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PIECEWISE CATENARIAN AND GOING BETWEEN RINGS

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The main purpose of this paper is to prove the following theorem. Let R be a noetherian ring and n a nonnegative integer. Then $R[X_1, \cdots, X_n]$ is a going-between ring (=GB) iff R is GB and is $(n+1)$-piecewise catenarian.

In [7] Ratliff proved that all polynomial rings over an unitary commutative noetherian going-between-(= GB)-ring R are again GB iff R is catenarian (thus universally catenarian by [6, (3.8)] and [5, (2.6)]). (Recall that R is called a GB-ring if for any integral extension R' of R each adjacent pair of Spec (R') retracts to an adjacent pair of Spec (R).)

In the meantime we showed that there are noetherian GB-rings which are not catenarian, thus giving a negative answer to a corresponding question of [7] (s. [2]). So it may be interesting to know more about the relations between the GB-property of polynomial rings and the chain structure of Spec (R). In this note we shall investigate such a relation, thereby improving Ratliff's above result.

To formulate our statement, let us give the following

DEFINITION 1. R is called *n-piecewise catenarian* ($= C_n$). If $(R/P)_{\mathscr{Q}}$ is catenarian for any pair P, Q of Spec (R) related by a saturated chain $P = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_i = Q$ of length $i \leqq n$.

Our main goal is to prove

THEOREM 2. *Let R be a noetherian ring and n a nonnegative integer. Then $R[X_1, \cdots, X_n]$ is* GB *iff R is* GB *and satisfies the property C_{n+1}.*

Noticing that R is catenarian iff it is C_n for all $n > 1$, this gives immediately the quoted result of Ratliff.

To prove 2, let us introduce the following notations

3. (i) $c(R)$ = set of lengths of maximal chains $P_0 \subsetneqq P_1 \subsetneqq \cdots$ of Spec (R) (s. [3], where $c(R)$ was investigated).

(ii) If R is semilocal with Jacobson radical J, put $\hat{d}(R) = \min\{\dim(\hat{R}/\hat{P})$, where $\hat{P}$ is a minimal prime of $\hat{R}\}$, $\hat{R}$ denoting the J-adic completion of R (s. [1]).

We also shall use the following characterization of GB-rings, whose proof is immediate from the basic results of [6] and [7].

PROPOSITION 4. *For a noetherian ring R the following statements are equivalent:*

(i) *R is* GB.

(ii) *For all $P, Q \in \mathrm{Spec}(R)$ with $P \subseteq Q$ the ring $T = (R/P)_Q$ is* GB.

(iii) *For all T as in* (ii) *we have $c(T) = c(\hat{T})$.*

(iv) *For all T as in* (ii) *we have $\min \mathrm{c}(\hat{T}) = \hat{d}(T) = \min \mathrm{c}(T)$.*

(v) *For all T as in* (ii) *which moreover are of dimension > one, we have $\hat{d}(R) > 1$.*

To prove 2 we start with the case $n = 1$.

LEMMA 5. *Let R be a noetherian ring. Then $R[X]$ is* GB *iff R is* GB *and satisfies C_2.*

Proof. "$\Leftarrow$" Let $R[X]$ be GB. Then so obviously is $R = R[X]/(X)$.

To show that R satisfies C_2 let $P \subsetneqq Q \subsetneqq S$ be a saturated chain of $\mathrm{Spec}(R)$ such that $ht(S/P) > 2$. We have to prove that $R[X]$ fails to be GB under this assumption. In replacing R by $(R/P)_S$ we may restrict ourselves to show that $R[X]$ is not GB, where (R, M) is a local domain of dimension $> 2 = \min \mathrm{c}(R)$, which moreover is GB.

Let $\hat{P}$ be a minimal prime of $\hat{R}$ whose dimension is 2 (such a $\hat{P}$ exists by (4)). Choose $a \in M - (O)$ and let $b \in M$ be outside of all minimal prime divisors of aR and of $a\hat{R} + \hat{P}$. Put $f = aX + b$. Then we first have the inclusion $f\hat{R}[X] + \hat{P}\hat{R}[X] \subseteq M\hat{R}[X]$ showing that there is a minimal prime $\tilde{Q}$ of $f\hat{R}[X] + \hat{P}\hat{R}[X]$ with $\tilde{Q} \subseteq M\hat{R}[X]$. As $ht(\tilde{Q}/\hat{P}\hat{R}[X]) = 1$, we have the following two possibilities for $\hat{Q} = \tilde{Q} \cap R$:

