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INVARIANT MEANS AND ANALYTIC ACTIONS

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Let $T \times D \to D$ be a separately continuous analytic action of a semitopological semigroup T on D, the open unit disk in the complex plane, and let K be a compact T-invariant subset of D. The chief result of this paper is that if $AP(T)$, the space of almost periodic functions on T, has a left invariant mean, then D contains a common fixed-point of T. As a special case, we show that a finite group of analytic self maps of D has a common fixed-point in D.

M. M. Day [1] pioneered the investigation of the relationship between fixed-point properties of affine actions of a semigroup T to the existence of an invariant mean on T.

A. T. Lau [9] obtained the result that if $AP(T)$ has a left invariant mean, then every equicontinuous affine separately continuous action $T \times K \to K$ of a semitopological semigroup T on a compact convex subset K of a locally convex (separated) linear topological space has a common fixed-point in K. (A converse to this result was also obtained in [9].) Variants of this result concerning the fixed-point properties of actions of semitopological semigroups T, when $AP(T)$ has a left invariant mean, were studied later by J. C. S. Wong [14] and by H. D. Junghenn [7]. The results alluded to in this paragraph are united by a common theme in their proofs; they were shown essentially by "pulling back" various spaces of real valued functions on K into the space $AP(T)$.

However, there are certain fixed-point theorems about actions of T, when $AP(T)$ has a left invariant mean, which are not obtained by pull-back arguments alone, but which rely on some special structure theorem of the space on which T acts. As an example, in [10] Lau showed that if $T \times I \to I$ is an equicontinuous action of T on I, the closed unit interval of the real line, then I contains a common fixed-point of T. But the proof of this makes use of a fixed-point theorem of T. Mitchell [11, Theorem 2, p. 149], which in turn rests on the fact that a compact group of continuous self-maps of I has a common fixed-point in I (see [5, 3.24, p. 333]). However, this last fact no longer holds if I is replaced by an arbitrary disk K in n-space, even if the group is required to be finite and Abelian. (For counter-examples, see R. Oliver [12, Theorem 7, p. 174].) Moreover, the result of Lau in [10] no longer holds if I is replaced by such an arbitrary K.

Let f, g be two continuous commuting self-maps of $\bar{D}$, the closed unit disk in the complex plane, and let f, g be analytic on D.

Shields [13] has shown that f and g have a common fixed-point in $\bar{D}$. We would like to generalize Shields' result to a statement of the following type:

(*) Let T be a semitopological semigroup, and let $T \times \bar{D} \to \bar{D}$ be a separately continuous action for which each $t \in T$ is analytic on D. If $AP(T)$ has a left invariant mean, then T has a common fixed-point in $\bar{D}$.

Unfortunately, the statement (*) is false, as the counterexample to Theorem 1 which is given after Corollary 2 will show. We shamelessly rectify this by imposing a new condition in the hypothesis of (*); that there exists a T-invariant compact subset of D. With this new condition, we can now drop the requirement that each $t \in T$ is defined, let alone continuous, on $\bar{D} - D$. However, the modified and now correct statement (Theorem 2 in Section 4) is not proved by pull-back arguments alone, but makes use of a special case, viz: A compact group of analytic self-maps of D has a common fixed-point in D. Since, unlike the case of [5, 3.24, p. 333], we could not find any such result in the literature, we were forced to prove this special case (Lemma 2 in §3) before we could obtain our other results.

2. Preliminaries. A *semigroup* is a set equipped with an associative binary product called the semigroup product. A *topological semigroup* is a semigroup with a Hausdorff topology in which the product st is jointly continuous. A *topological group* is a group having a Hausdorff topology for which the product st^{-1} is jointly continuous.

