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ISOMORPHISMS OF THE FOURIER ALGEBRAS IN CROSSED PRODUCTS

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Let (M, G, α), (N, H, β) be W^*-systems, $F_\alpha(G; M_*)$ and $F_\beta(H; N_*)$, their Fourier algebras. The main result is that $F_\alpha(G; M_*)$ and $F_\beta(H; N_*)$ are isometrically isomorphic as Banach algebras if and only if M (resp. G) is isomorphic to N (resp. H) by θ (resp. I) such that $\beta_{I(g)}\circ\theta=\theta\circ\alpha_g$ for all $g \in G$, or M (resp. G) is anti-isomorphic to N (resp. H) such that $\beta_{I(g^{-1})}\circ\theta=\theta\circ\alpha_g$ for all $g \in G$.

1. Introduction. For locally compact abelian groups G and H, Pontryagin's duality theorem mentions that $L^1(G)$ is isomorphic to $L^1(H)$ if and only if G is isomorphic to H. Y. Kawada [4] and J. G. Wendel [11] proved the same statement for arbitrary locally compact groups.

When G is a locally compact abelian group, $L^1(G)$ is isometrically isomorphic to the Fourier algebra $A(G)$ in [7]. Therefore $A(G)$ is isomorphic to $A(H)$ as Banach algebras if and only if G is isomorphic to H.

P. Eymard [1], on the other hand, defined the Fourier algebra $A(G)$ of a locally compact group G and showed that it is isomorphic to the predual $m(G)_*$ of the von Neumann algebra $m(G)$ generated by the left regular representation of G.

M. E. Walter [10] showed that $A(G)$ and $A(H)$ are isometrically isomorphic as Banach algebras if and only if G and H are isomorphic.

Recently for W^*-system (M, G, α), the Fourier space $F_\alpha(G; M_*)$ was defined in [8] such that $F_\alpha(G; M_*)$ is isometrically isomorphic to the predual of the crossed product $G \otimes_\alpha M$ as Banach spaces.

M. Fugita [2] quite recently defined the Banach algebra structure in the Fourier space $F_\alpha(G; M_*)$. Then he showed that the group of all characters $\widehat{F_\alpha(G; M_*)}$ of $F_\alpha(G; M_*)$ is isomorphic to G and studied the support of the operators in $G \otimes_\alpha M$.

In this paper we generalize the Walter's result for W^*-system (M, G, α).

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2. Notations and preliminaries. Let M be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and G be a locally compact group. The triple (M, G, α) is said to be a W^*-system if the mapping α of G into the group $\mathrm{Aut}(M)$ of all automorphisms of M is a homomorphism and the function $g \mapsto \omega \circ \alpha_g(x)$ is continuous on G for all $x \in M$ and $\omega \in M_*$ where M_* is the predual of M.

The crossed product $G \otimes_\alpha M$ of M by α is the von Neumann algebra generated by the family of operators $\{\pi_\alpha(x), \lambda_G(g);\ x \in M, g \in G\}$;

$$(2.1) \qquad \begin{aligned} (\pi_\alpha(x)\xi)(h) &= \alpha_h^{-1}(x)\xi(h) \\ (\lambda_G(g)\xi)(h) &= \xi(g^{-1}h) \end{aligned}$$

for $\xi \in L^2(G; \mathfrak{H})$.

Each element ω in the predual $(G \otimes_\alpha M)_*$ of $G \otimes_\alpha M$ may be regarded as an element u_ω of $C^b(G; M_*)$;

$$(2.2) \qquad u_\omega[g](x) = \langle \pi_\alpha(x)\lambda_G(g), \omega \rangle$$

for all $x \in M$, $g \in G$ where $C^b(G; M_*)$ is the space of all bounded continuous M_*-valued functions on G. We denote $F_\alpha(G; M_*) = \{u_\omega; \omega \in (G \otimes_\alpha M)_*\} \subset C^b(G; M_*)$. A norm $\|\ \|$ is defined on $F_\alpha(G; M_*)$ by

