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ON SPACES WHOSE STONE-ČECH COMPACTIFICATION IS OZ

TOSHIJI TERADA

A Tychonoff space X is called Oz if every open subset is z-embedded in X. In this paper we characterize a class of spaces whose Stone-Čech compactifications are Oz. Especially it is shown that for a realcompact Oz-space of countable type, βX is Oz if and only if X is expressed as the union of an extremally disconnected subset and a compact subset.

1. Introduction. All spaces considered here are Tychonoff. A subset S of a space X is z-embedded in X in case each zero-set of S is the restriction to S of a zero-set of X. A space X is called an Oz-space if every open subset of X is z-embedded in X. Perfectly normal spaces and extremally disconnected spaces are Oz. For basic results of Oz-spaces, see [2] and [6]. Especially R. L. Blair [2] showed the following result: A space X is an Oz-space if and only if υX is Oz, where υX is the Hewitt realcompactification of X. However it is unknown whether the Stone-Čech compactification βX of an Oz-space X is Oz.

The purpose of this paper is to characterize a class of spaces whose Stone-Čech compactifications are Oz. As an application of our characterizations it will be shown that both βR and βQ are not Oz, where R is the space of all real numbers and Q is the space of all rational numbers. In § 2, we will show formal characterizations. In § 3, structural characterizations will be studied. For example, it will be shown that for a realcompact Oz-space X of countable type, βX is Oz if and only if X can be expressed as the union of an extremally disconnected subset and a compact subset.

2. Formal characterizations. The following lemmas are basic for our studies.

LEMMA 1 (*R. L. Blair* [2]). *A space X is an Oz-space if and only if every regular closed subset of X is a zero-set in X.*

LEMMA 2. *Let X be a dense subspace of a space Y.*

(1) *If A is a regular closed subset of X, then $\mathrm{Cl}_Y A$ is a regular closed subset of Y.*

(2) *If B is a regular closed subset of Y, then $B \cap X$ is a regular closed subset of X.*

Lemma 2 is well-known. Let U be an open subset of a space X. Then $\beta X - \mathrm{Cl}_{\beta X}(X - U)$ is denoted by U^β in this paper.

LEMMA 3 (*E. G. Skljarenko* [5]). *For any open subset U of a space X, the equality* $\mathrm{Bd}_{\beta X}(U^\beta) = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X U)$ *holds.*

The following lemma is used only once for the proof of Theorem 1.

LEMMA 4 (*D. Rudd* [4]). *For a zero-set Z of a space X the following are equivalent.*

(1) $\mathrm{Cl}_{\beta X} Z$ *is a zero-set of* βX.

(2) *There exists a real-valued continuous function f on X with the following properties*; (a) $Z = f^{-1}(0)$.

(b) *If a subset A of X is completely separated from Z, then* $\inf\{f(a): a \in A\} > 0$.

The following theorem can be established by a routine argument relying on Lemmas 1, 2, and 4.

THEOREM 1. *For an Oz-space X the following are equivalent.*

(1) βX *is Oz.*

(2) *For each regular closed subset A of X there is a sequence $\{U_i: i < \omega\}$ of cozero-sets of X with the following properties*; (a) $A \subset U_i$ *for each* $i < \omega$. (b) *For any cozero-set U of X containing A there is some U_i such that $U_i \subset U$.*

Another formal characterization is given as follows. This characterization is useful for the studies in § 3.

THEOREM 2. *For an Oz-space X the following are equivalent.*

(1) βX *is Oz.*

(2) *For each regular closed subset A of X there is a sequence $\{U_i: i < \omega\}$ of regular open subsets of X with the following properties*; (a) $A \subset U_i$ *for each* $i < \omega$. (b) *For any regular open subset U of X containing A there is some U_i such that $U_i \subset U$.*

Proof. (1)→(2). Let A be a regular closed subset of X. Then by Lemma 2 $\mathrm{Cl}_{\beta X} A$ is a regular closed subset of βX. Hence $\mathrm{Cl}_{\beta X} A$ has a countable neighborhood basis $\{V_i: i < \omega\}$ consisting of regular open subsets of βX since βX is a compact Oz-space. For each $i < \omega$ let $U_i = V_i \cap X$. Then it will be shown that $\{U_i: i < \omega\}$ has the properties (a) and (b). (a) is obviously satisfied. Let U be a regular open subset of X containing A. Then A and $X - U$ are completely separated since A and $X - U$ are regular closed subsets of an Oz-space X. Hence $\mathrm{Cl}_{\beta X}A \subset U^\beta$. Therefore, for some i, $\mathrm{Cl}_{\beta X}A \subset V_i \subset U^\beta$. Thus $U_i \subset U$ for some i. Hence (b) is satisfied.