$\hat{Q} = \hat{P}$, or $ht(\hat{Q}/\hat{P}) = 1$ and a and b belong to $\hat{Q}$. By our choice of a and b we may exclude the second case. So, as $ht(\tilde{Q}/\hat{P}\hat{R}[X]) = 1$, $\tilde{Q}$ is a minimal prime of $f\hat{R}[X]$. But now $ht(M\hat{R}[X]/\hat{P}\hat{R}[X]) = ht(M\hat{R}/\hat{P})$ implies that $ht(M\hat{R}[X]/\tilde{Q}) = 1$. From this we conclude that $f(\hat{R}[X]_{M\hat{R}[X]})$ has a minimal prime divisor of dimension one. On the other hand we have a canonical isomorphism of $R[X]$-algebras

$$(\hat{R}[X]_{M\hat{R}[X]})^{\wedge} \simeq (R[X]_{MR[X]})^{\wedge} ,$$

which shows that $f(R[X]_{MR[X]})^{\wedge}$ has a minimal prime divisor of dimension one.

Let us denote this prime divisor by S' and put $S = S' \cap R[X]_{MR[X]}$.

Then, by the flatness of completion, S' is a minimal prime divisor of $SR[X]_{MR[X]}$ and S is a minimal prime divisor of $fR[X]_{MR[X]}$. Our choice of a and b implies that $S'' = R[X] \cap (R-(O))^{-1}fR[X]$ is the unique minimal prime divisor of $fR[X]$. Thus $S = S''R[X]_{MR[X]}$ is the unique minimal prime divisor of $fR[X]_{MR[X]}$. This implies that $T = R[X]_{MR[X]}/S$ is of dimension $ht(M)-1 > 1$ but such that $\hat{d}(T) \leqq \dim(S') = 1$. So, by (i) $\Rightarrow$ (v) of (4) $R[X]$ is not GB.

"$\Longrightarrow$" By 4 we may restrict ourselves to prove

6. *Let (R, M) be a noetherian local domain which is* GB *and C_2 and let U be a simply generated extension domain of R. Let $N \in \operatorname{Spec}(U)$ such that $N \cap R = M$ and $ht(N) > 1$. Then it holds $\hat{d}(U_N) > 1$.*

Put $U_N = T$. If $\hat{d}(R) \leqq 2$ 4 shows that $\min c(R) \leqq 2$. Thus the C_2 property of R and 4 imply that $\hat{d}(R) = \dim(R)$, hence that R is quasiunmixed. But then T is also quasinmixed ([5], Cor. (2.6)] and therefore satisfies $\hat{d}(T) = ht(N) > 1$.

If $\hat{d}(R) > 2$ we use the inequality

$$\hat{d}(T) - \hat{d}(R) \geqq \operatorname{deg\,trans}(T\colon R) - \operatorname{deg\,trans}(U/N\colon R/M)$$

(s. [1, (4.4) (i)]), which gives the result as both of its right hand terms are 0 or 1.

Next we give two results which deal with the C_n property of polynomial rings.

LEMMA 7. *Let (R, M) be a noetherian local domain and let $(O) = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = M (n \geqq 2)$ be a maximal chain of* $\operatorname{Spec}(R)$ *such that $ht(M/P_{n-2}) = 2$. Then there is a saturated chain $Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_{n-2} \subsetneqq Q_{n-1} = MR[X]$ satisfying:*

$$Q_i \cap R = P_i \text{ and } ht(MR[X]/Q_i) = ht(M/P_i) - 1 \text{ for } i = 1, \cdots, n-2\,.$$

Proof. Choose $a \in M - P_{n-2}$ and let $b \in M$ be outside of all minimal prime divisors of $aR + P_i$ for $i = 1, \cdots, n-2$. Put $f = aX + b$. Then for all indices i in question $fR[X] + P_iR[X]$ has exactly one minimal prime divisor, say Q_i. This implies that $Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_{n-2} \subsetneqq MR[X]$, $Q_i \cap R = P_i$ and $ht(MR[X]/Q_i) = ht(M/P_i) - 1$ for $i = 1, \cdots, n-2$.

Thus it remains to prove that $ht(Q_i/Q_{i-1}) \leqq 1$ for $1 \leqq i \leqq n-2$. But this is immediately clear from $ht(Q_i/P_{i-1}R[X]) \leqq 2$, a relation due to $Q_i \cap R = P_i$ and the fact that R is noetherian.