$$\|u_\omega\| = \|\omega\| .$$

Then $\|u\|_\infty \leqq \|u\|$ for all $u \in F_\alpha(G; M_*)$ where $\|\ \|_\infty$ is the sup-norm on $C^b(G; M_*)$. We define a product on $F_\alpha(G; M_*)$ by

$$(2.3) \qquad (u*v)[g](x) = u[g](x)v[g](1)$$

for all $u, v \in F_\alpha(G; M_*)$, $x \in M$ and $g \in G$. Then $F_\alpha(G; M_*)$ is a Banach algebra ([2] Theorem 3.5). So that we can define products between $G \otimes_\alpha M$ and $F_\alpha(G; M_*)$;

$$\begin{aligned} \langle uT, v \rangle &= \langle T, v*u \rangle \\ \langle Tu, v \rangle &= \langle T, u*v \rangle \end{aligned}$$

for $T \in G \otimes_\alpha M$, $u, v \in F_\alpha(G; M_*)$((3.7), (3.9) in [2]).

Let T be an operator in $G \otimes_\alpha M$. Then the $\mathrm{supp}(T)$ of T is the set of all $g \in G$ satisfying the condition that $\lambda_G(g)$ belongs to the σ-weak closure of $TF_\alpha(G; M_*)$ [See [2] Proposition 4.1].

THEOREM 1. *Let (M, G, α), (N, H, β) be W^*-systems and $F_\alpha(G; M_*)$, $F_\beta(H; N_*)$ their associated Fourier algebras. Let ϕ be an isometric isomorphism of $F_\alpha(G; M_*)$ onto $F_\beta(H; N_*)$ as Banach algebras.*

Then we have five elements (k, p, q, I, θ) with the following properties:

(1) $k \in G$ such that $\lambda_G(k) = {}^t\phi(\lambda_H(e))$ where ${}^t\phi$ is the transposed map of ϕ and e is the identity of H.

(2) I is an isomorphism or anti-isomorphism of H onto G.

(3) p (resp. q) is a projection of $Z_M \cap M^G$ (resp. $Z_N \cap N^H$) where Z_M (resp. Z_N) is the center of M (resp. N) and $M^G = \{x \in M: \alpha_g(x) = x$ for all $g \in G\}$, $N^H = \{x \in N: \beta_h(x) = x$ for all $h \in H\}$.

(4) θ is an isometric linear map of N onto M such that

θ is an isomorphism of N_q onto M_p,

θ is an anti-isomorphism of N_{l-q} onto M_{l-p}.

(5) $\phi(u)[h](y) = ({}_ku)[I(h)](\theta(y)p) + ({}_ku)[I(h)]](\alpha_{I(h)}(\theta(y)(l-p)))$ for all $y \in N$, $h \in H$ and $u \in F_\alpha(G; M_*)$, where $({}_ku)[g](y) = u[kg](\alpha_k(y))$.

(6) $\theta[\beta_h(y)] = [\alpha_{I(h)}\theta(y)]p + [\alpha^{-1}_{I(h)}\theta(y)](l-p)$ for all $y \in N$, $h \in H$.

Proof. The transposed map ${}^t\phi$ of ϕ is an isometric linear map of $H \otimes_\beta N$ onto $G \otimes_\alpha M$. Using [3] Theorem 7, 10, we get;

$$ {}^t\phi = {}^t\phi(\lambda_H(e))(\gamma_I + \gamma_A) $$

where γ_I is an isomorphism of $(H \otimes_\beta N)_{z'}$ onto $(G \otimes_\alpha M)_z$, γ_A is an anti-isomorphism of $(H \otimes_\beta N)_{(l-z')}$ onto $(G \otimes_\alpha M)_{(l-z)}$, z (resp. z') being a central projection of $G \otimes_\alpha M$ (resp. $H \otimes_\beta N$). (2.4)

