, Lecture Notes in Mathematics #497. (Analyse Harmonique sur les Groupes de Lie, Sem. Nancy-Strasbourg 1973–1975), Springer-Verlag.

Received November 19, 1977.

BUCKNELL UNIVERSITY
LEWISBURG, PA 17837