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SYMMETRIC TWINS AND COMMON TRANSVERSALS

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In this paper, we study the properties of certain families of sets on the circle and use the result to obtain a theorem on common transversals for sets in the plane.

1. Introduction. The standard Helly type results (see [2]) are essentially of the following nature:

If each subfamily of a given size of a family of sets has a certain property, then the whole family has the same property.

Our results in this paper are in a different form:

Let $\mathscr{F}$ be a family of n sets where n is sufficiently large. For any constant c, $0 < c < 1$, there exists an integer $k = k(c)$, $1<k<n$, such that if each subfamily of $\mathscr{F}$ of size k has a certain property, then some subfamily of $\mathscr{F}$ of size at least cn has the same property.

A symmetric twin (see [3] for other kinds of twins) is a subset of a circle which consists of two closed arcs symmetric about the center of the circle. We shall also consider the whole circle as a degenerate symmetric twin. The property of interest here is that of having nonempty intersection. Our result is:

THEOREM A. *Let $\mathscr{F}$ be a family on n symmetric twins on the same circle and let k be an integer, $1 < k < n$. If each subfamily of $\mathscr{F}$ of size k has nonempty intersection, then some subfamily of $\mathscr{F}$ of size at least $n(k-2)/(k+1)$ has nonempty intersection.*

We point out that given $0 < c < 1$, we can choose k so that $(k-2)/(k+1) > c$ provided that n is sufficiently large.

For families of connected closed sets in the plane, the property of interest here is that of having a common transversal (see [4]), which is a straight line interesting all members of the family. Our result is:

THEOREM B. *Let $\mathscr{F}$ be a family of n connected closed sets in the plane where n is sufficient large. For any constant c, $0<c<1$, there exists an integer $k = k(c)$, $1 < k < n$, such that if each subfamily of $\mathscr{F}$ of size k has a common transversal, then some subfamily of $\mathscr{F}$ of size at least n has ca common transversal.*

To prove Theorem B, we shall make use of Theorem A as well as yet another result of similar nature, proved in different terminology by Abbott and Katchalski ([1]):

THEOREM C. *Let $\mathscr{G}$ be a family of n closed intervals on the line where n is sufficiently large. Let α be any constant, $0<\alpha<1$. If at least $\alpha\binom{n}{2}$ of the pairs of intervals have nonempty intersections, then some subfamily of $\mathscr{G}$ of size at least $(1-\sqrt{1-\alpha})n$ has nonempty intersection.*

2. Proof of Theorem A. We may assume that $k \geqq 3$. Since $n(k-2)/(k+1)$ is an increasing function of k, we may assume that $\mathscr{F}$ has a subfamily $\mathscr{B}=\{B_1, B_2, \cdots, B_{k+1}\}$ with empty intersection. We may also assume that none of the B's is the whole circle.

For $1 \leqq i \leqq k+1$, choose antipodal points a_i and a_{i+k+1} on the circle belonging to $\cap(\mathscr{B}-\{B_i\})$. Relabelling if necessary, assume that $a_1, a_2, \cdots, a_{2k+2}$ are in clockwise order on the circle. The arc from a_u to a_v will be denoted by $[a_u, a_v]$, and all subscripts are to be reduced mod $(2k+2)$.

Let $1 \leqq i \leqq k+1$. Since B_i is a symmetric twin, we have

$$[a_{i+1}, a_{i+k}] \cup [a_{i+k+2}, a_{i-1}] \subset B_i \,.$$

Thus $x \in B_i$ if $x \notin [a_{i-1}, a_{i+1}] \cup [a_{i+k}, a_{i+k+2}]$. Consequently,

$$\cap(\mathscr{B}-\{B_{i+1}, B_{i+2}\}) \subset [a_i, a_{i+3}] \cup [a_{i+k+1}, a_{i+k+4}] \,.$$

For any $F \in \mathscr{F}-\mathscr{B}$, $\{F\} \cup (\mathscr{B}-\{B_{i+1}, B_{i+2}\})$ is a subfamily of $\mathscr{F}$ of size k and has nonempty intersection. Hence for $1 \leqq i \leqq k+1$,

$$F \cap [a_i, a_{i+3}] \neq \phi$$

as F is a symmetric twin.

It follows that each $F \in \mathscr{F}-\mathscr{B}$, being a symmetric twin, contains all of the points $a_1, a_2, \cdots, a_{2k+2}$ with the possible exception of 6. Hence one of these points, say a, belongs to at least

$$\frac{(2k+2)-6}{2k+2}\,|\mathscr{F}-\mathscr{B}| = \frac{k-2}{k+1}(n-k-1)$$

members of $\mathscr{F}-\mathscr{B}$. The point a also belongs to k members of $\mathscr{B}$. The theorem follows since $(k-2)/k+1)(n-k-1)+k > n(k-2)/(k+1)$.

3. Proof of Theorem B. Let $\mathscr{F}=\{F_1, F_2, \cdots, F_n\}$. For $0<c<1$, choose k so that

$$c = 1-\sqrt{1-\alpha}$$

with

$$\alpha = \left(\left[\frac{k}{2}\right] - 2\right)\Big/\left(\left[\frac{k}{2}\right] + 1\right).$$

Let C be a fixed circle in the plane. For $1 \leqq i, j \leqq n, i \neq j$, let A_{ij} be the set of all points on C which lie on straight lines which pass through the center of C and are parallel to some common transversal of F_i and F_j. Clearly A_{ij} is a symmetric twin on C. Let $\mathscr{A}$ denote the collection of all these A's.

Since every subfamily of $\mathscr{F}$ of size k has a common transversal, every subfamily of $\mathscr{A}$ of size $[k/2]$ has nonempty intersection. By Theorem A, $\mathscr{A}$ has a subfamily of size at least $\alpha\binom{n}{2}$ with nonempty intersection. Let x be a point in this intersection.

Let L be a fixed straight line perpendicular to the straight line joining x and the center of C. For $1 \leqq i \leqq n$, let G_i be the projection of F_i onto L. Clearly G_i is a closed interval on L. Let $\mathscr{G}$ denote the collection of all these G's.

For $1 \leqq i, j \leqq n, i \neq j, G_i \cap G_j \neq \phi$ if $x \in A_{ij}$. Hence at least $\alpha\binom{n}{2}$ of the pairs of intervals have nonempty intersection. By Theorem C, $\mathscr{G}$ has a subfamily of size at least cn with nonempty intersection. Let y be a point in this intersection.

The theorem now follows as the straight line passing through y and perpendicular to L is a common transversal of a subfamily of $\mathscr{F}$ of size at least cn.

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REFERENCES

1. H. L. Abbott and M. Katchalski, *A Turan type problem for interval graphs*, Discrete Mathematics, **25** (1979), 85-88.

2. L. Danzer, B. Grunbaum and V. Klee, *Helly's theorem and its relatives*, Proceedings of Symposia in Pure Mathematics, **7**, "Convexity", Amer. Math. Soc., (1962), 101-180.

3. B. Grunbaum and T. Motzkin, *On components in some families of sets*, Proc. Amer. Math. Soc., **12** (1961), 607-613.

4. H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, 1964.

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