Let C denote the complex plane, and let D be the open unit disk $D = \{z \in C; |z| < 1\}$. If $A \subseteqq C$, then $\operatorname{Int}(A)$ denotes the set of all interior points of A. The space of all analytic maps $f: D \to C$ is designated by $E(D)$, where $E(D)$ has the (metric) topology of uniform convergence on compact subsets of D. The subspace of all analytic self-maps of D is designated by $H(D)$. When $H(D)$ is given the semigroup product of functional composition, it becomes a topological semigroup in the topology of $E(D)$ [13, Lemma 1, p. 703]. If $S \subseteqq E(D)$, then $\bar{S}$ denotes the closure of S in $E(D)$.

A set $F \subseteqq C$ for which there exist $w \in C$ and $\alpha > 0$ such that $F = \{z \in C; |w - z| \leqq \alpha\}$, is called a closed disk. Let $K \subseteqq D$, where K is compact. Then $\mathscr{F}(K)$ is the family of sets $\mathscr{F}(K) = \{F \subseteqq D; K \subseteqq F$ and F is a closed disk$\}$. The set $CIR(K) = \cap \{F; F \in \mathscr{F}(K)\}$ is called the *circular hull* of K. The set K is said to be *circular hulled* if $K = CIR(K)$. If $SK \subseteqq K$, where $S \subseteqq E(D)$, then K is called *invariant* under S.

3. Fixed-points.

LEMMA 1. *Let $K \subseteq D$, where K is compact.*

(a) *K is circular hulled iff K is the intersection of a family of closed disks in D.*

(b) *$CIR(K)$ is circular hulled.*

(c) *K is circular hulled iff K is the intersection of a family of circular hulled sets in S.*

Proof. (a) If K is circular hulled, then K is the intersection of the family of closed disks $\mathscr{F}(K)$.

Now let $K = \cap\{F_t; t \in I\}$, where F_t is a closed disk in D for each $t \in I$, an index set. Then $K \subseteq F_t$, hence $F_t \in \mathscr{F}(K)$ for each $t \in I$. Thus we get

$$CIR(K) = \cap\{F: F \in \mathscr{F}(K)\} \subseteq \cap\{F_t; t \in I\} = K \subseteq CIR(K) ,$$

which shows (a).

(b), (c). These follow immediately from (a).

LEMMA 2. *Let $G \subseteq H(D)$, where G is a compact topological group. Then D contains a common fixed-point of G. If, in addition, K is a compact G-invariant subset of D, then $CIR(K)$ contains a common fixed-point of G.*

Proof. Since the semigroup product on $H(D)$ is jointly continuous, so is the map $H(D) \times D \to D$ given by $(f, z) \to f(z)$ for $f \in H(D)$, $z \in D$. (To see this, identify each $z \in D$ with the constant z function on D.) For each $z \in D$, it follows by compactness of G that $G(z)$ is a compact G-invariant subset of D.

Let K be a compact G-invariant subset of D; by the paragraph above, such a set K exists. Let e be the identity element of G.

Case 1. The function e is a constant, $e(z) \equiv z_0$. Then $e = eg = g$ for all $g \in G$. For any $k \in K$, we get $z_0 = e(k) \in K \subseteq CIR(K)$, hence the lemma holds trivially for this case.

Case 2. The function e is not a constant. Then $e(D)$ is open by the interior mapping theorem [3, p. 92]. For all $w \in D$, $e(w) = (e \cdot e)(w) = e(e(w))$. So $e(z) = z$ for all $z \in e(D)$, hence for all $z \in D$ by uniqueness of analytic continuation [3, p. 199]. (This part of the proof of Case 2 repeated [13, Lemma 2, p. 704].) Thus e is the identity map on D, so G is a group of analytic bijections of D, hence a group of Möbius transformations of D by [4, 15.1.2, p. 236]. But a Möbius transformation that is analytic on a closed disk $F \subseteq C$

maps F onto another closed disk (see [3, Theorem 3.2.1, p. 51]). So if $F \in \mathscr{F}(K)$ and $g \in G$, then $g(F) \in \mathscr{F}(K)$ by G-invariance of K. Thus g permutes the closed disks $\mathscr{F}(K)$, since G is a group. This yields

$$g(CIR(K)) = g(\cap\{F; F \in \mathscr{F}(K)\}) = \cap\{g(F); F \in \mathscr{F}(K)\}$$
$$= \cap\{F; F \in \mathscr{F}(K)\} = CIR(K)$$

for all $g \in G$, hence $CIR(K)$ is G-invariant, and by Lemma 1(c), also circular hulled.

