PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

RECURSION FORMULAS FOR THE HOMOLOGY OF $\Omega(X \vee Y)$

G. DULA AND E. KATZ

A recursion formula for $H(\Omega(X \vee Y))$, the homology of the loop space of the wedge of the spaces X and Y is established when ΩX and ΩY are connected, and have finite dimensional homology. The recursion formula is expressed in terms of $H(\Omega X)$ and $H(\Omega Y)$, and applies to dimensions higher than a fixed integer which depends on the dimension of the highest nonvanishing homologies of ΩX and ΩY. A similar but much simpler recursion formula for $H(\Omega X) \amalg H(\Omega Y)$, the co-product of the two algebras $H(\Omega X)$ and $H(\Omega Y)$ is also formulated. If G_1 and G_2 are topological groups and $G_1 * G_2$ is their co-product in the category, then our results definitely hold for $H(G_1 * G_2)$ by replacing ΩX by G_1, ΩY by G_2, and $\Omega(X \vee Y)$ by $G_1 * G_2$.

1. Introduction. Over a field $H(\Omega(X \vee Y))$ equals $H(\Omega X) \amalg H(\Omega Y)$ [1] [2], a fact which substantially simplifies the problem of computing the homology of $\Omega(X \vee Y)$. Over a Dedekind domain a torsion factor is added [5] [3] which significantly complicates the situation. Taking a principal ideal domain as the coefficient ring, $H(\Omega(X \vee Y))$ was computed in [3]. However, even if ΩX and ΩY are finite dimensional, those computations call for an increasing number of manipulations as the dimension of the homology to be computed gets higher. If n_1 and n_2 are the highest dimensions of non vanishing homologies of ΩX, ΩY, then for any $k > 3(n_1 + n_2) + 4$ we introduce a recursion formula which expresses $H_k(\Omega(X \vee Y))$ in terms of $H_i(\Omega(X \vee Y))$ $i < k$. The number of computations does not increase with k. Of course $H_i(\Omega(X \vee Y))$ with $i \leqq 3(n_1 + n_2)_{+4}$ has to be computed independently, for example by the method of [5].

In §2 we state the result of [5] in a generalized form which will be used here. We also present in this section most of the relevant notation of this paper. Recursion formulas in general are introduced in §3. The recursion formula for the free component of $H(\Omega(X \vee Y))$ is presented in §4. In §5 we derive a recursion formula for $H(\Omega X) \amalg H(\Omega Y)$. The main result which is a recursion formula for the torsion component of $H(\Omega(X \vee Y))$ is proved in §6. We close with an application by computing $H(SO_3 * SO_3)$.

The ring R will always be a principal ideal domain. The notation and terminology are those of [5].

2. The holomogy of $\Omega(X \vee Y)$ in dimension k. Let L^j be

free resolutions of the modules A^j, $j = 1, 2, \cdots, n$. Define $\operatorname{mult}_i^n(A^1, \cdots, A^n) = H_i(L^1 \otimes \cdots \otimes L^n)$. We have [3]:

$$\tilde{H}_k(\Omega(X \vee Y)) = \sum_{n=1}^{\infty}\sum_{i=1}^{n-1}\sum_{\Sigma r_t + i = k} \operatorname{mult}_i^n(\tilde{H}_{r_1}(\Omega X_1), \cdots, \tilde{H}_{r_n}(\Omega X_n))$$

where $r = (r_1, \cdots, r_n)$ is a sequence of nonnegative integers, $j = (j_1, \cdots, j_n)$ is a sequence alternating on 1, 2, and $\Omega X_2 = \Omega Y$. Thus the next step is to express explicitly the elements in the above summation. However, we first introduce some extra notation:

(i) $\operatorname{mult}_i^n(j, r) = \operatorname{mult}_i^n(\tilde{H}_{r_1}(\Omega X_{j_1}), \cdots, \tilde{H}_{r_n}(\Omega X_{j_n}))$.

(ii) $R(M) =$ the number of R direct summands in the module M.

$R_{p^h}(M) =$ the number of R_{p^h} direct summands in M where p is a prime in R and h is a nonnegative integer.