It follows from (2.3) that for all $u, v \in F_\alpha(G; M_*)$,

$$ \begin{aligned} \langle {}^t\phi(\lambda_H(h)),\ u*v\rangle &= \langle \lambda_H(h), \phi(u*v)\rangle \\ &= \langle \lambda_H(h), \phi(u)*\phi(v)\rangle \\ &= \langle \lambda_H(h) \otimes \lambda_H(h), \phi(u) \otimes \phi(v)\rangle \\ &= \langle {}^t\phi(\lambda_H(h)), u\rangle\langle {}^t\phi(\lambda_H(h)), v\rangle\ . \end{aligned} $$

Therefore ${}^t\phi(\lambda_H(h))$ is a character of $F_\alpha(G; M_*)$ for all $h \in H$, which implies that ${}^t\phi(\lambda_H(H)) \subseteqq \lambda_G(G)$ because the group of all characters $\widehat{F_\alpha(G; M_*)}$ is isomorphic to G ([2] Theorem 3.14), moreover since ϕ is an isomorphism,

$$ {}^t\phi(\lambda_H(H)) = \lambda_G(G)\ . $$

We denote $\lambda_G(k) = {}^t\phi(\lambda_H(e))$.

By the same argument in [10] Theorem 2, we get that

$$ \gamma \equiv {}^t\phi(\lambda_H(e))^{-1t}\phi = \gamma_I + \gamma_A \tag{2.5} $$

is a C^*-isomorphism in Kadison's sense [3] and $\gamma(\lambda_H(h_1)\lambda_H(h_2))$ is either $\gamma(\lambda_H(h_1))\gamma(\lambda_H(h_2))$ or $\gamma(\lambda_H(h_2))\gamma(\lambda_H(h_1))$, moreover if we put $\lambda_G(I(h)) = \gamma(\lambda_H(h))$,

(2.6) then I is either an isomorphism or an antiisomorphism of H onto G.

The transposed map ψ of γ is also an isometric isomorphism of $F_\alpha(G; M_*)$ onto $F_\beta(H; N_*)$. Then we get;

$$\begin{aligned}\langle\gamma(\pi_\beta(y)),\ u*v\rangle &= \langle\pi_\beta(y),\ \psi(u*v)\rangle\\ &= \langle\pi_\beta(y),\ \psi(u)*\psi(v)\rangle\\ &= \langle\pi_\beta(y)\otimes 1,\ \psi(u)\otimes\psi(v)\rangle\\ &= \langle\gamma(\pi_\beta(y)),\ u*v\rangle\end{aligned}$$

for all $y\in N$, $u, v\in F_\alpha(G; M_*)$.

By [5] Proposition 2.3, we obtain $\gamma(\pi_\beta(y))$ is an element of $\pi_\alpha(M)$, so that we can define an isometric surjective linear map θ of N onto M by $\theta=\pi_\alpha^{-1}\circ\gamma\circ\pi_\beta$.

Since γ is a Jordan isomorphism,

$$\gamma(T)\gamma(z')+\gamma(z')\gamma(T)=\gamma([T, z'])=2\gamma(Tz')$$

for all $T\in H\otimes_\beta N$, therefore we get $\gamma(Tz')=\gamma(T)z$.

Hence $\gamma(\pi_\beta(xy))z=\gamma(\pi_\beta(x))\gamma(\pi_\beta(y))z$ for all $x, y\in N$.

Since z is a central projection of $G\otimes_\alpha M$, z is also a projection of $\pi_\alpha(M)'$, then we get;

$$\gamma(\pi_\beta(xy))p=\gamma(\pi_\beta(x))\gamma(\pi_\beta(y))p \tag{2.7}$$

for all $x, y\in N$ where p is the central support of z in $\pi_\alpha(M)'$.

We denote by q the central support of z' in $\pi_\beta(N)'$, then the equations $\gamma(q)z=\gamma(qz')=\gamma(z')=z$ imply that $\gamma(q)p=p$, similarly we obtain $\gamma^{-1}(p)q=q$ so that $\gamma(q)=\gamma(\gamma^{-1}(p)q)=\gamma(\gamma^{-1}(p))\gamma(q)p=p\gamma(q)=p$.