Let $\mathscr{V}$ be the family of all G-invariant circular hulled (hence compact) subsets of $CIR(K)$ ordered downwards by inclusion. The intersection of any linearly ordered subset of $\mathscr{V}$ is compact, G-invariant, and by Lemma 1(c), circular hulled. By Zorn's lemma, there exists a minimal $P \in \mathscr{V}$.

Suppose P contains more than one point; we shall show this leads to a contradiction. Let $p, q \in P$ where $p \neq q$. Let F_1, F_2 be the two closed disks of unit radius in C whose boundary circles intersect at both p and q. Let L be the lens shaped space $L = F_1 \cap F_2$, and let $F \in \mathscr{F}(P)$. Then $p, q \in P \subseteq F$. Further, the radius α of the closed disk F satisfies $\alpha < 1$, since $F \subseteq D$.

We now need a little plane geometry. If two circles in C of different radii intersect, a major arc of the larger circle cannot lie on or inside the smaller circle. So the portion of the larger circle that lies outside the smaller circle is a major arc. Hence a minor arc of unit radius connecting the two points p and q must lie in the closed disk F. Thus $L \subseteq F$.

Since $L \subseteq F$ for all $F \in \mathscr{F}(P)$, we have

$$L \subseteq \cap\{F; F \in \mathscr{F}(P)\} = CIR(P) = P\,,$$

where the last equality follows from the fact that $P \in \mathscr{V}$. Let $v = (p + q)/2$. Then $v \in \operatorname{Int}(L)$, so $v \in \operatorname{Int}(P)$. Therefore $G(v) \subseteq \operatorname{Int}(P)$, since each $g \in G$ is a homeomorphism and P is G-invariant. But $G(v)$ is compact and G-invariant, so it follows by the first paragraph of Case 2 above that $CIR(G(v))$ is G-invariant and circular hulled. Also, $CIR(G(v)) \subseteq CIR(P) = P$. Let A be the smallest closed disk with center at zero such that $P \subseteq A \subseteq D$. Since $G(v) \subseteq \operatorname{Int}(P) \subseteq \operatorname{Int}(A)$, the compact $G(v)$ is bounded away from the boundary circle of A. Thus there exists B, an even smaller closed disk than A, such that B has center at zero, and $G(v) \subseteq B \subseteq \operatorname{Int}(A)$. Clearly, $B \in \mathscr{F}(G(v))$ but $B \notin \mathscr{F}(P)$. Thus $CIR(G(v)) \neq CIR(P) = P$, hence $CIR(G(v))$ is a proper G-invariant circular hulled subset of P, a contradiction of the minimality of $P \in \mathscr{V}$, which proves Lemma 2.

We remark that some restriction must be placed on the group

$G \subseteq H(D)$ to guarantee the existence of a common fixed-point, since the family of all analytic homeomorphisms of D serves as a well known example of a group with no common fixed-points (see [4, Theorem 15.1.2, p. 236]). Although the compactness condition on G cannot be dropped, it can be relaxed. By the use of Lemma 2, it can be shown that a group $G \subseteq H(D)$ has a common fixed-point in D iff $\bar{G} \subseteq H(D)$, but we shall not need this in what follows. (Recall that $\bar{G}$ is the closure of G in $E(D)$.)

COROLLARY 1. *Let G be a finite group of analytic self-maps of D. Then D contains a common fixed-point of G.*

Proof. G is compact.

We suspect that Corollary 1 is known, but we do not know of an explicit reference.