(iii) $a_i = R(\tilde{H}_i(\Omega X))$, $b_i = R(H_i(\Omega Y))$,
$c_i = \sum_{h'>h} R_{p^{h'}}(\tilde{H}_i(\Omega X))$, $\bar{c}_i = \sum_{h' \geqq h} R_{p^{h'}}(\tilde{H}_i(\Omega X))$,
$d_i = \sum_{h'>h} R_{p^{h'}}(\tilde{H}_i(\Omega Y))$, $\bar{d}_i = \sum_{h' \geqq h} R_{p^{h'}}(\tilde{H}_i(\Omega Y))$.

(iv) $m_t(l) =$ the number of times that $H_l(\Omega X_t)$ appears in $\operatorname{mult}_i^n(j, r)$, $t = 1, 2$, $l = 1, 2, \cdots, k$.

(v) $\phi(s_1, \cdots, s_k, t_1, \cdots, t_k) = \prod_{l=1}^k \bar{c}_l^{s_l} \bar{d}_l^{t_l} - \prod_{l=1}^k c_l^{s_l} d_l^{t_l}$,
$\psi(s_1, \cdots, s_k, t_1, \cdots, t_k) = \prod_{l=1}^k \binom{m_1(l)}{s_l} a_l^{(m_1(l)-s_l)} \cdot \prod_{l=1}^k \binom{m_2(l)}{t_l} b_l^{(m_2(l)-t_l)}$
where $0 \leqq s_l \leqq m_1(l)$, $0 \leqq t_l \leqq m_2(l)$ and

$$\binom{p}{q} = \begin{cases} 0 & q > p \text{ or } q < 0 \\ 1 & q = p \text{ or } 0 = q < p \\ \dfrac{p!}{(p-q)!\,q!} & \text{otherwise}. \end{cases}$$

With this notation we have [3]:

THEOREM 1. $R(\operatorname{mult}_0^n(j, r)) = \prod_{l=1}^k a_l^{m_1(l)} \cdot b_l^{m_2(l)}$,

$$R_{p^h}(\operatorname{mult}_i^n(j, r)) = \sum_{\substack{0 \leqq s_l \leqq m_1(l) \\ 0 \leqq t_l \leqq m_2(l) \\ 0 \leqq l \leqq k}} \psi(s_1, \cdots, t_k) \cdot \phi(s_1, \cdots, t_k) \binom{\sum_{l=1}^k (s_l + t_l) - 1}{i}.$$

We close this section with some further notation:

$$\tilde{H}(\Omega(X \vee Y)) = \operatorname{mult}^0(\Omega X, \Omega Y) \oplus \operatorname{mult}^1(\Omega X, \Omega Y)$$

where $\operatorname{mult}^0(\Omega X, \Omega Y) = \sum_{n,j,r} \operatorname{mult}_0^n(j, r)$

$$\operatorname{mult}^1(\Omega X, \Omega Y) = \sum_{n=1}^{\infty}\sum_{i=1}^{n-1}\sum_{j,r} \operatorname{mult}_i^n(j, r).$$

Note that $\operatorname{mult}^0(\Omega X, \Omega Y)$ is exactly $H(\Omega X) \coprod H(\Omega Y)$.

3. Recursion formulas. In this section we will make the general preparation for setting up the recursion formulas mentioned in the introduction.

Let $\{c_r\}_{r=1}^{\infty}$ be a sequence of numbers, and $q(x) = 1 - u_1x - u_2x^2 - \cdots - u_lx^l$ a polynomial. We define a new sequence $\{c'_r\}_{r=1}^{\infty}$ as follows:

$$c'_t = q_t\{c_r\} = c_t - u_1c_{t-1} - \cdots - u_lc_{t-l} .$$

The sequence $\{c_r\}$ satisfies the recursion formula corresponding to the polynomial $q(x)$ at t if $c'_t = q_t\{c_r\} = 0$.

The following results will be very useful for the sequel:

LEMMA 1. *Let $p(x)$, $q(x)$ be polynomials and $\{c_r\}_{r=1}^{\infty}$ a sequence of numbers. Then*:

$$q_t\{p_s\{c_r\}\} = (pq)_t\{c_r\} ,$$

where $(pq)(x) = p(x) \cdot q(x)$, the product of the two polynomials.

Proof. For $p(x) = \sum_{i=0}^{k} u_ix^i$ and $q(x) = \sum_{j=0}^{l} v_jx^j$ we have:

$$\begin{aligned} q_t\{p_s\{c_r\}\} &= \sum_{j=0}^{l} v_jp_{t-j}\{c_r\} = \sum_{j=0}^{l} v_j \sum_{i=0}^{k} u_ic_{t-j-i} \\ &= \sum_{h=0}^{l+k} \sum_{i+j=h} u_iv_jc_{t-h} = (pq)_t\{c_r\} , \end{aligned}$$

which completes the proof.

