PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 1, 1980

TIME VARYING LINEAR DISCRETE-TIME SYSTEMS: II: DUALITY

BOSTWICK F. WYMAN

Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

This paper treats questions of duality for time-varying linear systems defined on a locally finite partially ordered time set. Difference equations are studied by considering the state space of the system as a module over the incidence algebra of the poset, and dual systems can be described abstractly. The resulting dual system gives the evolution equations (in reverse time) for the Lagrange multipliers associated to standard linear-quadratic optimal control problems.

1. Introduction.

Our goal is to study duality aspects of the algebraic theory of time-varying, discrete-time linear dynamical systems. In the most familiar case of systems on the line, these systems are given by

$$\begin{aligned} x(t) &= x(t-1)F(t) + u(t)G(t) \\ y(t) &= x(t)H(t) \end{aligned} \tag{1.1}$$

where X, U, and Y are vector spaces over a field k and $F(t)$, $G(t)$, $H(t)$ are time-varying matrices. The standard dual of system (1.1) is given by

$$\begin{aligned} \xi(t) &= F(t)\xi(t+1) + H(t)\eta(t) \\ \varphi(t) &= G(t)\xi(t) \end{aligned} \tag{1.1*}$$

with $\xi(t)$ in $X^* = \mathrm{Hom}_k(X, k)$, $\eta(t)$ in Y^* and $\varphi(t)$ in U^*. (Compare [9, p. 263]). Duality is important for some problems of optimal control (discussed in §3 below) and estimation and filtering theory (not treated in this paper).

We treat here a generalization of these ideas to linear systems defined over a locally finite partially ordered time set $(T, \leqq)$. See [13] for motivation and examples and [14] for details. We include a partial summary for reference.

Suppose given a partially ordered set ("poset")$(T, \leqq)$ which is *locally finite* in the sense that segments $[s, t] = \{r : s \leqq r \leqq t\}$ are finite sets. Let k be a fixed field of scalars and denote by $\mathscr{A}$ the *incidence algebra* of T with coefficients in k [10, 4]. An incidence function α in $\mathscr{A}$ is a partial function $\alpha: T \times T \to k$ with $\alpha(s, t)$ defined only for $s \leqq t$ in T. Addition in $\mathscr{A}$ is functional addition, and multiplication in $\mathscr{A}$ is *convolution*:

$$(\alpha\beta)(s, t) = \sum_{r: s \leqq r \leqq t} \alpha(s, r)\beta(r, t) .$$

Next, we define a subring of $\mathscr{A}$ which is suitable for our applications to system theory.

DEFINITION 3.1. Let T be a locally finite poest with incidence algebra $\mathscr{A}$. An incidence function α in $\mathscr{A}$ is said to have *finite memory* if for every t in T,

$$\{s: s \leqq t \quad \text{and} \quad \alpha(s, t) \neq 0\}$$

is a finite set. The subring of all functions with finite memory is denoted by $\mathscr{A}_f$ or $\mathscr{A}_f(T)$ and is called the *finite memory algebra* of T.

Let $\mathscr{K} = k^T = \{a: T \to k\}$ be the ring of k-valued time functions with the usual definitions of addition and multiplication. The embedding $\mathscr{K} \subset \mathscr{A}_f$ given by

$$a(s, t) = \begin{cases} a(t) & \text{if} \quad s = t \\ 0 & \text{otherwise} \end{cases}$$

makes $\mathscr{K}$ into a subring of $\mathscr{A}_f$. The ring $\mathscr{K}$ is of course commutative, but it does not lie in the center of $\mathscr{A}_f$ since $(\beta a)(s, t) = \beta(s, t)a(t)$, while $(a\beta)(s, t) = a(s)\beta(s, t) = \beta(s, t)a(s)$, for a in $\mathscr{K}$.

We will be mostly concerned with right $\mathscr{A}_f$-modules $\mathscr{X}$, called *dynamical*, which have the following special form:

(a) $\mathscr{X} \cong \prod_{t \in T} X_t$.

(b) For each $s \leqq t$, there exists a k-linear map $\Phi(s, t)$: $X_s \to X_t$ such that for all $\alpha \in \mathscr{A}_f$:

$$(x\alpha)(t) = \sum_{s \leqq t} x(s)\Phi(s, t)\alpha(s, t) .$$

(c) The maps $\Phi(s, t)$ satisfy the following "coherence" assumptions:

1. $\Phi(t, t) = I$, the identity of X_t, for all $t \in T$.
2. $\Phi(s, r)\Phi(r, t) = \Phi(s, t)$ whenever $s \leqq r \leqq t$.

It is easy to check that if e_t in $\mathscr{A}_f$ is defined by $e_t(u, v) = 1$ for $u = v = t$, else 0, then $X_t = \mathscr{X} e_t$. The space X_t is called the *local state space* at time t, and $\Phi(s, t)$ is the *state-transition matrix*.

A *linear system on the poset* T consists of $\mathscr{K}$-modules $\mathscr{U} = U^T$ and $\mathscr{Y} = Y^T$ (U and Y are vector space over k), together with a dynamical state module $\mathscr{X}$ and $\mathscr{K}$-linear maps G: $\mathscr{U} \to \mathscr{X}$, H: $\mathscr{X} \to \mathscr{Y}$. A system on T gives an *input/output diagram* of right $\mathscr{A}_f$-modules

$$\begin{tikzcd} \Omega\mathscr{U} \arrow[rr, "f"] \arrow[dr, "\tilde{G}"'] & & \Gamma\mathscr{Y} \arrow[dl, "\tilde{H}"] \\ & \mathscr{X} & \end{tikzcd}$$

where $\Omega\mathscr{U}=\mathscr{U}\otimes_{\mathscr{X}}\mathscr{A}_f$ and $\Gamma\mathscr{Y}=\mathrm{Hom}_{\mathscr{X}}(\mathscr{A}_f, \mathscr{Y})$. This construction, introduced in [12], generalizes R. E. Kalman's algebraic treatment of classical transfer functions and z-transforms ([6]). Note that Ω and Γ are right—and left—adjoints to the forgetful functor:

$$((\mathrm{Mod}\text{-}R)) \longrightarrow ((\mathrm{Mod}\text{-}\mathscr{X})) .$$

Our treatment in this paper is somewhat similar to the earlier work of Arbib and Manes [2] on the realization and duality of time-varying systems on $(\boldsymbol{Z}, \leqq)$. Since their work is based on their extensive categorical theory of machines (see e.g., [1, 8]), it has more general applicability then the present set-up, including nonlinear and group machines. Arbib and Manes also remark that their theory can be generalized to more complex underlying sets by introducing functor categories [2, p. 1267], and this generalization would include the locally finite posets discussed here.

