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ONE-PARAMETER SEMIGROUPS OF ISOMETRIES INTO H^p

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In this paper we explicitly describe all strongly continuous one-parameter semigroups $\{T_t\}$ of isometries of $H^p(D)$ into $H^p(D)$, where $1 \leqq p < \infty$, $p \neq 2$, and D is the unit disc $|z| < 1$ in the complex plane C. It turns out (Theorem (1.6)) that for each t, $T_t = \phi_t U_t$, where U_t is a surjective isometry and ϕ_t is an inner function (the families $\{\phi_t\}$ and $\{U_t\}$ are uniquely determined provided $\{U_t\}$ is suitably normalized). The nature of the family $\{\phi_t\}$ depends on the set of common fixed points of the family of Möbius transformations of the disc associated with the family $\{U_t\}$. If there is exactly one common fixed point in D, then $\{T_t\}$ must consist of surjective isometries (§ 4); otherwise $\{T_t\}$ consists of surjective isometries only in very special cases (§§ 2, 5). The families $\{\phi_t\}$ are explicitly described in this paper.

1. **Preliminaries.** The linear isometries of H^p into H^p were characterized by Forelli [7, Theorem 1]. For convenience we quote here a part of the statement of Forelli's theorem.

THEOREM. *Let T be a linear isometry of H^p into H^p, $1 \leqq p < \infty$, $p \neq 2$. Then T has a unique representation*

$$(1.1) \qquad Tf = Ff(\phi), \text{ for all } f \in H^p ,$$

where F is analytic on D, and ϕ is a nonconstant inner function.

Let $\boldsymbol{R}$ be the set of real numbers, and $\boldsymbol{R}^+$ be $\{t \in \boldsymbol{R}: t \geqq 0\}$. Let $\{T_t\}$, $t \in \boldsymbol{R}^+$, be a strongly continuous one-parameter semigroup of isometries of H^p into H^p, $1 \leqq p < \infty$, $p \neq 2$. For each $t \in \boldsymbol{R}^+$, let F_t and ϕ_t be as in the representation (1.1) for T_t. From the identity $T_{s+t} = T_s T_t$ we get for all $s, t \in \boldsymbol{R}^+$:

$$(1.2) \qquad \phi_{s+t} = \phi_s \circ \phi_t$$

$$(1.3) \qquad F_{s+t} = F_s F_t(\phi_s) ,$$

where "$\circ$" denotes composition of maps. Let Z be the identity map, $Z(z) = z$. Obviously $F_t = T_t 1$, and $T_t Z = F_t \phi_t$. It follows by strong continuity that if $u \in \boldsymbol{R}^+$, $z_0 \in D$, and $F_u(z_0) \neq 0$, then $\phi_t(z_0) \to \phi_u(z_0)$ as $t \to u$. From this and the fact that $\{\phi_t: t \in \boldsymbol{R}^+\}$ is normal, we find that $t \mapsto \phi_t$ is continuous from $\boldsymbol{R}^+$ to the usual metric space of all analytic functions on D. Using this and the pointwise equicontinuity of $\{\phi_t: t \in \boldsymbol{R}^+\}$, we infer that $\phi_t(z)$ is a continuous function of (t, z)