LEMMA 2. *Let $\{c_r\}_{r=1}^{\infty}$ satisfy the polynomial $p(x) = \sum_{i=0}^{k} u_ix^i$ at $t, t-1, \cdots, t-l$, and $\{d_r\}_{r=1}^{\infty}$ satisfy the polynomial $q(x) = \sum_{j=0}^{l} v_jx^j$ at $t, t-1, \cdots, t-k$. Then the sequence $\{c_r + d_r\}_{r=1}^{\infty}$ satisfies the polynomial $q(x) \cdot p(x)$ at t.*

Proof.

$$\begin{aligned} (qp)_t\{c_r + d_r\} &= (qp)_t\{c_r\} + (pq)_t\{d_r\} \\ &= q_t\{p_t\{c_r\}\} + p_t\{q_t\{d_r\}\} \\ &= \sum_{j=0}^{l} v_jp_{t-j}\{c_r\} + \sum_{i=0}^{k} u_iq_{t-i}\{d_r\} = 0 . \end{aligned}$$

We are now ready for the construction of the recursion formulas.

4. A recursion formula for the free part of $H(\Omega(X \vee Y))$. Our interest in this section is focused on the sequence $\{\alpha_k\}$ where

$$\alpha_k = R(\tilde{H}_k(\Omega(X \vee Y))) .$$

Since $\tilde{H}(\Omega(X \vee Y)) = \tilde{H}(\Omega X) \amalg \tilde{H}(\Omega Y) \oplus \mathrm{mult}^1(\Omega X, \Omega Y)$ and

$$R(\mathrm{mult}^1(\Omega X, \Omega Y)) = 0$$

we actually have $\alpha_k = R(\tilde{H}(\Omega X) \amalg \tilde{H}(\Omega Y)_k)$ $k \geqq 1$.

THEOREM 2. *Let*

$$R(H_i(\Omega X)) = \begin{cases} 0 & i > n_1 \\ a_i & i \leqq n_1 \end{cases}$$
$$R(H_i(\Omega Y)) = \begin{cases} 0 & i > n_2 \\ b_i & i \leqq n_2 . \end{cases}$$

Then the sequence $\{\alpha_k\}_{k=0}^{\infty}$ *satisfies the recursion formula*:

$$q_1(x) = 1 - \sum_{i=1}^{n_1}\sum_{j=1}^{n_1} a_i b_j x^{i+j} , \qquad \text{for any} \quad k > n_1 + n_2 .$$

The proof of this theorem derives from the following:

PROPOSITION 1. $\alpha_k = a_k + b_k + 2\sum_{i+j=k} a_i b_j + \sum_{i+j<k} a_i b_j \alpha_{k-i-j}$.

Proof. According to the definition of α_k we have

$$\alpha_k = \sum_{n\ j, \Sigma r_t = k} \mathrm{mult}_0^n(j, r) .$$

We can split up this sum into, $\alpha_k = A + B + C$, where:

$$A = \sum_{n=1, j, r=(k)} R(\mathrm{mult}_0^1(j, r))$$
$$B = \sum_{n=2, j, \Sigma r_t = k} R(\mathrm{mult}_0^2(j, r))$$
$$C = \sum_{n \geqq 3, j, \Sigma r_t = k} R(\mathrm{mult}_0^n(j, r)) .$$

Next we compute each term separately:

$$A = R(\mathrm{mult}_0^1(\tilde{H}_k(\Omega X))) + R(\mathrm{mult}_0^1(\tilde{H}_k(\Omega Y))) = a_k + b_k$$
$$\begin{aligned} B &= \sum_{r_1+r_2=k} R(\mathrm{mult}_0^2((1, 2), r)) + \sum_{r_1+r_2=k} R(\mathrm{mult}_0^2((2, 1), r)) \\ &= \sum_{r_1+r_2=k} a_{r_1} \cdot b_{r_2} + \sum_{r_1+r_2=k} b_{r_1} \cdot a_{r_2} = 2 \sum_{i+j=k} a_i b_j . \end{aligned}$$

The computation of C is somewhat more complicated. Let $\mathrm{mult}_0^n(j, r)$ be a direct summand of $(\mathrm{mult}^0(\Omega X, \Omega Y))_k$, with $n \geqq 3$. We denote $\hat{j} = (j_1, \cdots, j_{n-2})$ and $\hat{r} = (r_1, \cdots, r_{n-2})$. Then it is not difficult to see that:

$$R(\mathrm{mult}_0^n(j, r)) = R(\Omega X_{j_{n-1}}) \cdot R(\Omega X_{j_n}) \cdot R(\mathrm{mult}_0^n(\hat{j}, \hat{r})) .$$

Summing up the last equality on the proper possibilites of j and r we get the desired equality for A.