On the other hand, the restriction to linear systems and the use of incidence algebras allows a more detailed study. In many cases, only a few of which are included here, explicit calculations are possible. In particular, the relationship to combinatorics and the mobius function seems to be useful and somewhat unexpected, and have not appeared in the more general categorical theory so far.

2. Dual modules. Suppose given a locally finite poset $(T, \leqq)$, base field k, and finite memory algebra $\mathscr{A}_f(T)$ with coefficients in k. Since a *dual system* as defined below will evolve in reverse time, we consider the *opposite poset* $(T^{op}, \leqq^{op})$ given by $T^{op} = T$(as sets) and $t \leqq^{op} s$ if and only if $s \leqq t$. The full incidence algebra $\mathscr{A}(T^{op}, \leqq^{op})$ is isomorphic to the opposite ring $\mathscr{A}(T, \leqq)^{op}$ (where $\alpha\cdot\beta$ in $\mathscr{A}^{op}$ is $\beta\alpha$ in $\mathscr{A}$).

One definition of dual system would involve right modules over the finite memory algebra of $(T^{op}, \leqq^{op})$. However, we prefer to consider functions defined on $(T, \leqq)$ exclusively, and consider left modules instead.

DEFINITION 2.1. The *finite predictor algebra* of the poset $(T, \leqq)$ is the subring of $\mathscr{A}(T)$ consisting of all α such that for all s in T

$$\{t: s \leqq t \quad \text{and} \quad \alpha(s, t) \neq 0\}$$

is finite.

To simplify notation, from now on let $R = \mathscr{A}_f(T)$ be the finite memory algebra of T and R^0 be the finite predictor algebra of T. (Note that R^0 is usually different from R^{op}. The opposite ring will not be used in this paper.)

Our goal is to define a duality theory $\mathscr{X} \leftrightarrow \mathscr{X}^*$, where $\mathscr{X}$ is a right R-module. On the $\mathscr{K}$-module level we can take $\mathscr{X}^* = \mathrm{Hom}_{\mathscr{K}}(\mathscr{X}, \mathscr{K})$, or equivalently $\mathscr{X}^* = \mathrm{Hom}_R(\mathscr{X}, \Gamma\mathscr{K})$ (where Hom(,) indicates right module homomorphisms, and $\Gamma\mathscr{K} = \mathrm{Hom}_{\mathscr{K}}(R, \mathscr{K})$). Since R is noncommutative, there is no straightforward way to put a module structure on $\mathscr{X}^*$, and we are led to the following constructions. For s, t in T, denote by e_{st} the incidence function $e_{st}(u, v) = 1$ for $s = u$, $t = v$, else 0.

PROPOSITION 2.2. *Let $\mathscr{K} = k^T$. Then $\Gamma\mathscr{K} = \mathrm{Hom}_{\mathscr{K}}(R, \mathscr{K})$ is a left R^0-module under the action*

$$(\alpha f)(\theta)(s) = \sum_{t:s\leqq t} \alpha(s, t) f(\theta e_{st})(t) ,$$

with α in R^0, f in $\Gamma\mathscr{K}$, and θ in R.

Proof. The sum is finite since α is a finite predictor. We must verify that αf is right $\mathscr{K}$-linear (by definition of $\mathrm{Hom}_{\mathscr{K}}$). Additivity is clear. Let a be in $\mathscr{K}$. Then for any s in T

$$(\alpha f)(\theta a)(s) = \sum_t \alpha(s, t) f(\theta a e_{st})(t) .$$

Since f is right R-linear, we need a trick. An easy convolution shows that if b_t in $\mathscr{K}$ is defined by $b_t(t) = a(s)$, and $b_t(u)$ arbitrary for $u \neq t$, then $a e_{st} = e_{st} b$. This gives

$$(\alpha f)(\theta a)(s) = \sum_t \alpha(s, t) f(\theta e_{st} b_t)(t) = \sum_t \alpha(s, t) f(\theta e_{st})(t) b_t(t) ,$$

since f is right R-linear. But $b_t(t) = a(s)$, for all t, and by definition of αf, we have

$$(\alpha f)(\theta a)(s) = (\alpha f)(\theta)(s) \cdot a(s) ,$$

and since s is arbitrary, this gives $(\alpha f)(\theta a) = (\alpha f)(\theta) a$ is required.

Next we have to that $f \to \alpha f$ is a left R^0-module action, and the only difficulty is $\alpha(\beta f) = (\alpha\beta) f$. We have, for any s in T,

$$\begin{aligned}
\alpha(\beta f)(\theta)(s) &= \sum_t \alpha(s, t)(\beta f)(\theta e_{st})(t) \\
&= \sum_t \alpha(s, t) \sum_u \beta(t, u)(\theta e_{su})(u) \\
&= \sum_u (\alpha\beta)(s, u)(\theta e_{su})(u) \\
&= (\alpha\beta) f(\theta)(s) ,
\end{aligned}$$

where we used the definition of $\alpha\beta$ and the fact that $e_{st}e_{tu} = e_{su}$ for $s \leqq t \leqq u$.

Now we are ready to define the dual of a right R-module $\mathscr{X}$.

DEFINITION 2.3. The *dual* of a right R-module $\mathscr{X}$ is the left R^0-module $\mathscr{X}^* = \mathrm{Hom}_R(\mathscr{X}, \Gamma\mathscr{K})$, where for $\tilde{\xi}: \mathscr{X} \to \Gamma\mathscr{K}$ and $r \in R^0$, $x(r\tilde{\xi}) = r(x\tilde{\xi})$(computed in $\Gamma\mathscr{K}$).

It is straightforward to verify that Definition 2.3 really gives a left R^0-action on $\mathscr{X}^*$. Note carefully that $\alpha(x\tilde{\xi}) \neq (x\alpha)\tilde{\xi}$ in general, for α in R^0, x in $\mathscr{X}$, and $\tilde{\xi}$ in $\mathscr{X}^*$. The attempt $x(\alpha\tilde{\xi}) = (x\alpha)\tilde{\xi}$ fails, leading to the present complicated method.