on $\boldsymbol{R}^+ \times D$. It follows by [4, Proposition (2.2)] that ϕ_t is univalent for all $t \in \boldsymbol{R}^+$. Since the singular factor of a univalent function on D is trivial [6, Theorem 3.17], a univalent inner function (in particular, ϕ_t for $t \in \boldsymbol{R}^+$) is a Möbius transformation of the disc. Thus $|\phi_t(0)| = |\phi_t^{-1}(0)|$ for $t \in \boldsymbol{R}^+$. If $\{t_n\} \subseteqq \boldsymbol{R}^+$, and $t_n \to t_0$, it follows that there is an $r \in (0, 1)$ such that $|\phi_{t_n}^{-1}(0)| < r$ for $n = 0, 1, 2 \cdots$. A standard integral representation for the inverse of a conformal map [5, Prop. 3.7] coupled with the uniform convergence of $\{\phi_{t_n}\}$ to ϕ_{t_0} on compact subsets of D gives the conclusion that $\phi_{t_n}^{-1}(0) \to \phi_{t_0}^{-1}(0)$. Hence $\phi_t^{-1}(0)$ is a continuous function of t on $\boldsymbol{R}^+$. For each t let a_t and b_t be constants, with $|b_t| < 1 = |a_t|$, such that $\phi_t(z) = a_t(z - b_t)/(1 - \bar{b}_t z)$. We have just shown that $t \mapsto b_t$ is continuous on $\boldsymbol{R}^+$, and it follows that $t \mapsto a_t$ is also continuous. Direct computation now shows that $\phi_t^{-1}(z)$ is continuous in (t, z) on $\boldsymbol{R}^+ \times D$. For $t < 0$ define ϕ_t to be ϕ_{-t}^{-1}. It is easy to see that $\{\phi_t\}$, $t \in \boldsymbol{R}$, is a one-parameter group of Möbius transformations of D, i.e., $t \mapsto \phi_t$ is a homomorphism of the additive group of $\boldsymbol{R}$ into the group of all Möbius transformations of D, and $\phi_t(z)$ is continuous in (t, z) on $\boldsymbol{R} \times D$. In particular [1, p. 231], $\phi_t'(z)$ has a unique continuous logarithm $l(t, z)$ on $\boldsymbol{R} \times \bar{D}$ such that $l(0, 0) = 0$; moreover, $l(t, z)$ is analytic in z for each t, and, if (as will be done henceforth), we standardize the branch of $(\phi_t')^{1/p}$, for each t, by taking $(\phi_t')^{1/p}$ to be $\exp[l(t, \cdot)/p]$, then for all $s, t \in \boldsymbol{R}$, $(\phi_{s+t}')^{1/p}/(\phi_t')^{1/p} = (\phi_s')^{1/p} \circ \phi_t$. Application of [7, (16)] to the family $\{F_t\}$, $t \in \boldsymbol{R}^+$, defined earlier, shows that for each $t \in \boldsymbol{R}^+$, $|F_t| = |(\phi_t')^{1/p}|$ a.e., on $|z| = 1$. Since $(\phi_t')^{1/p}$ is outer, F_t can be written as a product $F_t = (\phi_t')^{1/p}\psi_t$, where ψ_t is inner. (Compare [7, p. 727] where this last technique is employed under different hypotheses.) From (1.3) we have:

(1.4) $$\psi_{s+t}(z) = [\psi_s(z)][\psi_t(\phi_s(z))] \text{ for all } s, t \in \boldsymbol{R}^+, z \in D\,.$$

From the strong continuity of $\{T_t\}$ we easily infer:

(1.5) The function $t \mapsto \psi_t$ is continuous from $\boldsymbol{R}^+$ into $H^1(D)$.

We summarize the foregoing with:

THEOREM 1.6. *Suppose* $1 \leqq p < \infty$; $p \neq 2$. *If* $\{T_t\}$, $t \in \boldsymbol{R}^+$, *is a strongly continuous one-parameter semigroup of isometries of* $H^p(D)$ *into* $H^p(D)$, *then there are a unique one-parameter group of Möbius transformations of the disc,* $\{\phi_t\}$, $t \in \boldsymbol{R}$, *and a unique family* $\{\psi_t\}$, $t \in \boldsymbol{R}^+$, *of inner functions such that*

(1.7) $$(T_t f)(z) = \psi_t(z)[(\phi_t')^{1/p}(z)]f(\phi_t(z)) \text{ for all } t \in \boldsymbol{R}^+, f \in H^p, z \in D\,.$$

The families $\{\phi_t\}$ *and* $\{\psi_t\}$ *satisfy* (1.4) *and* (1.5). *Conversely, given* *a one-parameter group* $\{\phi_t\}$ *of Möbius transformations of the disc and a family of inner functions* $\{\psi_t\}$, $t \in \boldsymbol{R}^+$, *such that* (1.4) *and* (1.5) *hold,* (1.7) *defines a strongly continuous one-parameter semigroup of isometries of* H^p *into* H^p.