Proof of Theorem 2. If $k > n_1 + n_2$ then each one of a_k, b_k and $a_i b_j$ with $i + j = k$, equals zero. Thus for $k > n_1 + n_2$ the equation of Proposition 2 reduces to:

$$\alpha_k = \sum_{i+j<k} a_i b_j \alpha_{k-i-j} = \sum_{\substack{i \leqq n_1 \\ j \leqq n_2}} a_i b_j \alpha_{k-i-j} ,$$

which is exactly the result of Theorem 2.

5. **A recursion formula for the torsion component of $H(\Omega X) \amalg H(\Omega Y)$.** In this section we want to find a convenient way of expressing the k dimensional part of $H(\Omega X) \amalg H(\Omega Y)$. We do it by forming a recursion formula for the number of R_{p^h} direct summands in each dimension, for each R_{p^h}, which is a direct summand of either $\tilde{H}(\Omega X)$ or $\tilde{H}(\Omega Y)$. If R_{p^h} is one of these modules, we denote:

$$\beta_k = R_{p^h}(\tilde{H}(\Omega X) \amalg \tilde{H}(\Omega Y)) .$$

THEOREM 3. *Let n_3 and n_4 be integers such that: $a_k + \bar{c}_k > 0$ implies that $k \leqq n_3$ and $b_k + \bar{d}_k > 0$ implies that $k \leqq n_4$. Consider the polynomials*:

$$q_2(x) = 1 - \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} (a_i + \bar{c}_i)(b_j + \bar{d}_j) x^{i+j}$$

$$q_3(x) = 1 - \sum_{i=1}^{n_3} \sum_{j=1}^{n_4} (a_i + c_i)(b_j + d_j) x^{i+j} .$$

Then for $k > 2(n_3 + n_4)$ the polynomial $q_2(x) \cdot q_3(x)$ corresponds to the recursion formula for $\{\beta_k\}_{k=1}^{\infty}$.

For the proof we need some intermediate result as well as some auxiliary functions. The following functions are similar to functions introduced in §2.

(i) $$\phi^1(s_1, \cdots, s_k, t_1, \cdots, t_k) = c_1^{s_1} \cdots c_k^{s_k} \cdot d_1^{t_1} \cdots d_k^{t_k}$$

(ii) $$R^1_{p^h}(\mathrm{mult}^n_i (j, r)) = \sum_{\substack{0 \leqq s_l \leqq m_1(l) \\ 0 \leqq t_l \leqq m_2(l) \\ 1 \leqq l \leqq k}} \psi(s_1, \cdots, t_k) \cdot \phi^1(s_1, \cdots, t_k) \binom{\sum_{l=1}^k s_l + t_l - 1}{i} .$$

(iii) $$\beta^1_k = \sum_{n, j, \sum r_l = k} R^1_{p^h}(\mathrm{mult}^n_0 (j, r)) .$$

Note that in the expression $R_{p^h}(\mathrm{mult}^n_0(j, r))$ the binomial term $\binom{\sum_{l=1}^k s_l + t_l - 1}{0}$ can be omitted. For if $\sum_{l=1}^k s_l + t_l \geqq 1$ the binomial term equals 1, and if $\sum_{l=1}^{k} s_l + t_l = 0$ the function $\phi(s_1, \cdots, t_k)$ is zero.