We frequently identify $\mathrm{Hom}_R(\mathscr{X}, \Gamma\mathscr{K})$ with $\mathrm{Hom}_{\mathscr{K}}(\mathscr{X}, \mathscr{K})$ according to the adjunction diagram

$$\begin{tikzcd} & \Gamma\mathscr{K} \arrow[d] \\ \mathscr{X} \arrow[ur, dashed, "\tilde{\xi}"] \arrow[r, "\xi"'] & \mathscr{K} \end{tikzcd}$$

with $(x\tilde{\xi})(r) = (xr)\xi$ and $x\xi = (x\tilde{\xi})(1)$.

We can also dualize morphisms. Let $T: \mathscr{X}_1 \to \mathscr{X}_2$, $x \mapsto xT$, be a right R-module homomorphism. If $\tilde{\xi}: \mathscr{X}_2 \to \Gamma\mathscr{K}$ (equivalently, $\xi: \mathscr{X}_2 \to \mathscr{K}$) lies in $\mathscr{X}_2^*$ define $T^*\tilde{\xi} = T \circ \tilde{\xi}$ as usual, giving a map $T^*: \mathscr{X}_2^* \to \mathscr{X}_1^*$. We must verify that T^* is R^0-linear, but this follows from $x(T \circ \alpha\tilde{\xi}) = (xT)(\alpha\tilde{\xi}) = \alpha((xT)\tilde{\xi}) = \alpha(x(T^*\tilde{\xi})) = x(\alpha(T^*\tilde{\xi}))$ for all x in $\mathscr{X}_1$ and α in R^0.

We leave to the reader the thankless task of dualizing left R^0-modules $\mathscr{M}$ to obtain right R-modules $\mathscr{M}^*$. In detail, this involves construction of a right adjoint $\Gamma^0(-) = {}_{\mathscr{K}}\mathrm{Hom}(R^0, -)$ of left $\mathscr{K}$-linear maps, adapting Proposition 2.2 to define a right R-module structure on $\Gamma^0(\mathscr{K})$, and eventually obtaining a right R-module structure on ${}_{\mathscr{K}}\mathrm{Hom}(\mathscr{M}, \mathscr{K}) = {}_R\mathrm{Hom}(\mathscr{M}, \Gamma^0\mathscr{K})$. It is unpleasant but essentially trivial to verify the following result, whose proof is omitted:

PROPOSITION 2.4. *Let $\mathscr{X}$ be a right R-module, so that $\mathscr{X}^*$ is a left R^0-module and $\mathscr{X}^{**}$ is again a right R-module. Then the map $\mathscr{X} \xrightarrow{\iota} \mathscr{X}^{**}$ given by $(x\iota)(\tau) = x\tau$ for x in $\mathscr{X}$ and $\tau: \mathscr{X} \to \mathscr{K}$ in $\mathscr{X}^*$, is a right R-module homomorphism.*

A really desirable duality theory should satisfy "reflexivity" and "exactness" properties. Reflexivity means that the map of Proposition 2.4 above is an isomorphism, and exactness means that injections dualize to surjections (equivalently, certain linear functionals can be extended). We restrict our attention to a class of R-modules for which these pleasant results are available and which is nevertheless large enough for our system-theoretic applications. From now, we consider only *dynamical modules*

$$\mathscr{X} \cong \prod_{t \in T} X_t$$

as defined in §1, including in the adjective the further assumption that each local space X_t is finite dimensional over the base field k.

PROPOSITION 2.6. *Let* $0 \to \mathscr{X}_1 \xrightarrow{S} \mathscr{X}_2 \xrightarrow{T} \mathscr{X}_3 \to 0$ *be a short exact sequence of dynamical right R-modules. Then the dual sequence*

$$0 \longrightarrow \mathscr{X}_3^* \xrightarrow{T^*} \mathscr{X}_2^* \xrightarrow{S^*} \mathscr{X}_1^* \longrightarrow 0$$

is exact, identifying

$$\mathscr{X}_3^* = \mathscr{X}_1^\perp = \{\xi \in \mathscr{X}_1^* : x\xi = 0 \textit{ for all } x \textit{ in } \mathscr{X}_1\} .$$

Furthermore $(\mathscr{X}_1^\perp)^\perp = \mathscr{X}_1$.

Proof. All this follows from reflexivity and left exactness of Hom once it is established that S^* is surjective, and this is clear because $S = \prod S(t)$, where $S(t): (\mathscr{X}_1)_t \to (\mathscr{X}_2)_t$ is a vector space map.

Next we describe explicitly the dual of a dynamical module. Consider $\mathscr{X} = \prod X_t$ with state-transition matrices $\Phi(s, t): X_s \to X_t$ for $s \leqq t$. We need the R^0-module action on $\mathscr{X}^* = \prod X_t^*$. Identify $\xi: \mathscr{X} \to \mathscr{K}$ with $\tilde{\xi}: \mathscr{X} \to \Gamma\mathscr{K}$ by $(x\tilde{\xi})(\theta) = (x\theta)\xi$ as usual. We want to give an explicit formula for $x(\beta\xi)$ for any x in $\mathscr{X}$ and any β in R^0. Start with

$$x(\beta\tilde{\xi})(\theta)(s) = \beta(x\tilde{\xi})(\theta)(s) = \sum_{t: s \leqq t} \beta(s, t)(x\tilde{\xi})(\theta e_{st})(t)$$

using Proposition 2.2. Next, set $\theta = 1$ to recover $\beta\xi = (\beta\tilde{\xi})(1)$:

$$\begin{aligned} x(\beta\xi)(s) &= \sum_t \beta(s, t)(x\tilde{\xi})(e_{st})(t) \\ &= \sum_t \beta(s, t)(xe_{st})\xi(t) \\ &= \sum_t \beta(s, t)(xe_{st})(t)\xi(t) . \end{aligned}$$

Using the right R-module structure on $\mathscr{X}$ in terms of $\Phi(s, t)$, we have $(xe_{st})(t) = x(s)\Phi(s, t)$, so that

$$x(\beta\xi)(s) = \sum_{t: s \leqq t} x(s)\Phi(s, t)\xi(t)\beta(s, t) . \tag{$*$}$$