DEFINITION. For the semigroup $\{T_t\}$ in Theorem 1.6 we shall call $\{\phi_t\}$, $t \in \boldsymbol{R}$, and $\{\psi_t\}$, $t \in \boldsymbol{R}^+$, the *conformal group* and the *inner coefficient* of $\{T_t\}$, respectively.

2. The case of trivial conformal group. Henceforth let K be the unit circle $|z| = 1$. A singular measure will be a measure on K singular with respect to Lebesgue measure.

THEOREM 2.1. *If* $\{T_t\}$ *is a strongly continuous semigroup of isometries of* H^p *into* H^p, $1 \leqq p < \infty$, $p \neq 2$, *whose conformal group is trivial* (i.e., $\phi_t = Z$ *for all* $t \in \boldsymbol{R}$), *then there are a unique real number* δ *and a unique positive singular measure* λ *such that*

$$T_t f = e^{i\delta t} S^t f \text{ for all } t \in \boldsymbol{R}^+, f \in H^p \,, \tag{2.2}$$

where S *is the singular inner function induced by* λ. *Conversely, for* δ *and* λ *as above,* (2.2) *defines a strongly continuous semigroup of isometries with trivial conformal group.*

Proof. Let $\{T_t\}$ be given with trivial conformal group, and let $\{\psi_t\}$ be its inner coefficient. If, for some $u > 0$ and $z_0 \in D$, $\psi_u(z_0) = 0$, then by (1.4) for each positive integer n, $\psi_{u/n}(z_0) = 0$. Thus $(T_{u/n}1)(z_0) = 0$. Letting $n \to \infty$ gives a contradiction, and so $\psi_t(z) \neq 0$ for $t \in \boldsymbol{R}^+$, $z \in D$. Thus each ψ_t can be written $\psi_t = \alpha_t S_t$, where α_t is a unimodular constant and S_t is a singular inner function. It follows from (1.4) that $\alpha_{s+t} = \alpha_s \alpha_t$ and $S_{s+t} = S_s S_t$ for $s, t \in \boldsymbol{R}^+$. For $t \in \boldsymbol{R}^+$, $(T_t 1) = \alpha_t S_t$; in particular, $|(T_t 1)(0)| = S_t(0)$. Thus for $u \in \boldsymbol{R}^+$, $\lim_{t \to u} S_t(0) = S_u(0)$, and $\lim_{t \to u} \alpha_t S_t(0) = \alpha_u S_u(0)$. It follows that there is a real number δ such that $\alpha_t = e^{i\delta t}$ for $t \in \boldsymbol{R}^+$. We also have

$$||S_t - S_u||_p \longrightarrow 0 \text{ as } t \longrightarrow u \,. \tag{2.3}$$

For each $t \in \boldsymbol{R}^+$, let λ_t be the singular measure corresponding to S_t. For each positive integer n, $S_t = (S_{t/n})^n$, and so $\lambda_{t/n} = n^{-1}\lambda_t$. It follows that for each positive rational r, $\lambda_r = r\lambda_1$. By (2.3) $S_t = (S_1)^t$ for $t \in \boldsymbol{R}^+$. This proves (2.2), uniqueness being evident. Conversely, it is clear that (2.2) defines a semigroup of isometries. Let $A(z) = (2\pi)^{-1}\int_K (w + z)(w - z)^{-1} d\lambda(w)$, for $z \in D$. Then ([6, Theorem 3.2]) $A \in H^s(D)$ for $s < 1$. In particular, the boundary function of S^t is e^{-tB}, where B is the boundary function of A. Strong continuity of

$\{T_t\}$ is readily obtained from the Lebesgue dominated convergence theorem.

REMARK. It is known that for $1 \leqq p < \infty$, $p \neq 2$, the only one-parameter semigroups of isometries of H^p into itself, continuous in the uniform operator topology, are the semigroups $\{e^{i\delta t}I\}$, $t \in \boldsymbol{R}^+$, where δ is a real constant and I is the identity operator. For, by general semigroup theory, such a semigroup automatically extends to a one-parameter group with the same continuity. Now apply [2, Theorem (2.8)].