PROPOSITION 2.

$$\begin{aligned}\beta_k = &(\bar{c}_k - c_k) + (\bar{d}_k - d_k)\\ &+ 2\sum_{i+j=k}[(a_i + \bar{c}_i)(b_j - \bar{d}_j) - (a_i + c_i)(b_j + d_j)]\\ &+ \sum_{i+j<k}(a_i + \bar{c}_i)(b_j + \bar{d}_j)\beta_{k-i-j}\\ &+ \sum_{i+j=k}[(a_i + \bar{c}_i)(b_k + \bar{d}_j) - (a_i + c_i)(b_j + d_j)]\beta^1_{k-i-j}\,.\end{aligned}$$

Proof. We split up β_k into three, $\beta_k = \sum_{n,j,\cdot r_t = k} R_{p^h}(\mathrm{mult}_0^n(j, r)) = A + B + C$, and compute each term separately:

$$\begin{aligned}A &= R_{p^h}(\mathrm{mult}_0^1((1), (k))) + R_{p^h}(\mathrm{mult}_0^1((2), (k)))\\ &= (\bar{c}_k - c_k) + (\bar{d}_k - d_k)\,,\end{aligned}$$

$$\begin{aligned}B &= \sum_{j\ \Sigma r_t = k} R_{p^h}(\mathrm{mult}_0^2(j, r)) = 2\sum_{i+k=k} R_{p^h}(\tilde{H}_i(\Omega X) \otimes \tilde{H}_j(\Omega Y))\\ &= 2\sum_{i+j=k}[(a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)]\,.\end{aligned}$$

The last term is more complicated, and needs some preliminary computations. For $n \geqq 3$ and $j = (j_1, \cdots, j_n)$, $r = (r_1, \cdots, r_n)$ we denote $\hat{j} = (j_1, \cdots, j_{n-2})$, $\hat{r} = (r_1, \cdots, r_{n-2})$. If $j_{n-1} = 1$ and $j_n = 2$ we denote the following:

$$\hat{m}_1(i) = \begin{cases} m_1(i) & i \neq r_{n-1} \\ m_1(i) - 1 & i = r_{n-1} \end{cases}$$

$$\hat{m}_2(i) = \begin{cases} m_2(i) & i \neq r_n \\ m_2(i) - 1 & i = r_n\,. \end{cases}$$

Consider the following:

$$\begin{aligned}R_{p^h}(\mathrm{mult}_0^n(j, r)) = &\sum_{\substack{0 \leqq s_l \leqq m_1(l)\\ 0 \leqq t_l \leqq m_2(l)\\ 1 \leqq l \leqq k}} \prod_{l \neq r_{n-1}} \left\{\binom{m_1(l)}{s_l} a^{(m_1(l) - s_l)}\right\}\\ &\times \prod_{l \neq r_n}\left\{\binom{m_2(l)}{t_l} b^{(m_2(l) - t_l)}\right\} \cdot \binom{\hat{m}_1(r_{n-1}) + 1}{s_{r_{n-1}}}\binom{\hat{m}_2(r_n) + 1}{t_{r_n}} \cdot a_{r_n}^{(m_1(r_{n-1}) - s_{r_{n-1}})}\\ &\times b_{r_n}^{m_2(r_n) - t_{r_n}} \cdot \phi(s_1, \cdots, t_k) = U_1 + U_2 + U_3 + U_4\end{aligned}$$

where each of the U_i equals the previous sum except that $\binom{\hat{m}_1(r_{n-1}) + 1}{s_{r_{n-1}}}\binom{\hat{m}_2(r_n) + 1}{t_{r_n}}$ is replaced by one of the terms

$$\binom{\hat{m}_1(r_{n-1})}{s_{r_{n-1}}}\binom{\hat{m}_2(r_n)}{t_{r_n}}, \quad \binom{\hat{m}_1(r_{n-1})}{s_{r_{n-1}}}\binom{\hat{m}_2(r_n)}{t_{r_n}-1}, \quad \binom{\hat{m}_1(r_{n-1})}{s_{r_{n-1}}-1}\binom{\hat{m}_2}{t_{r_n}},$$
$$\binom{\hat{m}_1(r_{n-1})}{s_{r_{n-1}}-1}\binom{m_2(r_n)}{t_{r_n}-1}$$

respectively. Now we compare $\mathrm{mult}_0^n(j, r)$ with $\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})$:

$$\begin{aligned}
U_1 &= a_{r_{n-1}} b_{r_n} \sum_{\substack{0 \leqq s_l \leqq \hat{m}_1(l) \\ 0 \leqq t_l \leqq \hat{m}_2(l) \\ 1 \leqq l \leqq k}} \prod_{l=1}^{k} \left\{ \binom{m_1(l)}{s_l} a_l^{(\hat{m}_1(l)-s_l)} \right\} \\
&\quad \times \prod_{l=1}^{k} \binom{m_2(l)}{t_l} b_l^{(m_2(l)-t_l)} \cdot \phi(s_1, \cdots, t_k) \\
&= a_{r_{n-1}} b_{r_n} R_{p^h} \,\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r}) ,
\end{aligned}$$