Hence θ is an isomorphism of N_q onto M_p and θ is an anti-isomorphism of $N_{(1-q)}$ onto $M_{(1-p)}$.

The projection p (resp. q) is G-invariant (resp. H-invariant) since $\pi_\alpha(M)'=\lambda_G(g)\pi_\alpha(M)'\lambda_G(g)^*$ and $\lambda_G(g)z\lambda_G(g)^*=z$.

Now we have already proved (1) $\sim$ (4) and the statements (5), (6) still remain to prove.

For all $y\in N$, $h\in H$ we get,

$$\begin{aligned}\{\pi_\alpha\circ\theta(\beta_h(y))\}z &= \gamma(\lambda_H(h)\pi_\beta(y)\lambda_H(h)^*z')\\ &= \lambda_G(I(h))\pi_\alpha\circ\theta(y)\lambda_G(I(h)^{-1})z\\ &= \{\pi_\alpha\circ\alpha_{I(h)}\circ\theta\}(y)z\ ,\end{aligned}$$

hence

$$\theta\circ\beta_h=\alpha_{I(h)}\circ\theta \text{ on } N_q\ ,$$

and similarly

$$\theta\circ\beta_h=\alpha_{I(h^{-1})}\circ\theta \text{ on } N_{(1-q)}\ .$$

Therefore $\theta\circ\beta_h(y)=\alpha_{I(h)}\circ\theta(y)p+\alpha_{I(h^{-1})}\circ\theta(y)(1-p)$ for all $y\in N$ and $h\in H$. To prove the statement (5), we shall show first,

$$\operatorname{supp}\gamma(\pi_\beta(y)\lambda_H(h))=\{I(h)\}\ .$$

For since $\gamma(\pi_\beta(y)\lambda_H(h))u = \gamma(\pi_\beta(y)\lambda_H(h)\psi(u))$ for all $u \in F_\alpha(G; M_*)$ and ψ is surjective,

$$[\gamma(\pi_\beta(y)\lambda_H(h))F_\alpha(G; M_*)]^{-\sigma-w} = \gamma[\pi_\beta(y)\lambda_H(h)F_\beta(H; N_*)]^{-\sigma-w}$$

where $[\cdots]^{-\sigma-w}$ means a σ-weak closure, on the other hand,

$$[\pi_\beta(y)\lambda_H(h)F_\beta(H; N_*)]^{-\sigma-w} \cap \lambda_H(H) = \boldsymbol{C}\lambda_H(h)$$

because of $\operatorname{supp} \pi_\beta(y)\lambda_H(h) = \{h\}$, so that we obtain;

$$[\gamma(\pi_\beta(y)\lambda_H(h))F_\alpha(G; M_*)]^{-\sigma-w} \cap \lambda_G(G) = \boldsymbol{C}\lambda_G(I(h))$$
$$\operatorname{supp} \gamma(\pi_\beta(y)\lambda_H(h)) = \{I(h)\} .$$

By [2] Theorem 4.4 or [6] Proposition 6.1, there exists an element x of M such that $\gamma(\pi_\beta(y)\lambda_H(h)) = \pi_\alpha(x)\lambda_G(I(h))$.

$$\begin{aligned}\pi_\alpha(x)\lambda_G(I(h))z &= \gamma(\pi_\beta(y)\lambda_H(h))z \\ &= \gamma(\pi_\beta(y))\gamma(\lambda_H(h))z \\ &= \pi_\alpha(\theta(y))\lambda_G(I(h))z\end{aligned}$$

then

$$xp = \theta(y)p, \text{ and similarly } x(1-p) = \alpha_{I(h)}\theta(y)(1-p) .$$

We get;

$$x = \theta(y)p + \alpha_{I(h)}\theta(y)(1-p) ,$$

$$\gamma(\pi_\beta(y)\lambda_H(h)) = \pi_\alpha(\theta(y)p)\lambda_G(I(h)) + \pi_\alpha(\alpha_{I(h)}\theta(y)(1-p))\lambda_G(I(h)) .$$

By (2.2), $\phi(u) = \psi({}_k u)$ for $u \in F_\alpha(G; M_*)$ and the above equation, we can get the statement (5).