3. Some properties of inner coefficients. In this section we obtain properties of inner coefficients needed to find explicit representations. Let $\{\phi_t\}$, $t \in \boldsymbol{R}$, be a nontrivial one-parameter group of Möbius transformations of the disc, and $\{\psi_t\}$, $t \in \boldsymbol{R}^+$, be a family of inner functions such that (1.4) and (1.5) hold. For the purpose of classification we reproduce here [2, Proposition (1.5)]:

SCHOLIUM 3.1. *Let Ω be the set of common fixed points in the extended plane of the functions ϕ_t, $t \in \boldsymbol{R}$. Ω must be one of the following*:

(i) *A doubleton set consisting of a point $\tau \in D$ and $\bar{\tau}^{-1}$ (the latter to be ∞ if $\tau = 0$),*

(ii) *a singleton subset of the unit circle K, or*

(iii) *a doubleton subset of K.*

If u is any real number such that ϕ_u is not the identity function, then Ω coincides with the set of fixed points (in the extended plane) of ϕ_u.

We describe $\{\phi_t\}$ as being of type (i), (ii), or (iii) in accordance with the condition which holds in (3.1). Explicit characterizations of the groups of each type are in [1, Theorem (1.6)]. It will sometimes be convenient to write $\phi_t(z)$ as $\phi(t, z)$. In the latter notation partial differentiation will be indicated by numerical subscripts (analogous comments apply to $\psi_t(z)$). We recall some basic facts from [2, §1]. For each $z \in \boldsymbol{C}$, $\phi_1(0, z)$ exists; moreover, $\phi(\cdot, \cdot)$ has continuous partial derivatives of all orders on $\boldsymbol{R} \times D$. The function $\phi_1(0, z)$ on $\boldsymbol{C}$ is a polynomial of degree 1 or 2 whose set of zeros is $\Omega \cap \boldsymbol{C}$. We denote this polynomial by q, and call it the invariance polynomial of $\{\phi_t\}$. For $t \in \boldsymbol{R}$, $z \in \bar{D}$, $\phi_1(t, z) = q(\phi(t, z))$ [1, Theorem (1.5)].

If we form the semigroup $\{T_t\}$ in (1.7), then standard differentiation theory of semigroups can be used to show that $\psi(\cdot, \cdot)$ has continuous first partial derivatives on $\boldsymbol{R}^+ \times D$, and that $\psi_1(0, z)$ is an analytic function of z on D. We omit the details. Differentiate (1.4) with respect to s, set $s = 0$, and denote $\psi_1(0, z)$ by $W(z)$ to get:

$$\psi_1(t, z) = \psi_2(t, z)q(z) + \psi(t, z)W(z) \text{ for } t \in \boldsymbol{R}^+, z \in D . \tag{3.2}$$

For arbitrary (temporarily fixed) $z \in D$, we have by direct differentiation followed by substitution using (3.2)

$$d\psi(t, \phi_{-t}(z))/dt = \psi(t, \phi_{-t}(z))W(\phi_{-t}(z)) \text{ for } t \in \boldsymbol{R}^+ . \tag{3.3}$$

By using the usual type of integrating factor in (3.3) and noting that $\psi_0 \equiv 1$, we see that

$$\psi(t, \phi_{-t}(z)) = \exp\left[\int_0^t W(\phi_{-u}(z))du\right] \text{ for } t \in \boldsymbol{R}^+, z \in D . \tag{3.4}$$

Replace z by $\phi_t(z)$ in (3.4) and transform the variable of integration to get

$$\psi(t, z) = \exp\left[\int_0^t W(\phi_s(z))ds\right] \text{ for } t \in \boldsymbol{R}^+, z \in D . \tag{3.5}$$

In particular, $\psi(t, z)$ never vanishes. Denote the exponent on the right of (3.5) by $L(t, z)$. Obviously L, and also $\psi(\cdot, \cdot)$, have continuous partial derivatives of all orders on $\boldsymbol{R}^+ \times D$. Moreover,

$$\frac{\partial L}{\partial t} = W(\phi_t(z)) \text{ for } t \in \boldsymbol{R}^+, z \in D . \tag{3.6}$$