$$\begin{aligned}
&U_2 - a_{r_{n-1}} \bar{d}_{r_n} \cdot R_{p^h} \,\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r}) \\
&= a_{r_{n-1}} \cdot \sum_{\substack{0 \leqq s_l \leqq m_1(l) \\ 0 \leqq t_l \leqq m_2(l) \\ 1 \leqq l \leqq k}} \prod_{l=1}^{k} \left\{ \binom{\hat{m}_1(l)}{s_l} a_l^{(\hat{m}_1(l)-s_l)} \right\} \prod_{l=1}^{k} \left\{ \binom{\hat{m}_2(l)}{t_l} b_l^{(m_2(l)-t_l)} \right\} \\
&\quad \times \phi(s_1, \cdots, t_{r_n}+1, \cdots, t_k) - a_{r_{n-1}} \bar{d}_{r_n} R_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) \\
&= a_{r_{n-1}} \sum_{\substack{0 \leqq s_l \leqq \hat{m}_1(l) \\ 0 \leqq t_l \leqq \hat{m}_2(l) \\ 1 \leqq l \leqq k}} \prod_{l=1}^{k} \left\{ \binom{m_2(l)}{s_l} \binom{m_2(l)}{t_l} \cdot a_l^{(m_1(l)-s_l)} b_l^{(m_2(l)-t_l)} \right\} \\
&\quad \times [\phi(s_1, \cdots, t_{r_n}+1, \cdots, t_k) - \bar{d}_{\lambda_n} \phi(s_1, \cdots, t_k)] \\
&= a_{r_{n-1}} (\bar{d}_{r_n} - d_{r_n}) R^1_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) ,
\end{aligned}$$

$$U_3 - b_{r_n} \bar{c}_{r_{n-1}} R_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) = b_{r_n}(\bar{c}_{r_n} - c_{r_n}) R^1_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r}))$$

$$U_4 - \bar{c}_{r_{n-1}} d_{r_n} R_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) = (\bar{c}_{r_{n-1}} \bar{d}_{r_n} - c_{r_{n-1}} d_{r_n}) R^1_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) .$$

Adding up the last equations we get:

$$\begin{aligned}
&R_{p^h}(\mathrm{mult}_0^n(j, r)) - (a_{r_{n-1}} + \bar{c}_{r_{n-1}})(b_{r_n} + \bar{d}_{r_n}) R_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) \\
&\quad = [(a_{r_{n-1}} + \bar{c}_{r_n})(b_{r_n} + \bar{d}_{r_n}) - (a_{r_{n-1}} + c_{r_{n-1}})(b_{r_n} + d_{r_n})] \\
&\quad\quad \times R^1_{p^h}(\mathrm{mult}_0^{n-2}(\hat{j}, \hat{r})) .
\end{aligned}$$

We observe that the equation holds also when $j_{n-1} = 2$, $j_n = 1$ and $r = (r_1, \cdots, r_{n-2}, r_n, r_{n-1})$. Summing up the later equation for all j and r, we get:

$$\begin{aligned}
C = &\sum_{i+j<k} (a_i + \bar{c}_i)(b_j + \bar{d}_j)\beta_{k-i-j} + \sum_{i+j<k} (a_i + \bar{c}_i)(b_j + \bar{d}_j) \\
&- (a_i + c_i)(b_j + d_j)\beta^1_{k-i-j} .
\end{aligned}$$

This concludes the proof of Proposition 2:

PROPOSITION 3.

$$\beta_k^1 = (a_k + c_k) + (b_k + d_k) + 2 \sum_{i+j=k} (a_i + c_i)(b_j + d_j) + \sum_{i+j>k} (a_i + c_i)(b_j + d_j)\beta^1_{k-i-j} .$$

Although the proof of this proposition is lengthy, it is similar to the proof of Proposition 2 and will therefore be skipped.