REMARK 2. Theorem 1 is a generalization of [10] Theorem 2.

COROLLARY 3. *Let (M, G, α), (N, H, β) be W^*-systems and the two actions α and β are ergodic on their centers (that is $Z_M \cap M^G = Z_N \cap N^H = \boldsymbol{C}$).*

The following statements are equivalent;

(1) $F_\alpha(G; M_*)$ is isomorphic to $F_\beta(H; N_*)$ in the sense of Banach algebra

(2) there exists either an isomorphism I of H onto G, an isomorphism θ of N onto M such that $\theta\circ\beta_h = \alpha_{I(h)}\circ\theta$ for all $h \in H$, or an anti-isomorphism I of H onto G, an anti-isomorphism θ of N onto M such that $\theta\circ\beta_h = \alpha_{I(h^{-1})}\circ\theta$ for all $h \in H$.

Proof. Suppose ϕ is an isometric isomorphism of $F_\alpha(G; M_*)$ onto $F_\beta(H; N_*)$ and we use the same notations in Theorem 1. The projection p in (3) of Theorem 1 must be zero or 1 by the ergodicity of the action α, then θ is either an isomorphism or an anti-isomorphism of N onto M.

When G is a locally compact abelian group (it follows from (2.6) that H is a locally compact abelian group), I in (2.6) can be regarded as both an isomorphism and an anti-isomorphism, therefore the statement (2) follows from Theorem 1 when G is abelian. Hence we may assume that G is non-abelian.

When I is an anti-isomorphism of H onto G, the projection $(1-z)$ in (2.4) must be nonzero. For if the projection z is the identity in $G \otimes_\alpha M$, then γ in (2.5) is an isomorphism of $H \otimes_\beta N$ onto $G \otimes_\alpha M$, so I is an isomorphism, which is a contradiction. Taking the central support of $(1-z)$ in $\pi_\alpha(M)'$ as (2.7), θ is an anti-isomorphism of H onto G such that $\alpha_{I(h^{-1})}\circ\theta = \theta\circ\beta_h$ for all $h \in H$. If I is an isomorphism, θ is an isomorphism such that $\alpha_{I(h)}\circ\theta = \theta\circ\beta_h$ for all $h \in H$.

Conversely suppose I is an isomorphism of H onto G such that $\theta\circ\beta_h = \alpha_{I(h)}\circ\beta_h$ for all $h \in H$. Then there exists an isomorphism Γ of $H \otimes_\beta N$ onto $G \otimes_\alpha M$ such that $\Gamma(\pi_\beta(y)) = \pi_\alpha(\theta(y))$ for all $y \in N$ and $\Gamma(\lambda_H(h)) = \lambda_G(I(h))$ for all $h \in H$ (cf. [9] Proposition 3.4). Then the transposed map ϕ of Γ is an isometric isomorphism of $F_\alpha(G; M_*)$ onto $F_\beta(H; N_*)$.

Suppose I is an anti-isomorphism of H onto G such that $\theta\circ\beta_h = \alpha_{I(h^{-1})}\circ\theta$ for all $h \in H$. Considering the opposite von Neumann algebra M° of M and the isomorphism J of H onto G by $J(h) = I(h^{-1})$ for all $h \in H$, there exists an isomorphism Γ of $H \otimes_\beta N$ onto $G \otimes_\alpha M^\circ$ such that $\Gamma(\pi_\beta(y)) = \pi_\alpha(\theta(y))$ for all $y \in N$, $\Gamma(\lambda_H(h)) = \lambda_G(J(h))$ for all $h \in H$. On the other hand, $G \otimes_\alpha M^\circ$ is isometrically isomorphic to $G \otimes_\alpha M$ as Banach spaces, therefore Γ is a σ-weakly continuous isometric linear map of $H \otimes_\beta N$ onto $G \otimes_\alpha M$. Then the transposed map ϕ of Γ is an isometric isomorphism of $F_\alpha(G; M_*)$ onto $F_\beta(H; N_*)$.

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