(2) $\rightarrow$ (1). Let B be a regular closed subset of βX. Then $A = B \cap X$ is a regular closed subset of X. Hence there is a sequence $\{U_i: i < \omega\}$ of regular open subsets of X with the properties (a) and (b). Then it is obvious that $\mathrm{Cl}_{\beta X}A = B = \cap \{U_i^\beta: i < \omega\}$. Hence B is a zero-set of βX since βX is normal. This completes the proof.

COROLLARY 1. *For a normal space X the following are equivalent.*

(1) *βX is Oz.*

(2) *Every regular closed subset of X has a countable neighborhood basis.*

COROLLARY 2. *βR, βQ and $\beta(R - Q)$ are not Oz.*

3. Structural characterizations. A subset S of a space X is called relatively pseudocompact if every real-valued continuous function f on X satisfies the condition that the restriction $f|S$ is bounded.

THEOREM 3. *If βX is Oz, then for any regular closed subset A of X, $\mathrm{Bd}_X A$ is relatively pseudocompact.*

Proof. Let A be a regular closed subset of X. Assume that $\mathrm{Bd}_X A$ is not relatively pseudocompact. Then it will be proved that condition (2) of Theorem 2 is not satisfied. Let $\{U_i: i < \omega\}$ be a sequence of regular open subsets of X containing A. Since $\mathrm{Bd}_X A$ is not relatively pseudocompact, $\mathrm{Cl}_{\beta X}(\mathrm{Bd}_X A) \cap (\beta X - \upsilon X)$ is nonempty. Let y be a point of $\mathrm{Cl}_{\beta X}(\mathrm{Bd}_X A) \cap (\beta X - \upsilon X)$. Then it is obvious that $y \in \mathrm{Cl}_{\beta X}(U_i - A)$ for each $i < \omega$. Since $y \notin \upsilon X$, there is a discrete sequence $\{F_i: i < \omega\}$ of regular closed subsets of X such that $F_i \subset U_i - A$ for each $i < \omega$. Now let $U = X - \cup \{F_i: i < \omega\}$. Then U is a regular open subsets of X containing A. But U contains no member of $\{U_i: i < \omega\}$ by the construction.

COROLLARY 3. *If βX is Oz, then the following hold.*

(1) $\mathrm{ind}(\beta X - \nu X) \leqq 0$.

(2) *For any regular open subset U of νX, $\mathrm{Bd}_{\nu X} U$ is compact.*

A space X is called of countable type if, for any compact subset C of X, there is a compact subset C' such that $C \subset C'$ and C' has a countable neighborhood basis (see [1]).

THEOREM 4. *If υX is of countable type, then the following are equivalent.*

(1) *βX is Oz.*

(2) *For any regular closed subset A of X, $\mathrm{Bd}_X A$ is a relatively pseudocompact zero-set.*

Proof. (1) $\rightarrow$ (2). Since X must be Oz, $\mathrm{Bd}_X A$ is a zero-set for any regular closed subset A of X. Then by Theorem 3 this implication is obvious.

(2) $\rightarrow$ (1). Let B be a regular closed subset of βX. Then $B \cap X$ is a regular closed subset of X. Hence $\mathrm{Bd}_X(B \cap X)$ is a relatively pseudocompact zero-set of X. Since $\mathrm{Bd}_{\beta X} B = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_x(B \cap X))$, $\mathrm{Bd}_{\beta X} B$ is a compact zero-set of υX. By the assumption that νX is of countable type, $\mathrm{Bd}_{\beta X} B$ is a G_δ-set of βX. Thus B is G_δ in βX.

Next, we will show that, in Corollary 1, the normality of X can be replaced by the realcompactness of X.