Proof of Theorem 3. Under the conditions of the theorem, the formulas of β_k and β_k^1 reduce to the following:

$$\beta_k = \sum_{i=1}^{n_3}\sum_{j=1}^{n_4} (a_i + c_i)(b_j + \bar{d}_j)\beta_{k-i-j} + \sum_{i=1}^{n_3}\sum_{j=1}^{n_4} [(a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)]\beta^1_{k-i-j} .$$

We observe that:

$$(q_2\{\beta_l\})_k = \sum_{i=1}^{n_3}\sum_{j=1}^{n_4} [(a_i + \bar{c}_i)(b_i + \bar{d}_j) - (a_i + c_i)(b_j + d_j)]\beta^1_{k-i-j}$$

$$(q_3\{\beta^1_l\})_k = 0 .$$

As a consequence of the results of §3 the proof of the theorem is obtained.

6. A recursion formula for the torsion component of $H(\Omega(X \vee Y))$. We denote the number of R_{p^h} direct summands in $H_k(\Omega(X \vee Y))$ by γ_k. A recursion formula for $\{\gamma_k\}$ will be stated next precisely:

THEOREM 4. *Let n_3, n_4 satisfy: $a_i + \bar{c}_i > 0$ and $b_j + \bar{d}_j > 0$ imply that $i \leqq n_3$, $j \leqq n_4$.*

Denote:

$$q_4(x) = 1 - \sum_{i=1}^{n_3}\sum_{j=1}^{n_4} [a_i b_j x^{i+j} + (a_i\bar{d}_j + b_j\bar{c}_i)x^{i+j}(1 + x) + \bar{c}_i\bar{d}_j x^{i+j}(1 + x)^2]$$

$$q_5(x) = 1 - \sum_{i=1}^{n_3}\sum_{j=1}^{n_4} [a_i b_j x^{i+j} + (a_i d_j + b_j c_i)x^{i+j}(1 + x) + c_i d_j x^{i+j}(1 + x)^2] .$$

Then $\{\gamma_k\}_{k=1}^{\infty}$ satisfies the recursion formula corresponding to $q_4(x) \times q_5(x) \cdot q_1(x)$ at any $k > 3(n_3 + n_4) + 4$.

The proof of Theorem 4 is much more complicated than the proofs of the previous theorems. However, in principal it is similar to them. We state the intermediate results and leave the proofs for the reader. We denote:

$$\gamma_k^1 = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{j} \sum_{r_t+i=k} R_{ph}^1(\mathrm{mult}_i^n(j, r)) .$$

PROPOSITION 4.

$$\begin{aligned}\gamma_k = {} & [(\bar{c}_k - c_k) + (\bar{d}_k - d_k)] \\ & + 2 \sum_{i+j=k} \{(a_i + \bar{c}_i)(b_j + \bar{d}_j) - (a_i + c_i)(b_j + d_j)\} \\ & + 2 \sum_{i+j=k-1} \{\bar{c}_i \bar{d}_j - c_i d_j\} \\ & + \sum_{i+j<k} \{a_i b_j \gamma_{k-i-j} + a_i \bar{d}_j(\gamma_{k-i-j} + \gamma_{k-i-j-1}) \\ & + b_j \bar{c}_i(\gamma_{k-i-j} + \gamma_{k-i-j-1}) + \bar{c}_i \bar{d}_j(\gamma_{k-i-j} + 2\gamma_{k-i-j-1} + \gamma_{k-i-j-2})\} \\ & + \sum_{i+j<k} \{a_i(\bar{d}_j - d_j)(\gamma_{k-i-j}^1 + \gamma_{k-i-j-1}^1) + b_j(\bar{c}_i - c_i)(\gamma_{k-i-j}^1 + \gamma_{k-i-j-1}^1) \\ & + (\bar{c}_i \bar{d}_j - c_i d_j) \cdot (\gamma_{k-i-j}^1 + 2\gamma_{k-i-j-1}^1 + \gamma_{k-i-j-2}^1)\} .\end{aligned}$$

PROPOSITION 5.

$$\begin{aligned}\gamma_k^1 = {} & c_k + d_k + 2 \sum_{i+j=k} \{(a_i + c_i) \cdot (b_j + d_j) - a_i b_j\} \\ & + 2 \sum_{i+j=k-1} c_i d_j + \sum_{i+j<k} \{(a_i b_j \gamma_{k-i-j}^1 + a_i d_j)(\gamma_{k-i-j}^1 + \gamma_{k-i-j-1}^1) \\ & + b_j c_i(\gamma_{k-i-j}^1 + \gamma_{k-i-j-1}^1) + c_i d_j(\gamma_{k-i-j}^1 + 2\gamma_{k-i-j-1}^1 + \gamma_{k-i-j-2}^1) \\ & + (c_i b_i + d_j a_i + c_i d_j)\alpha_{k-i-j}\} .\end{aligned}$$