The next result is a complex analogue of a theorem of T. Mitchell [11, Theorem 2, p. 149] concerning common fixed-points of equicontinuous self-maps of the closed unit interval. A semigroup is called *left reversible* if for every pair of elements $a, b \in S$, there exists a pair $c, d \in S$ such that $ac = bd$.

THEOREM 1. *Let S be a left reversible semigroup of analytic self-maps of D, and let K be a compact S-invariant subset of D. Then $CIR(K)$ contains a common fixed-point of S.*

Proof. By [4, Theorem 15.2.3, p. 246], S is a normal family of analytic maps so $\bar{S}$ is a compact subset of $E(D)$. Each $s \in S$ maps D into D, so each $t \in \bar{S}$ maps D into $\bar{D}$, the closure of D in C.

Suppose that $\bar{S} \not\subseteq H(D)$; then there exists $t_0 \in \bar{S}$ for which $t_0(D) \not\subseteq D$. But each nonconstant function $t \in \bar{S}$ maps D into D by the interior mapping theorem [3, p. 92]. Hence $t_0(z) \equiv z_0$, where $z_0 \notin D$, for all $z \in D$. There exists a sequence $s_n \in S$ such that $s_n \to t_0$ in the topology of $E(D)$. Thus for each $k \in K$, $s_n(k) \to z_0$, a contradiction, since $s_n(k) \in K$ and $z_0 \notin K$. Hence $\bar{S} \subseteq H(D)$, so $\bar{S}$ is a compact topological semigroup.

We now repeat an argument used in the proof of [11, Theorem 1, p. 147]. Let $a, b \in \bar{S}$, then there exist sequences a_m and b_m in S such that $a_m \to a$ and $b_m \to b$. By left reversibility of S, there exist sequences c_m and d_m in S such that $a_m c_m = b_m d_m$. But $\bar{S}$ is compact, so there exist subsequences c_n and d_n for which $c_n \to c$ and $d_n \to d$ for some $c, d \in \bar{S}$. Hence

$$\begin{aligned} ac &= (\lim a_n)(\lim c_n) = \lim (a_n c_n) \\ &= \lim (b_n d_n) = (\lim b_n)(\lim d_n) = bd \ , \end{aligned}$$

where the second and fourth equalities hold by virtue of the joint continuity of the product in $\bar{S}$. Thus $\bar{S}$ is left reversible.

An induction argument shows that if $\{a_1, a_2, \cdots, a_r\}$ is any finite subset of $\bar{S}$, there exists a finite subset $\{b_1, b_2, \cdots, b_r\}$ of $\bar{S}$ such that $a_1b_1 = a_2b_2 = \cdots = a_rb_r$. Hence

$$\cap\{a_j\bar{S}; j = 1, \cdots, r\} \supseteqq \cap\{a_jb_j; j = 1, \cdots, r\} = \{a_1b_1\} \neq \varnothing .$$

Thus the family $\{a\bar{S}; a \in \bar{S}\}$ of compact, hence closed, subsets of $\bar{S}$ has the finite intersection property, so $\cap\{a\bar{S}; a \in \bar{S}\} \neq \varnothing$ by compactness of $\bar{S}$. But each right ideal of $\bar{S}$ contains some principal right ideal $a\bar{S}$; and the intersection of a family of right ideals of $\bar{S}$ is, if nonempty, also a right ideal of $\bar{S}$. Therefore $\cap\{a\bar{S}; a \in \bar{S}\}$ is the *unique* minimal right ideal of $\bar{S}$. By [5, Theorem 1, p. 57], it follows that $\bar{S}$ contains a compact group G for which $\bar{S}G \subseteqq G$. In particular, if e is the identity element of G, then $se \in G$ for all $s \in S$.

For any $g \in G$, there exists a sequence $s_n \in S$ such that $s_n \to g$. But for each $k \in K$, $s_n(k) \in K$, therefore $g(k) = \lim (s_n(k)) \in K$. Hence K is G-invariant, so $CIR(K)$ contains a common fixed-point p of G by Lemma 2. Thus for any $s \in S$, we have $s(p) = s(e(p)) = (se)(p) = p$, which proves Theorem 1.