To summarize:

PROPOSITION 2.7. *If $\mathscr{X}$ is a right R-module, then the dual module $\mathscr{X}^* = \mathrm{Hom}_{\mathscr{K}}(\mathscr{X}, \mathscr{K})$ can be given the structure of a left R^0-module. If $\mathscr{X} = \prod X_t$ is a dynamical R-module with R-action given by*

$$(x\alpha)(t) = \sum_{s:s\leqq t} x(s)\Phi(s, t)\alpha(s, t)\ ,$$

then $\mathscr{X}^* = \prod X_t^*$, with

$$(\beta\xi)(s) = \sum_{t:s\leqq t} \beta(s, t)\Phi(s, t)\xi(t)\ .$$

The duality theory presented here has a weakness which we have ignored up to now. The pairing

$$\mathscr{X} \times \mathscr{X}^* \longrightarrow \mathscr{K}$$
$$(x, \xi) = x\xi$$

does *not* satisfy the expected formula $(x\alpha, \xi) = (x, \alpha\xi)$ for α in R (or even α in $R \cap R^0$). In fact, for dynamical modules,

$$(xe_{st}, \xi)(t) = x(s)\Phi(s, t)\xi(t)$$

while

$$(x, e_{st}\xi)(t) = 0\ .$$

On the other hand, we do have

$$(xe_{st}, \xi)(t) = (x, e_{st}\xi)(s)$$

which ensures the following important result.

PROPOSITION 2.8. *Let $\mathscr{X}$ be a dynamical module, and suppose $\mathscr{W} \subseteqq \mathscr{X}$ is a right R-submodule. Then $\mathscr{W}^\perp \subseteqq \mathscr{X}^*$ is a left R^0-submodule.*

Proof. Check that $\mathscr{W}^\perp \subseteqq \mathscr{X}^*$ is a left R^0-submodule if and only if $e_{st}\mathscr{W}^\perp \subseteqq \mathscr{W}^\perp$ for all $s \leqq t$. Now $\xi \in \mathscr{W}^\perp$ if $(x, \xi)(t) = 0$ for all $x \in \mathscr{W}$ and all t in T. By the above remark, $(x, e_{st}\xi)(s) = (xe_{st}, \xi)(t) = 0$, since $xe_{st} \in \mathscr{W}$. Further, $(x, e_{st}\xi)(u) = 0$ for $u \neq s$ automatically. Hence, $e_{st}\xi$ lies in $\mathscr{W}^\perp$.

3. Dual systems and Lagrange multipliers. Suppose given a linear system $\sum = (\mathscr{X}, \mathscr{U}, \mathscr{Y}; G, H)$ on the poset T as defined in [14]. In detail, $\mathscr{U}$ and $\mathscr{Y}$ are right $\mathscr{K}$-modules, $\mathscr{X}$ is a dynamical right R-module, and $G: \mathscr{U} \to \mathscr{X}$, $H: \mathscr{X} \to \mathscr{Y}$ are right $\mathscr{K}$-module maps. The vector difference equations associated with $\sum$ are

$$x\mu = uG \qquad (3.1)$$
$$y = xH$$

or, more explicitly,

$$\sum_{s \leq t} x(s)\Phi(s, t)\mu(s, t) = u(t)G(t) \qquad (3.2)$$
$$y(t) = x(t)H(t)$$

where μ is the Möbius function of T and each $\Phi(s, t)$ is a state-transition map on $\mathscr{X}$. Here x lies in $\mathscr{X}$, u in $\mathscr{U}$, and y in $\mathscr{Y}$.

DEFINITION 3.1. The system $\sum^*$ *dual* to the system $\sum$ defined above is given by $\sum^* = (\mathscr{X}^*, \mathscr{Y}^*, \mathscr{U}^*; H^*, G^*)$, where $\mathscr{X}^*$ is the left R^0-module defined in §2, $\mathscr{Y}^*$, $\mathscr{U}^*$ are left $\mathscr{K}$-modules, and $H^*: \mathscr{Y}^* \to \mathscr{X}^*$; $G^*: \mathscr{X}^* \to \mathscr{U}^*$ are left $\mathscr{K}$-module maps.

Since $\mathscr{X}$ is dynamical, we can write explicit dual difference equations for $\sum^*$. These are:

$$\begin{aligned} \mu\xi &= H^*\eta \\ \varphi &= G^*\xi \end{aligned}\ , \quad \text{or} \qquad (3.3)$$
$$\sum_{t : s \leq t} \mu(s, t)\Phi(s, t)\xi(t) = H(s)\eta(s)$$
$$\varphi(s) = G(s)\xi(s)$$

for ξ in $\mathscr{X}^*$, η in $\mathscr{Y}^*$, φ in $\mathscr{U}^*$. Dualized mappings are written as matrices on the left.

The input/output analysis of $\sum^*$ is complicated by the fact that the functors change: we introduce $\Omega^0\mathscr{Y}^* = R^0 \otimes_{\mathscr{K}} \mathscr{Y}^*$ and $\Gamma^0\mathscr{U}^* = {}_{\mathscr{K}}\mathrm{Hom}(R^0, \mathscr{U}^*)$, (left $\mathscr{K}$-linear maps).

As usual, we introduce the i/o diagram

$$\begin{tikzcd} \Omega^0\mathscr{Y}^* \arrow[rr, "f_\Sigma"] \arrow[dr, "\tilde{H}^*"'] & & \Gamma^0\mathscr{U}^* \\ & \mathscr{X}^* \arrow[ur, "\tilde{G}^*"'] & \end{tikzcd}$$

and say that $\sum^*$ is *reachable* if $\tilde{H}^*$ is onto, *observable* is $\tilde{G}^*$ is one-to-one, and *canonical* if both conditions hold.

The expected result can be proved.

THEOREM 3.2. (*"Observability and reachability are dual."*)
(a) *The system $\sum^*$ is reachable if and only if $\sum$ is observable.*
(b) *The system $\sum^*$ is observable if and only if $\sum$ is reachable.*
(c) *The system $\sum^*$ is canonical if and only if $\sum$ is canonical.*

Proof. Part (c) follows from (a) and (b) by definition, and part

(b) follows easily from (a) since $\sum^{**} \cong \sum$. (Recall that we assume $\mathscr{X} \cong \prod X_t$ with each X_t finite dimensional.)