7. **An example.** In this section we apply the recursion formulas obtained, to compute the holomogy of the free product of two groups $G_1 * G_2$. Our method holds in this case, for $G_1 * G_2$ is of the homotopy type of $\Omega(BG_1 \vee BG_2)$ and ΩBG_i is of the homotopy type of G_i $i = 1, 2$, where BG_i is the classifying space of G_i, $i = 1, 2$, [4], [7]. We actually demonstrate our method of computation on the free product of the special orthogonal group SO_3 with itself. The homology of SO_3 is computed [6] and equals:

$$H_j(SO_3) = \begin{cases} Z & j = 0, 3 \\ Z_2 & j = 1 \\ 0 & \text{otherwise} . \end{cases}$$

We are content with this group, because though its homology is simple the homology of $\mathrm{SO}_3 * \mathrm{SO}_3$ is infinite dimensional and complicated.

The recursion formulas for $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\gamma_k\}$ can be applied only to $k > n_1 + n_2$, $2(n_3 + n_4)$, $3(n_3 + n_4) + 4$ respectively, when we know the sequences in lower dimensions. Of course $\{\beta_k\}$ and $\{\gamma_k\}$ correspond to the number of Z_2 summands. The sequences of $\{\alpha_k\}$ and $\{\gamma_k\}$ in the lower dimensions were computed in [3], essentially by the use of the formulas of §2. $\{\beta_k\}$ can be computed similarly.

We summarize these results in the following table:

	α	β	γ
1	0	2	2
2	0	2	2
3	2	2	4
4	0	6	10
5	0	8	16
6	2	10	30
7	0	18	58
8	0	26	104
9	2	36	192
10	0	56	356
11	0	82	652
12	2	118	1200
13	0	176	2210
14	0	258	4062
15	2	376	7472
16	0	554	13746
17	0	812	25280
18	2	1188	46498

Where the numbers beneath the heavy lines can be computed by the recursion formulas as will be seen presently.

The recursion formulas are the following:

$$\begin{aligned}
q_1(x) &= 1 - x^6\\
q_2(x) &= 1 - x^2 - 2x^4 - x^6\\
q_3(x) &= 1 - x^6\\
(q_2q_3)(x) &= 1 - x^2 - 2x^4 - 2x^6 + x^8 + 2x^{10} + x^{12}\\
q_4(x) &= 1 - x^2 - 2x^3 - 3x^4 - 2x^5 - x^6\\
q_5(x) &= 1 - x^6\\
(q_4q_5)(x) &= 1 - x^2 - 2x^3 - 3x^4 - 2x^5 - 2x^6 + x^8 + 2x^9 + 3x^{10} + 2x^{11} + x^{12}\,.
\end{aligned}$$

As to that $q_1(x) = q_5(x)$, $(q_4q_5)(x)$ expresses the recursion formula for γ.

For example, to obtain γ_{17} we substitute into:

$$\gamma_{17} = \gamma_{15} + 2\gamma_{14} + 3\gamma_{13} + 2\gamma_{12} + 2\gamma_{11} - \gamma_9 - 2\gamma_8 - 3\gamma_7 - 2\gamma_6 - \gamma_5 = 25280\,.$$

References

1. I. Berstein, *On co-groups in the category of graded algebras*, Trans. Amer. Math. Soc., **115** (1965), 257–269.

2. A. Clark and L. Smith, *The rational homotopy of a wedge*, Pacific J. Math., **24** (1968), 241-246.

3. G. Dula and E. Katz, *Computation of the homology of $\Phi(X \vee Y)$*, submitted.

4. D. M. Kan, *A combinational definition of homotopy groups*, Ann. of Math., (2) **67** (1958), 282-312.

5. E. Katz, *A Künneth formula for coproducts of simplicial groups*, Procr. Amer. Math. Soc., **61**, No. 1 (1976), 117-121.

6. C. Miller, *The topology of rotation groups*, Ann. of Math., (2) **57** (1953), 90-114.

7. J. W. Milnor, *Construction of universal bundles*, I, Ann. of Math., (2) **63** (1956), 272-284.

Received January 21, 1978.

UNIVERSITY OF HAIFA
MOUNT CARMEL, HAIFA, 31999, ISRAEL

Current address: Tel-Aviv University
Tel-Aviv, Israel