COROLLARY 2. *Let f be an analytic self-map of D, and let K be a compact f-invariant subset of D. Then $CIR(K)$ contain a fixed-point of f.*

Proof. The semigroup S generated by f is Abelian, hence left reversible.

We note that the fixed-point whose existence is asserted in Theorem 1 is unique if S contains an s which is not the identity map on D. For in such a case, if s is an analytic homeomorphism of D onto D, and D contains a fixed-point of s, then the fixed-point in D is unique by [13, Lemma 5, p. 705]. On the other hand, if s is an analytic self-map of D, but not a homeomorphism onto D, then there exists $z_0 \in \bar{D}$ such that $s^n(z) \to z_0$ for all $z \in D$ by [13, p. 705]. So if s has a fixed-point $p \in D$, then we have $p = s^n(p) \to z_0$, thus $p = z_0$, hence p is unique.

If in Theorem 1, we drop the assumption that there exists an S-invariant compact subset $K \subseteqq D$, then D need not contain a common fixed-point of S. Even if one also adds the requirement that each $s \in S$ is a continuous self-map of $\bar{D}$ and is analytic on D, a left reversible semigroup S need not have a common fixed-point in $\bar{D}$. The group of all analytic bijections of D, extended continuously to $\bar{D}$, serves as an easy counterexample.

Similarly, if we let $K = \{-1/2, 1/2\}$, $s_1(z) = -1/2$, and $s_2(z) = 1/2$ for all $z \in D$, then the semigroup $S = \{s_1, s_2\}$ has no common fixed-point in D, let alone in $CIR(K)$, which shows that one cannot entirely drop the assumption in Theorem 1 that S is left reversible.

4. **Actions of semitopological semigroup.** A. T.-M. Lau [9, Theorems 3.2, 4.1], [10, p. 381] has shown that three fixed-point theorems that were known to hold for left reversible semigroups could be extended to the case of actions of semitopological semigroups, S, for which the space $AP(S)$, the space of almost periodic functions on S, has a left invariant mean. (In [9], Lau also obtained converses to the extended forms of two of these theorems.) Lau's work suggests to us that Theorem 1 may also be generalizable to actions of such semitopological semigroups, which in fact, it is. We need first some terminology.

A *semitopological semigroup* is a semigroup with a Hausdorff topology in which the product st is separately continuous. If T is a semitopological semigroup, $CB(T)$ will denote the space of continuous bounded real-valued functions on T, where $CB(T)$ is given the supremum norm. A function $h \in CB(T)$ is *almost periodic* if $\{\ell_a h; a \in T\}$ is relatively compact in $CB(T)$, where $(\ell_a h)(s) = h(as)$ for all $a, s \in T$. Then $AP(T)$, the space of all almost periodic functions $h \in CB(T)$, is a norm closed linear subspace of $CB(T)$ which contains the constant functions, and which satisfies $\ell_s(AP(T)) \subseteqq AP(T)$ for all $s \in T$ (see [2, p. 80]).

Let W be any norm closed linear subspace of $CB(T)$ that contains the constant functions, and satisfies $\ell_s(W) \subseteqq W$ for all $s \in T$. A *mean* on W is an element $\mu \in W^*$, the continuous dual of W, for which $\|\mu\| = 1$ and $\mu(h) \geqq 0$ whenever $h \in W$ and $h \geqq 0$. If a mean μ on W satisfies $\mu(\ell_s h) = \mu(h)$ for all $s \in T$ and all $h \in W$, then μ is a LIM (*left invariant mean*) on W.