To prove (a), assume that $\tilde{G}^*$ is injective. Then $\ker G^*$ contains no nontrivial left R^0-submodules of $\mathscr{X}^*$, since $\xi \in \mathscr{X}^*$ with $R^0\xi \subseteq \ker G^*$ implies $\tilde{G}^*\xi = 0$. On the other hand, $(\operatorname{im} \tilde{G})^\perp \subseteq (\operatorname{im} G)^\perp \subseteq \ker G^*$, and $(\operatorname{im} \tilde{G})^\perp$ is a left R^0-module by Proposition 2.8. It follows that $(\operatorname{im} \tilde{G})^\perp = 0$, so that $\tilde{G}$ is surjective (see Proposition 2.6). The converse is similar.

We conclude this section with a study of the dual systems arising from Lagrange multipliers, verifying that Equations (3.3) occur naturally. This example is the classical linear-quadratic regulator problem. The base field is the real field $\boldsymbol{R}$.

Consider a state-input pair (i.e., no outputs) given by

$$\begin{aligned} x\mu &= uG \\ x(t) &= -\sum_{s<t} x(s)\Phi(s, t)\mu(s, t) + u(t)G(t) \end{aligned} \tag{3.4}$$

as above. One version of the *quadratic regulator problem* goes as follows.

Suppose given a finite poset T with initial set T_i. (That is, $t \in T_i$ if t has no predecessor in T.) For each t in T, let $R_1(t)$ be a nonnegative definite quadratic form and $R_2(t)$ be a positive definite quadratic form. For x in $\mathscr{X}$ and u in $\mathscr{U}$, define the objective function

$$\tau(x, u) = \sum_{t \in T} x(t)R_1(t)x^T(t) + u(t)R_2(t)u^T(t) .$$

Then our problem is:

(3.5) Given initial states $x(t_i)$, t_i in T_i choose $u(t)$ in $\mathscr{U}$ to minimize $\tau(x, u)$

subject to the system constraint (3.4).

We consider here only the naive necessary conditions arising from Lagrange multipliers.

Let $\lambda^T: \mathscr{X} \to \mathscr{K}$ be a linear functional in $\mathscr{X}^*$, and form the Lagrangian

$$L(x, u, \lambda^T) = \tau(x, u) + (x\mu - uG)\lambda^T . \tag{3.6}$$

An optimal input u^0 corresponds to a stationary point (x^0, u^0, λ^0) of the Lagrangian, see for example [15, Ch. 6], giving differential conditions

$$\frac{\partial L}{\partial x(t)} = 0$$

$$\frac{\partial L}{\partial u(t)} = 0 \quad \text{for all} \quad t \quad \text{in} \quad T .$$
$$\frac{\partial L}{\partial \lambda(t)} = 0 .$$

Explicitly, the first two give

$$\begin{aligned} &R_1(t)x^T(t) + \lambda^T(t) + \sum_{l:t<l} \Phi(t, l)\mu(t, l)\lambda^T(l) = 0 \\ &R_2(t)u^T(t) + G(t)\lambda^T(t) = 0 . \end{aligned} \tag{3.7}$$

Rewriting these in terms of the dual module action, and recalling that each $R_2(t) > 0$,

$$\begin{aligned} \mu \cdot \lambda^T &= -R_1 x^T \\ u^T &= -R_2^{-1} G \lambda T . \end{aligned} \tag{3.8}$$

That is, the original state-input pair (3.4), together with a optimization problem (3.5), leads to a state-output pair (3.6, 3.7). In particular, the abstract definitions of duality in Section 2 appear naturally here.

4. Systems on the line. In this section we treat time-varying systems on the ordinary discrete line $T = (\boldsymbol{Z}, \leqq)$. In this case, the input/output theory can be described in terms of noncommutative formal Laurent series and gives another approach to the definition of dual modules. The duality resulting from this approach (or, equivalently, from §2) reduces to the familar semilinear duality of [3], which was introduced into system theory by K. Hafiz [5] (in a slightly different form). In fact, many of the results of this section can be found in [5] and [7], which supplied a major inspiration for this line of research. See also [16].

Following [14, §6] we replace the finite memory algebra of $(\boldsymbol{Z}, \leqq)$ by the skew polynomial ring

$$\begin{aligned} &\mathscr{K}_\sigma[z] = \{a_0 + za_1 + \cdots + z^n a_n : a_i \in \mathscr{K}\} \\ &za = a^\sigma z;\ a^\sigma(t) = a(t+1) . \end{aligned}$$

The right module action on a state module $\mathscr{X}$ is given by

$$(xz)(t) = x(t-1)F(t)$$

where $F(t): X_{t-1} \to X_t$ is k-linear. Alternatively, we have state-transition matrices

$$\Phi(s, t) = \begin{cases} I & \text{if} \quad s = t \\ F(s)F(s+1)\cdots F(t) & \text{if} \quad s < t . \end{cases}$$

For a $\mathscr{K}$-module $\mathscr{U}$, we write the input functor $\Omega\mathscr{U}$ as $\Omega\mathscr{U} = \mathscr{U} \otimes_{\mathscr{K}} \mathscr{K}_\sigma[z] = \mathscr{U}_\sigma[z]$. The output functor $\Gamma\mathscr{K} = \mathrm{Hom}_{\mathscr{K}}(\mathscr{K}_\sigma[z], \mathscr{K})$ can be described using formal Laurent series.

Define the ring of *skew-formal Laurent series* over $\mathscr{K}$ by

$$\mathscr{K}_\sigma((z^{-1})) = \sum_{i=-N}^{\infty} a_i z^{-i}$$

with a_i in $\mathscr{K}$. Addition is obvious, multiplication is familiar, subject to relations $za = a^\sigma z$ or $az^{-1} = z^{-1}a^\sigma$. Note carefully the identity

$$\sum_{i=-N}^{\infty} a_i z^{-i} = \sum_{i=-N}^{\infty} z^{-i} a_i^{\sigma^i} .$$

PROPOSITION 4.1. *There is an isomorphism of right $\mathscr{K}_\sigma[z]$-modules*

$$\mathrm{Hom}_{\mathscr{K}}(\mathscr{K}_\sigma[z], \mathscr{K}) \longrightarrow \mathscr{K}_\sigma((z^{-1}))/z\mathscr{K}_\sigma[z] ,$$

given by $\varphi \mapsto [\sum_{i=0}^{\infty} \varphi(z^i)z^{-i}] \bmod z\mathscr{K}_\sigma[z]$.