COROLLARY 8. *Let R be a noetherian ring. Assume that for each maximal ideal M of R the ring $R[X]_{MR[X]}$ satisfies C_{n-1}, where n is an integer >2. Then R satisfies C_n.*

Proof. Let $P, Q \in \operatorname{Spec}(R)$ be such that $P \subset Q$ and such that $2 \leqq \min c(T = (R/P)_Q) = m \leqq n$. We have to show that $\dim(T) = m$. Obviously we may replace R by T, hence assume that (R, M) is a local domain with $\min c(R) = m \leqq n$, and restrict ourselves to prove that $ht(M) = m$.

Thus let $(O) = P_0 \subsetneqq \cdots \subsetneqq P_m = M$ be a maximal chain of Spec(R). Then it is clear that $P_{m-2}R[X] \subsetneqq P_{m-1}R[X] \subsetneqq MR[X]$ form a saturated chain of $\operatorname{Spec}(R[X])$, hence, by the C_2 property of $R[X]$, that $ht(MR[X]/P_{m-2}R[X]) = 2$. This shows that $ht(M/P_{m-2}) = 2$, and so we may choose a chain $Q_0 \subsetneqq Q_1 \subsetneqq \cdots Q_{m-2} \subsetneqq Q_{m-1} = MR[X]$ as in 7. Now $ht(MR[X]/Q_0) = ht(M) - 1$ and $ht(MR[X]/Q_0) = m - 1$ (this latter is implied by the C_{n-1} property of $R[X]_{MR[X]}$) prove the result.

LEMMA 9. *Let R be a noetherian* GB *ring which satisfies C_n for an integer $n \geqq 2$. Then $R[X]$ satisfies C_{n-1}.*

Proof. As each ring is C_1, we may assume that $n > 2$. Thus let $\tilde{P}, \tilde{Q} \subseteqq \operatorname{Spec}(R[X])$ such that $\tilde{P} \subset \tilde{Q}$, $2 \leqq m = \min c((R[X]/\tilde{P})_{\tilde{Q}}) \leqq n - 1$. Then we have, with $P = \tilde{P} \cap R$, $Q = \tilde{Q} \cap R$:

$$\min c\,((R[X]/(P))_{\tilde{Q}}) \leqq m + 1, \text{ if } \tilde{Q} \neq QR[X]\,,$$

and

$$\min c\,((R[X]/(P))_{(Q,X)}) \leqq m + 2, \text{ if } \tilde{Q} = QR[X]\,.$$

Applying [3, (3.7)] we get $\hat{d}((R/P)_Q) \leqq m + 1$. As R is GB, (i) $\Rightarrow$ (iv) of 4 shows that $\min c\,((R/P)_Q) \leqq m + 1 \leqq n$, and the fact that R is C_n implies that $T = (R/P)_Q$ is catenarian. As T is GB it therefore is even universally catenarian, and so finally $(R[X]/\tilde{P})_{\tilde{Q}}$ is catenarian.

REMARK 10. Noetherian C_n rings appearently never have been studied for their own sake. C_n seems to be related to GB in general, as the GB property of R is easily proved to be a necessary hypothesis in (9) if $n > 2$. Note also that in general the properties C_n and C_{n+1} are independent (s. [2]) even for quasiexcellent GB domains.

Now we may prove our final result, from which 2 follows cleraly.

PROPOSITION 11. *Let R be a noetherian ring and let $n \in N$. Then the following statements are equivalent*:

(i) *R is* GB *and satisfies C_n.*

(ii) *$R[X_1, \cdots, X_m]$ is* GB *and C_{n-m} for all $m < n$.*

(iii) *$R[X_1, \cdots, X_{n-1}]$ is* GB.

Proof. "(i) $\Rightarrow$ (ii)" is immediately proved by induction on m, in making use of 5 and 9.

"(ii) $\Rightarrow$ (iii)" is clear.

"(iii) $\Rightarrow$ (i)" Use 5 and 8 to make induction on n.

To conclude this paper, let us note that the arguments in 5 give rise to an easy proof of the following result of Ratliff [7].

COROLLARY 12. *Let R be a noetherian ring. Then $R[X]$ is* GB *iff $R[X]_{MR[X]}$ is* GB *for all maximal ideals M of R.*

Proof. If $R[X]_{MR[X]}$ is GB for all M in question, so is R_M, hence R. But to prove "$\Leftarrow$" of 5 we obviously only made use of the GB property of the rings $R[X]_{MR[X]}$. So we see that R is C_2 and 5 gives the result.

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