LEMMA 5. *Let X be a realcompact space and let A be a closed subset of X. If A has a countable neighborhood basis in X, then $\mathrm{Cl}_{\beta X} A$ is a zero-set of βX.*

Proof. Let $\{U_i : i < \omega\}$ be a countable neighborhood basis of A. Assume that $\mathrm{Cl}_{\beta X} A - U_{i_0}^\beta \neq \emptyset$ for some i_0. Then since $\mathrm{Cl}_{\beta X} A \subset (U_{i_0} \cap U_i)^\beta \cup \mathrm{Bd}_{\beta X}((U_{i_0} \cap U_i)^\beta) = (U_{i_0} \cap U_i)^\beta \cup \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X(U_{i_0} \cap U_i)) \subset U_{i_0}^\beta \cup \mathrm{Cl}_{\beta X}(U_i - A)$ for each $i < \omega$, $\mathrm{Cl}_{\beta X} A - U_{i_0}^\beta \subset \mathrm{Cl}_{\beta X}(U_i - A)$ for each $i < \omega$. If we take a point y in $\mathrm{Cl}_{\beta X} A - U_{i_0}^\beta$, then by the same argument in the proof of Theorem 3 it is shown that $\{U_i : i < \omega\}$ is not a neighborhood basis of A in X. This is a contradiction. Thus $\mathrm{Cl}_{\beta X} A \subset U_i^\beta$ for each $i < \omega$. Then it is obvious that $\mathrm{Cl}_{\beta X} A = \cap \{U_i^\beta : i < \omega\}$. Thus $\mathrm{Cl}_{\beta X} A$ is a zero-set of βX.

COROLLARY 4. *Let X be a realcompact space. If every closed subset of X has a countable neighborhood basis, then X is (perfectly) normal.*

THEOREM 5. *For a realcompact space X the following are equivalent.*

(1) *βX is Oz.*

(2) *Any regular closed subset A of X has a countable neighborhood basis in X.*

(3) *For any regular closed subset A of X, $\mathrm{Bd}_X A$ is a compact subset with a countable neighborhood basis in X.*

Proof. (1) $\to$ (3). By Lemma 3, for any regular closed subset A of X, $\mathrm{Cl}_{\beta X}A = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X A) \cup (\mathrm{Int}_X A)^\beta$ and $\mathrm{Cl}_{\beta X}(X-A) = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X(X-A)) \cup (X-A)^\beta = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X A) \cup (X-A)^\beta$. Thus $\mathrm{Cl}_{\beta X}A \cap \mathrm{Cl}_{\beta X}(X-A) = \mathrm{Cl}_{\beta X}(\mathrm{Bd}_X A)$. Therefore $\mathrm{Cl}_{\beta X}(\mathrm{Bd}_X A)$ is G_δ in βX since βX is Oz. By Theorem 3, $\mathrm{Bd}_X A$ is relatively pseudocompact in X. Since X is realcompact, $\mathrm{Bd}_X A$ must be compact. Hence $\mathrm{Bd}_X A$ has a countable neighborhood basis in X.

(3) $\to$ (2). This is obvious.

(2) $\to$ (1). By Lemma 5 it is proved that every regular closed subset of βX is a zero-set of βX.

A space X is called extremally disconnected if the closure of every open subset is open. If X is extremally disconnected or pseudocompact Oz, then βX is Oz (see [2]). Conversely we have the following.

THEOREM 6. *If βX is Oz, then for each discrete sequence $\{U_i: i < \omega\}$ of open subsets of X there exists i_0 such that U_j is extremally disconnected for each $j \geqq i_0$.*

Proof. Assume the contrary. Then there is a subsequence $\{U_{i_k}: k < \omega\}$ of $\{U_i: i < \omega\}$ such that U_{i_k} is not extremally disconnected for each k. For each k let V_k be an open subset of U_{i_k} such that $\mathrm{Cl}_{U_{i_k}} V_k$ is not open. Let $F = \cup\{\mathrm{Cl}_X V_k: k < \omega\}$. Then obviously F is regular closed. But we will show that condition (2) of Theorem 2 is not satisfied. Let $\{W_i: i < \omega\}$ be a sequence of regular open subsets of X containing F. Then, for each k, there is a regular closed subset S_k of X such that $S_k \subset (W_k \cap U_{i_k}) - F$. Let $U = X - \cup\{S_k: k < \omega\}$. Then U is a regular open subset of X which contains no member of $\{W_i: i < \omega\}$.