The relation (1.4) shows that $|\psi_t(z)|$ is a decreasing function of t for each $z \in D$. Thus

$$0 \geqq \frac{d}{dt}\bigg|_{t=0} \log|\psi_t(z)| = \operatorname{Re} L_1(0, z) = \operatorname{Re} W(z) .$$

PROPOSITION 3.7. *Let* $\{\phi_t\}$, $t \in \boldsymbol{R}$, *be a nontrivial one-parameter group of Möbius transformations of the disc, and let* $\{\psi_t\}$, $t \in \boldsymbol{R}^+$, *be a family of inner functions such that* (1.4) *and* (1.5) *hold. Then there is a function* $L(t, z)$ *with continuous partial derivatives of all orders on* $\boldsymbol{R}^+ \times D$ *such that* $\psi(t, z) = \exp[L(t, z)]$ *on* $\boldsymbol{R}^+ \times D$. *Furthermore there is an analytic function* $W(z)$ *on* D *with* $\operatorname{Re} W \leqq 0$ *such that* (3.6) *holds.*

4. Semigroups with conformal group of type (i).

THEOREM 4.1. *Let* $\{T_t\}$, $t \in \boldsymbol{R}^+$, *be a strongly continuous one-parameter semigroup of isometries of* H^p *into* H^p, $1 \leqq p < \infty$, $p \neq 2$, *with conformal group* $\{\phi_t\}$, $t \in \boldsymbol{R}$, *of type* (i). *Then there is a unique* *real constant* δ *such that*

$$T_t f = e^{i\delta t}(\phi_t')^{1/p} f(\phi_t) \text{ for } t \in \mathbf{R}^+,\ f \in H^p . \tag{4.2}$$

On the other hand, for any real constant δ *and any one-parameter group of Möbius transformations of the disc* $\{\phi_t\}$, $t \in \mathbf{R}$, (4.2) *can be used for all* $t \in \mathbf{R}$ *to define a strongly continuous one-parameter group of isometries of* H^p *onto* H^p.

Proof. Let $\{\psi_t\}$, $t \in \mathbf{R}^+$, be the inner coefficient of $\{T_t\}$. Let τ be the common fixed point in D of the group $\{\phi_t\}$. By [1, Theorem (1.6)] the invariance polynomial q, which has τ as its only zero in D, has a simple zero at τ. By (1.4) $t \mapsto \psi_t(\tau)$ is multiplicative, and, by (1.5), this function is continuous on $\mathbf{R}^+$. So there is $\alpha \in \mathbf{C}$ such that $\psi_t(\tau) = e^{\alpha t}$ for $t \in \mathbf{R}^+$. Let $L(\cdot, \cdot)$ and $W(\cdot)$ be as in Proposition 3.7. $L_1(0, \tau) = \psi_1(0, \tau)$. By (3.6) $W(\tau) = \alpha$. So there is an analytic function $A(z)$ on D such that $W - \alpha = qA$. Let G be an antiderivative of A on D.

$$\frac{\partial}{\partial t} G(\phi_t(z)) = A(\phi_t(z))q(\phi_t(z)) = W(\phi_t(z)) - \alpha = \frac{\partial}{\partial t}(L - \alpha t) .$$

So there is a function $k(z)$ on D such that

$$L(t, z) = G(\phi_t(z)) + \alpha t + k(z) \text{ for } t \in \mathbf{R}^+,\ z \in D .$$

If we take exponentials on both sides of this equation, and use the fact that $\psi_0 = 1$, we obtain on $\mathbf{R}^+ \times D$

$$\psi_t(z) = \exp\left[\alpha t + G(\phi_t(z)) - G(z)\right] . \tag{4.3}$$

Let u be a positive number such that $\phi_{t+u} = \phi_t$ for all $t \in \mathbf{R}$ (such a u exists because $\{\phi_t\}$ is of type (i) [1, (1.7)]). Taking $t = u$ in (4.3), we have $\psi_u(z) = e^{\alpha u}$ for $z \in D$. Since ψ_u is inner, $\alpha = i\delta$, where δ is the imaginary part of α. Since $\psi_t(\tau) = e^{i\delta t}$ for $t \in \mathbf{R}^+$, the maximum modulus theorem shows that $\psi_t(z) = e^{i\delta t}$ for $t \in \mathbf{R}^+$, $z \in D$.