An *action* of a topological semigroup T on D is a map $\theta: T \times D \to D$ for which $\theta(t_1 t_2, z) = \theta(t_1, \theta(t_2, z))$ for all $t_1, t_2 \in T$ and all $z \in D$. The action is *separately continuous* if θ is continuous in each variable when the other variable is held fixed. For an action $T \times D \to D$, define a map $\lambda: T \to D^D$ by $(\lambda t)(z) = \theta(t, z)$ for $t \in T$, $z \in D$. The action is *analytic* if $\lambda: T \to H(D)$, a subset $K \subseteqq D$ is *T-invariant* if $(\lambda T)K \subseteqq K$, and an element $p \in D$ is a *common fixed-point* of T if $(\lambda T)(p) = \{p\}$.

The extension from Theorem 1 to Theorem 2 below is based upon the known result, implicit in the work of Deleeuw and Glicksberg in [2], that if there exists a continuous homomorphism from a semitopological semigroup T onto a dense subset of a compact topological semigroup S, and if $AP(T)$ has a LIM, then S is left reversible. This result is obtained by collecting four items from [2]; these items are explicitly cited in the proof below.

THEOREM 2. *Let $T \times D \to D$ be a separately continuous analytic action of a semitopological semigroup T on D, and let K be a compact T-invariant subset of D. If $AP(T)$ has a* LIM, *then $CIR(K)$ contains a common fixed-point of T.*

Proof. For $t_1, t_2 \in T$, we have

$$(\lambda(t_1t_2))(z) = \theta(t_1t_2, z) = \theta(t_1, \theta(t_2, z)) = (\lambda t_1)(\theta(t_2, z)) \\ = (\lambda t_1)((\lambda t_2)(z)) = ((\lambda t_1)(\lambda t_2))(z)$$

for all $z \in D$. Thus $\lambda(t_1t_2) = (\lambda t_1)(\lambda t_2)$, so λ is a homomorphism of T into $H(D)$.

Let $t \in T$, and let $\{t_\gamma\}$ be a net in T for which $t_\gamma \to t$. Then for all $z \in D$,

$$\lim ((\lambda t_\gamma)(z)) = \lim (\theta(t_\gamma, z)) = \theta(t, z) = (\lambda t)(z) ,$$

thus λt_γ converges pointwise on elements of D to λt. In order to show that the homomorphism $\lambda: T \to H(D)$ is continuous, we must show that λt_γ converges uniformly on compact subsets of D to λt. Denote λT by S, then a repetition of the argument used in the first two paragraphs of the proof of Theorem 1 yields that $\bar{S}$ is a compact topological semigroup. On $H(D)$, the topology of pointwise convergence on elements of D is Hausdorff and is weaker than the topology of uniform convergence on compact subsets of D. But $\bar{S}$ is compact in the second topology, so the two topologies coincide on $\bar{S}$ by [8, Theorem 8, p. 141], hence λ is a continuous homomorphism of T onto S, a dense subset of the compact topological semigroup $\bar{S}$.

Let $k \in K$, then $Sk \subseteq K$ by hypothesis. Since the map $\bar{S} \to K$, given by $s \to sk$, is continuous, we get that the set $Z = \{s \in \bar{S}; sk \in K\}$ is a closed subset of $\bar{S}$ that contains S, so $Z = \bar{S}$, thus K is $\bar{S}$-invariant. But $AP(T)$ has a LIM, hence so does $CB(\bar{S})$, by [2, Lemma 5.2, p. 81] and [2, Lemma 2.10, p. 71], therefore the compact topological semigroup $\bar{S}$ is left reversible by [2, Corollary 2.4, p. 67] and [2, Lemma 2.8, p. 70]. It follows by Theorem 1 that $\bar{S}$, hence S, has a common fixed-point in $CIR(K)$, which proves Theorem 2.

It is known (see Holmes and Lau [6, Corollary 1, p. 333] and Lau [9, Theorem 4.1, p. 74]) that every left reversible semitopological semigroup T has the property that $AP(T)$ has a LIM. We also remark that there exist semitopological semigroups T which are not left reversible but for which $AP(T)$ has a LIM (see [6, p. 335]), hence Theorem 2 is a proper generalization of Theorem 1.

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