COROLLARY 5. *If every open subset of a space X is not extremally disconnected, then the following are equivalent.*

(1) *βX is Oz.*

(2) *X is pseudocompact and Oz.*

The fact that βR, βQ and $\beta(R-Q)$ are not Oz follows also from Corollary 5. The following is the main theorem in this section.

THEOREM 7. *Let X be an Oz-space whose Hewitt realcompactification υX is of countable type. Then the following are equivalent.*

(1) *βX is Oz.*

(2) *X is expressed as the union of an extremally disconnected open subset and a relatively pseudocompact (closed) subset.*

Proof. (1) $\to$ (2). Let $\mathscr{U}$ be the family of all extremally disconnected open subsets of X. Then $\mathscr{U}$ is partially ordered by the inclusion relation $\subset$. Let $\mathscr{U}'$ be a linearly ordered subset of $\mathscr{U}$. Then it is not so difficult to see that $\cup\{U: U \in \mathscr{U}'\}$ is also a member of $\mathscr{U}$. Hence by Zorn's lemma there exists a maximal member E of $\mathscr{U}$. Let $P = X - E$. Assume that P is not relatively pseudocompact. Then there is a discrete sequence $\{U_i: i < \omega\}$ of open subsets of X such that $U_i \cap P \neq \emptyset$ for each i. If U_i is extremally disconnected, then $U_i \cup E$ is also extremally disconnected. But this contradicts the maximality of E. Hence each U_i is not extremally disconnected. But this is a contradiction by Theorem 6. Thus P is relatively pseudocompact.

(2) $\to$ (1). Let $X = E \cup P$, where E is an extremally disconnected open subset and P is a closed relatively pseudocompact subset. We will show that for each regular closed subset A of X, $\mathrm{Bd}_X A$ is relatively pseudocompact. Then by Theorem 4 it is true that βX is Oz. It suffices to show that $\mathrm{Bd}_X A \subset P$. This follows from the following observation:

$$\begin{aligned}\mathrm{Bd}_X A &= \mathrm{Cl}_X(\mathrm{Int}_X A) - \mathrm{Int}_X A \\ &= \mathrm{Cl}_X(((\mathrm{Int}_X A) \cap E) \cup ((\mathrm{Int}_X A) \cap P)) - \mathrm{Int}_X A \\ &= (\mathrm{Cl}_X(\mathrm{Cl}_E((\mathrm{Int}_X A) \cap E)) - \mathrm{Int}_X A) \cup (\mathrm{Cl}_X((\mathrm{Int}_X A) \cap P) \\ &\qquad - \mathrm{Int}_X A) \\ &\subset P \cup P \\ &= P\,.\end{aligned}$$

This completes the proof.

COROLLARY 6. *Let X be a realcompact Oz-space of countable type. Then the following are equivalent.*

(1) *βX is Oz.*

(2) *X is expressed as the union of an extremally disconnected subset and a compact subset.*

EXAMPLE. In Theorem 4 and Theorem 7, the assumption that υX is of countable type can not be omitted. In fact, there is a realcompact Oz-space X with the following properties:

(a) $X = E \cup C$, where E is an extremally disconnected subset and C is a compact subset.

(b) βX is not Oz.

Let N be a countably infinite discrete space and let p be a point of $\beta N - N$. Then $N \cup \{p\}$ is realcompact as a subspace of βN. Let X be the quotient space of the topological sum of $N \cup \{p\}$ and the unit interval I obtained by identifying the point p of $N \cup \{p\}$

and the point 0 of I. Then X is realcompact and Oz since X is Lindelöf and perfectly normal. It is also obvious that X can be expressed as the union of a discrete subset and a compact subset. But βX is not Oz since the homeomorphic image of I is a regular closed subset of X which does not have a countable neighborhood basis in X (see Theorem 5).

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UNIVERSITY OF TSUKUBA
SAKURAMURA, JAPAN

Current address: Department of Mathematics
Faculty of Engineering
Yokohama National University
Yokohama, Japan