5. Semigroups with conformal group of type (ii) or type (iii). The author is indebted to R. Kaufman for the idea of the next proof. Before taking up the next theorem, let us observe that if $\{\phi_t\}$ is a group of type (ii) with common fixed point α, then there is a unique nonzero real number c such that the invariance polynomial of $\{\phi_t\}$ is given by $q(z) = ic\bar{\alpha}(z - \alpha)^2$ [1, (1.8)]. We shall call this c *the group constant*.

THEOREM 5.1. *Let* $\{T_t\}$ *be a strongly continuous one-parameter semigroup of isometries of* H^p *into* H^p, $1 \leqq p < \infty$, $p \neq 2$, *with* *conformal group* $\{\phi_t\}$ *of type* (ii). *Let* α *be the common fixed point and* c *the group constant of* $\{\phi_t\}$. *Then there are a unique real constant* δ *and a unique nonnegative constant* μ *such that*

$$T_t f = \{\exp [i\delta t - ic\mu t^2 - \mu t(\alpha + z)(\alpha - z)^{-1}]\}(\phi_t')^{1/p} f(\phi_t) , \quad \text{for } t \in \boldsymbol{R}^+, \ f \in H^p . \tag{5.2}$$

Conversely, if α *is a unimodular complex number,* c *is a nonzero real number, and* $\{\phi_t\}$ *is the type* (ii) *group having* α *as common fixed point and* c *as group constant, then* (5.2) *defines a strongly continuous semigroup of isometries of* H^p *into* H^p *for* $\delta \in \boldsymbol{R}$, $\mu \geqq 0$.

Proof. Let $\{\psi_t\}$ be the inner coefficient of $\{T_t\}$, let $L(\cdot, \cdot)$ and W be as in Proposition (3.7), and let G be a primitive of W/q. Then $\partial G(\phi_t(z))/\partial t = W(\phi_t(z)) = \partial L/\partial t$. It follows that

$$\psi_t(z) = \exp [G(\phi_t(z)) - G(z)] \text{ for } t \in \boldsymbol{R}^+, \ z \in D . \tag{5.3}$$

Denote $\operatorname{Re} G$ by U. Let Q be the linear fractional transformation given by $Q(z) = (\alpha + z)(\alpha - z)^{-1}$. In particular, Q maps D onto the right half plane P. Define g on P by $g(w) = G(Q^{-1}(w))$, and put $u = \operatorname{Re} g = U(Q^{-1}(w))$. It is easy to see that $ig'(w) = W(Q^{-1}(w))/(2c)$, and so, writing $w = x + iy$, we have $\partial u/\partial y = (2c)^{-1} \operatorname{Re} W(Q^{-1}(w))$. Let $\beta = -c/|c|$; then $\beta \partial u/\partial y \geqq 0$. Applying the Herglotz representation theorem (for P) to $\beta \partial u/\partial y$ gives:

$$\beta \partial u/\partial y = \sigma x + \int_{\boldsymbol{R}} x[x^2 + (y - r)^2]^{-1} d\nu(r) , \tag{5.4}$$

where σ is a nonnegative constant, and $\nu \geqq 0$ is a certain measure, which is finite on bounded Borel sets of $\boldsymbol{R}$.