Proof. Note that $z\mathscr{K}_\sigma[z]$ is in fact a right $\mathscr{K}_\sigma[z]$-module, so the factor module is defined. Every coset has a unique representative of the displayed form. Since $\{1, z, z^2, \cdots\}$ is a $\mathscr{K}$-basis for $\mathscr{K}_\sigma[z]$ (as a countably generated $\mathscr{K}$-module), it is easy to see that our mapping is bijective and additive. The only difficulty is to compare the $\mathscr{K}_\sigma[z]$-actions. By definition, $(\varphi p(z))(\alpha(z)) = \varphi(p(z)\alpha(z))$ for $p(z)$, $\alpha(z)$ in $\mathscr{K}_\sigma[z]$. It suffices to examine monomials $p(z) = z^r a$, a in $\mathscr{K}$:

$$\begin{aligned}
\varphi &\longrightarrow \sum_{i=0}^{\infty} \varphi(z^i)z^{-i} \\
\varphi(z^r a) &\longrightarrow \sum [\varphi(z^r a)](z^i)z^{-i} \\
&= \sum \varphi(z^r z^i a^{\sigma^i})z^{-i} \\
&= \sum \varphi(z^{r+i})a^{\sigma^i}z^{-i} \\
&= \sum \varphi(z^{r+i})z^{-i}a \\
&= \sum \varphi(z^{r+i})z^{-i-r}z^r a \\
&\equiv \left(\sum_{i=0}^{\infty} \varphi(z^i)\, z^{-i}\right)z^r a (\mathrm{mod}\; z\mathscr{K}^\sigma[z]) .
\end{aligned}$$

The last equality results from replacing $i + r$ with r, which is fair because the "missing terms" are all in $z\mathscr{K}_\sigma[z]$.

Since $z\mathscr{K}_\sigma[z]$ is in fact a two-sided $\mathscr{K}_\sigma[z]$-submodule of $\mathscr{K}_\sigma((z^{-1}))$, we get a byproduct.

COROLLARY 4.2. *The module*

$$\Gamma\mathscr{K} = \mathrm{Hom}_{\mathscr{K}}(\mathscr{K}_\sigma[z], \mathscr{K}) \cong \mathscr{K}_\sigma((z^{-1}))/z\mathscr{K}_\sigma[z]$$

is a two-sided $\mathscr{K}_\sigma[z]$*-module.*

From the incidence algebra point of view z is the incidence function defined by $z(s, t) = 1$ if $s = t - 1$, else 0. A polynomial $p(z)$ in $\mathscr{K}_\sigma[z]$ has finite memory and also finite predictive power (bounded by the degree of $p(z)$), so $\mathscr{K}_\sigma[z]$ can play the roles of both the rings R and R^0 introduced in §2. The reader may check that the left $\mathscr{K}_\sigma[z]$-structure of $\Gamma\mathscr{K}$ is consistent with the left R^0-structure defined in Proposition 2.2.

Just as in §2, the left $\mathscr{K}_\sigma[z]$-structure on $\Gamma\mathscr{K}$ yields a similar structure on $\mathscr{X}^* = \mathrm{Hom}_{\mathscr{K}}(\mathscr{X}, \Gamma\mathscr{K})$ by

$$x(z\tilde{\xi})(\alpha) = z(x\tilde{\xi})(\alpha)$$

for $\tilde{\xi}\colon \mathscr{X} \to \Gamma\mathscr{K}$, α in $\mathscr{K}_\sigma[z]$, ane x in $\mathscr{X}$. Now identifying across the isomorphism of Proposition 4.1, and using $(x\tilde{\xi})(\alpha) = (x\alpha)\xi$,

$$\begin{aligned} x\tilde{\xi} &= \sum_{i=0}^{\infty} (x\tilde{\xi})(z^i)z^{-i} \\ &= \sum_{i=0}^{\infty} (xz^i)\xi \cdot z^{-i} \, . \end{aligned}$$

Thus,

$$\begin{aligned} x(z\tilde{\xi}) &= z\Big(\sum_{i=0}^{\infty} (xz^i)\xi \cdot z^{-i}\Big) \\ &= \sum_{i=1}^{\infty} (xz^i)\xi^{\sigma-1} \cdot z^{-i+1} \, . \end{aligned}$$

The first term of this formula shows

$$x(z\xi)(t) = x(t)F(t+1)\xi(t+1)$$

for all x in $\mathscr{X}$, or

$$(z\xi)(t) = F(t+1)\xi(t+1)$$

as expected. This gives, in analogy with the discussion before Proposition 2.8,

$$[(xz)\xi](t) = [x(z\xi)](t-1)$$

which is the closest we can come to an adjointness formula.

We describe briefly the formulas for dual systems. Suppose $\sum = (\mathscr{X}, \mathscr{U}, \mathscr{Y}\colon F, G, H)$, with equations

$$\begin{aligned} x(t) &= x(t-1)F(t) + u(t)G(t) \\ y(t) &= x(t)H(t) \, . \end{aligned}$$

Our conventions give a dual system $\sum^* = (\mathscr{X}^*, \mathscr{U}^*, \mathscr{Y}^*; F^*, H^*, G^*)$, where $\mathscr{X}^*$ is the left $\mathscr{K}_\sigma[z]$-module discussed above. The dual equations are

$$\begin{aligned}\xi(t) &= F(t+1)\xi(t+1) + H(t)\eta(t)\\ \varphi(t) &= G(t)\xi(t)\end{aligned}$$

for ξ in $\mathscr{X}^*$, η in $\mathscr{Y}^*$, and φ in $\mathscr{U}^*$.

We conclude this section with one calculation. The modules

$$\mathscr{X} = \mathscr{K}_\sigma[z]/(z^n + z^{n-1}a_1 + \cdots + a_n)K_\sigma[z]$$

occur in the realization theory of autoregressive time-varying differences equations [14, §8]. Hafiz [5, Th. 5.3] computed the dual $\mathscr{X}^*$ using explicit matrix calculations and somewhat different conventions and applied the result to state estimators. We give here a new approach to his calculations.

Let

$$\psi(z) = z^n + z^{n-1}a_1 + \cdots + a_n \text{ be in } \mathscr{K}_\sigma[z]\,.$$

Consider the exact sequence

$$0 \longrightarrow \mathscr{K}_\sigma[z] \xrightarrow{\psi} \mathscr{K}_\sigma[z] \longrightarrow \mathscr{X} \longrightarrow 0$$

where ψ is the right $\mathscr{K}_\sigma[z]$-module homomorphism $b(z) \mapsto \psi(z)b(z)$. This yields an exact dual sequence

$$0 \longrightarrow \mathscr{X}^* \longrightarrow \Gamma\mathscr{K} \xrightarrow{\psi^*} \Gamma\mathscr{K} \longrightarrow 0$$

of left $\mathscr{K}_\sigma[z]$-modules, where ψ^* sends $\gamma \longrightarrow \gamma\psi(z)$. Since $\Gamma\mathscr{K} \cong \mathscr{K}_\sigma((z^{-1}))/z\mathscr{K}_\sigma[z]$, we get

$$\mathscr{X}^* = \{[p(z)] \quad \text{in} \quad \Gamma\mathscr{K}\colon p(z)\psi(z) \in z\mathscr{K}_\sigma[z]\}\,.$$

Here $p(z)$ is a power series in $\mathscr{K}_\sigma((z^{-1}))$ and [ ] denotes class modulo $z\mathscr{K}_\sigma[z]$. For $a(z)$ in $\mathscr{K}_\sigma[z]$, we write $z^{-1}a(z)z = a^\sigma(z)$. In fact, if

$$a(z) = z^n a_n + z^{n-1}a_{n-1} + \cdots + a_0\,,$$

then $a^\sigma(z) = z^n a_n^\sigma + z^{n-1}a_{n-1}^\sigma + \cdots + a_0^\sigma$. Similarly, $a^{\sigma^{-1}}(z) = za(z)z^{-1}$.

Now map $\mathscr{K}_\sigma[z]$ to $\mathscr{X}^*$ by $a(z) \mapsto a(z)z(\psi(z))^{-1}(\operatorname{mod} z\mathscr{K}_\sigma[z])$. This map is left $\mathscr{K}_\sigma[z]$-linear, and surjective by the description of $\mathscr{X}^*$ above. Since $a(z)\cdot z = za^\sigma(z)$, we see that $a(z) \mapsto 0$ if and only if $a^\sigma(z) = b(z)\psi(z)$ for some $b(z)$ in $\mathscr{K}_\sigma[z]$. That is $a(z)^{\sigma^{-1}}(z)\psi^{\sigma^{-1}}(z)$. This proves the next proposition.

PROPOSITION 4.3. (*Hafiz*). *If* $\mathscr{X} \cong \mathscr{K}_\sigma[z]/\psi(z)\mathscr{K}_\sigma[z]$, *then* $\mathscr{X}^* \cong$

$\mathscr{K}_\sigma[z]/\mathscr{K}_\sigma[z]\psi^{\sigma^{-1}}(z)$.

5. Finite posets. In this section we discuss modules and systems on a *finite* poset $(T, \leqq)$. Here substantial technical simplifications are possible, since the whole incidence algebra coincides with both the finite memory algebra and the finite predictor algebra. The full matrix algebra on the (underlying, unordered) set T serves as a substitute for the Laurent series of §4, and a similar explicit description of the dual module is available.

Throughout this section, let $(T, \leqq)$ be a finite poset with $|T| = n$. Let R be the incidence algebra of $(T, \leqq)$ over a fixed field k, and let

$$P = \{\alpha \text{ in } R: \alpha(t, t) = 0 \text{ for all } t \text{ in } T\} .$$

For any poset T, P is the Jacobson radical of R, and in particular is a two-sided ideal. Note that

$$R/P \cong \mathscr{K} = k^T$$

is a (finite) direct product of fields.

Define next

$$\mathscr{L} = \{\text{all functions } f: T \times T \longrightarrow k\}$$

with multiplication

$$(f^*g)(s, t) = \sum_{u \text{ in } T} f(s, u)g(u, t) .$$

Then $\mathscr{L}$ is isomorphic to the full matrix $M_n(k)$, $n = |T|$. The intuition here is that $\mathscr{L}$ is a sort of "Laurent-series ring," and we are trying to mimic the isomorphism $\Gamma\mathscr{K} \cong \mathscr{K}_\sigma((z^{-1}))/z\mathscr{K}_\sigma[z]$ of the preceding section.

LEMMA 3.1. *The ideal P of R is a two-sided R-submodule of $\mathscr{L}$, so that $\mathscr{L}/P$ is an $R - R$ bimodule.*

Proof. Trivial, since $P \subset R \subset \mathscr{L}$ is a two-sided ideal of R.

For any s, t in T (not assuming $s \leqq t$) let e_{st} in $\mathscr{L}$ be defined as $e_{st}(u, v) = 1$ if $s = u$, $t = v$, else 0. It is easy to see that

$$\{e_{ts}: t \not< s \quad \text{in} \quad T\}$$

forms a k-basis for $\mathscr{L}/P$.

We proceed to define a mapping from $\Gamma\mathscr{K} = \operatorname{Hom}_{\mathscr{K}}(R, \mathscr{K})$ into $\mathscr{L}/P$. Let γ in $\Gamma\mathscr{K}$ be a right K-linear homomorphism. Then for any θ in R and t in T

$$\gamma(\theta e_t) = \gamma(\theta)e_t\ ,$$

where e_t is short for e_{tt} and $\gamma(\theta)e_t$ lies in $\mathscr{K}e_t = k$. We use the notation $\gamma(\theta)e_t = \gamma(\theta)(t)$.

Given γ in $\Gamma\mathscr{K}$, define $\hat{\gamma}$ in $\mathscr{L}$ by

$$\hat{\gamma}(t, s) = \begin{cases} (\gamma e_{st})(t) & \text{if} \quad s \leqq t \\ 0 & \text{if} \quad s \not< t\ . \end{cases}$$

This makes sense since $e_{st}e_t = e_{st}$.

THEOREM 5.2. *The map* $\wedge\colon \Gamma\mathscr{K} \to \mathscr{L}/P$ *given by*

$$\gamma \longrightarrow \hat{\gamma}(\text{mod } P)$$

defines an injective homomorphism of $R - R$ *bimodules which is surjective if and only if* $(T, \leqq)$ *is totally ordered.*

Proof. Any α in R is a finite sum $\alpha = \sum_{s\leqq t}\alpha(s, t)e_{st}$, and $\gamma\alpha = \sum_{s\leqq t}\alpha(s, t)\gamma e_{st}$. Therefore $\gamma e_{st} = (\gamma e_{st})(t) = 0$ for all $s \leqq t$ implies $\gamma = 0$. The map is clearly additive, hence injective. Now $\mathscr{L}/P$ is spanned by $e_{ts}(t \nleqq s)$, so it is easy to see that the map is surjective if and only if $(T, \leqq)$ is totally ordered. (An incomparable pair u, v gives e_{uv} not in P not hit by any γ.) It remains to check the module structures.

Consider the right R-module structures first. The right action of R on $\Gamma\mathscr{K}$ is given by $(\gamma\alpha)(\theta) = \gamma(\alpha\theta)$ for α, θ in R, so $(\gamma\alpha)(e_{st}) = \gamma(\alpha e_{st})$. Write

$$\begin{aligned} \alpha &= \sum_{u\leqq v}\alpha(u, v)e_{uv} \\ \alpha e_{st} &= \sum_{u\leqq v}\alpha(u, v)e_{uv}e_{st} \\ &= \sum_{u:u\leqq s}\alpha(u, s)e_{ut}\ . \end{aligned}$$

Therefore

$$(\gamma\alpha)(e_{st}) = \sum_{u:u\leqq s}\alpha(u, s)\gamma e_{ut}\ .$$

Therefore, for $s \leqq t$ in T

$$\begin{aligned} \widehat{\gamma\alpha}(t, s) &= (\gamma\alpha)(e_{st}) \\ &= \sum_{u:u\leqq s}(\gamma e_{ut})\alpha(u, s) \\ &= \sum_{u:u\leqq s}\hat{\gamma}(t, u)d(u, s) \end{aligned}$$

which exhibits the correct right action of R on $\mathscr{L}/P$.

For left action, refer to Proposition 2.2 for the formula

$$(\alpha\gamma)(\theta)(t) = \sum_{u:t \leqq u} \alpha(t, u)\gamma(\theta e_{tu})(u) .$$

For $\theta = e_{st}$, $s \leqq t$, this

$$(\alpha\gamma)(e_{st})(t) = \sum_{u:t \leqq u} \alpha(t, u)\gamma(s, u)(u) ,$$

$$\widehat{\alpha\gamma}(t, s) = \sum_{u:t \leqq u} \alpha(t, u)\hat{\gamma}(u, s)$$

which gives the correct left action of α in $\mathscr{L}/P$.

Theorem 5.2 provides an alternative approach to the duality theory. That is, $\Gamma\mathscr{K}$ inherits an $R - R$ bimodule structure from $\mathscr{L}/P$, so that each $\mathscr{X}^* = \mathrm{Hom}_{\mathscr{K}}(\mathscr{X}, \Gamma\mathscr{K})$ is an $R - R$ bimodule. The theory of dual modules and systems proceeds just as before.

6. Conclusion. In this foundational paper we have defined a duality theory for dynamical modules over a (finite-memory) incidence algebra and for the corresponding linear systems.

In general, if $\mathscr{X}$ is a right module over a finite memory algebra, $\mathscr{X}^*$ becomes a left module over the corresponding finite predictor algebra, and the corresponding dual system evolves in reverse time. The definition proceeds with the construction of on (apparently ad hoc) left action on $\Gamma\mathscr{K} = \mathrm{Hom}_{\mathscr{K}}(R, \mathscr{K})$, the left-adjoint of the forgetful functor. Evidence that the definition is a reasonable one includes applications to Lagrange multiplier theory. Consistent natural alternatives in the special cases $(\boldsymbol{Z}, \leqq)$ and finite posets are presented.

References

1. M. A. Arbib and E. G. Manes, *Machines in category: An expository introduction*, SIAM Rev., **16** (1974), 163-192.

2. ———, *Time-varying systems*, SIAM J. Control, **13** (1975), 1252-1270.

3. Bourbaki, *Algebra I*, Ch. 1-3, Ch. II, Sec 2.5, Addison-Wesley, 1974.

4. P. Doubilet, G.-C. Rota and R. Stanley, *On the Foundations of Combinatorial Theory VI: The Idea of Generating Function*, Proc. Sixth Berkeley Symp. Math. Stat. and Prob., Vol. II, 267-318. U. Calif. Press, 1972.

5. K. M. Hafiz, *New Results on Discrete-Time-Varying Linear Systems*, Thesis: Georgia Institute of Technology, March, 1975.

6. R. E. Kalman, P. Falb and M. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, 1969.

7. E. W. Kamen, *A new algebraic approach to linear time-varying systems*, preprint: March, 1974.

8. E. G. Manes, ed., *Category Theory Applied to Computations and Control*, Lecture Notes in Computer Science, Vol. **25**, Springer-Verlag.

9. L. Padulo and M. Arbib, *System Theory*, Hemisphere Publishing Co., 1974.

10. G.-C. Rota, *On the foundations of combinatorial theory I: Theory of Möbius*

functions, Z. Wahrscheinlichkeitstheorie and Verw. Geb., **2** (1964), 340-368.
11. B. F. Wyman, *Linear systems over commutative rings*, Unpublished lecture notes.
12. ———, *Linear systems over rings of operators*, in [8, pp. 218-223].
13. ———, *Linear Difference Systems on Partially Ordered Sets, in*: *Mathematical Systems Theory* (Udine 1975), Lec. Notes in Econ. and Math. Systems, No. 131, Springer-Verlag, 1976.
14. ———, *Time varying linear systems on partially ordered sets*: *Realization theory*, Adv. in Math. Supp. Studies, **2** (1978), 233-258,
15. P. Varaiya, *Notes on Optimization*, Van Nostrand Reinhold, 1972.
16. E. M. Kamer and K. Hafez, *Algebra Theory of Linear Time-Varying Systems*, SIAM J. Cont. Opt., **17** (1979), 500-510.

Received April 12, 1978 and in revised form September 5, 1978.

OHIO STATE UNIVERSITY
COLUMBUS, OH 43210