In order to make effective use of (5.4), we now examine the boundary behavior of U and u. Since $\operatorname{Re} W \leqq 0$, W belongs to the Hardy spaces H^s for $s < 1$. Obviously $(1/q)$ belongs to H^s for $s < 1/2$. Use of the Cauchy-Schwarz inequality now shows that $G' = W/q$ belongs to H^s for $s < 1/4$. By a theorem of Hardy and Littlewood [6, Theorem 5.12] it follows that G belongs to H^s for some $s > 0$. Accordingly, for any $t \in \boldsymbol{R}^+$, we can pass to the boundary of D in (5.3) to conclude that, with respect to Lebesgue measure m on K,

$$U(\phi_t(z)) - U(z) = 0 \text{ for almost all } z . \tag{5.5}$$

If $t < 0$, we can replace t by $(-t)$ and z by $\phi_t(z)$ in (5.5) and get equality for almost all z. Thus for each $t \in \boldsymbol{R}$, (5.5) holds for m-almost all z. It follows that for m-almost all z, (5.5) holds for almost all t (with respect to linear Lebesgue measure). In particular, pick $z_0 \in K \backslash \{\alpha\}$ so that with z replaced by z_0, (5.5) holds for almost all t. It follows that there is a real constant $\gamma(=U(z_0))$ such that for m-almost all z, the function U on D has a nontangential limit equal to γ at z. Hence for almost all $y \in \boldsymbol{R}$ (with respect to linear Lebesgue measure) u on P has a nontangential limit equal to γ at iy. If y_1, y_2 are any two such values of y, with $y_1 < y_2$, then integration of (5.4) with respect to y between y_1 and y_2, followed by interchanging the order of integration, readily gives the inequality:

$$\begin{aligned}\beta[u(x, y_2) - u(x, y_1)] \geqq \int_{y_1<r<y_2} &\{\arctan[x^{-1}(y_2 - r)] \\ &- \arctan[x^{-1}(y_1 - r)]\} d\nu(r) .\end{aligned} \tag{5.6}$$

Now, keeping y_1 and y_2 fixed, let x in (5.6) tend to 0 through a sequence of positive values. At each r on the interval of integration the integrand tends to π. Using Fatou's lemma (or bounded convergence) and the fact that the majorant in (5.6) tends to 0, we infer that the interval $y_1 < r < y_2$ is ν-null. Hence $\nu(\boldsymbol{R})=0$. Making use of this, integration of (5.4) now gives:

$$\beta u(x, y) = \sigma xy + \beta u(x, 0) . \tag{5.7}$$

In particular the Laplacian of $u(x, 0)$ must vanish. So there are real constants a, b such that

$$\beta u(x, y) = \sigma xy + ax + b . \tag{5.8}$$

Since $u(Q(z)) = \operatorname{Re} G(z)$ for $z \in D$, we have from (5.8) and (5.3) that for $t \in \boldsymbol{R}^+$, $z \in D$

$$\psi_t(z) = \exp\{-\sigma_0 i\beta[\{Q(\phi_t(z))\}^2 - \{Q(z)\}^2] + a_0[Q(\phi_t(z)) - Q(z)]\} , \tag{5.9}$$

where σ_0, a_0 are real constants, with $\sigma_0 \geqq 0$. Since $\partial\phi(t, z)/\partial t = q(\phi(t, z))$, it is easy to see that $(\alpha - \phi_t(z))^{-1} = ic\bar{\alpha}t + (\alpha - z)^{-1}$. Using this, we get the desired representation (5.2) from (5.9). Since the exponential expression on the right of (5.2) defines a family $\{\psi_t\}$, $t \in \boldsymbol{R}^+$, of inner functions for arbitrary $\delta \in \boldsymbol{R}$, $\mu \geqq 0$, uniqueness of δ and μ for the given semigroup $\{T_t\}$ is obvious from Theorem 1.6. For the converse, one sees easily that the exponential expression on the right of (5.2) satisfies (1.4) and (1.5).

Before taking up the type (iii) case, it will be convenient to introduce some further notation. If α and β are unimodular complex numbers, let $\sigma_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ be, respectively, the linear fractional transformations $(z - \alpha)/(z - \beta)$ and $(\bar{\alpha}\beta - 1)\sigma_{\alpha,\beta}$. If $\{\phi_t\}$ is a group of type (iii), then ([1, (1.9)]) it has a unique representation of the form

$$\phi_t(z) = \sigma_{\alpha,\beta}^{-1}(e^{ct}\sigma_{\alpha,\beta}(z)) , \tag{5.10}$$

where α and β are unimodular complex numbers and $c > 0$. The invariance polynomial in this instance is $q(z) = c(\alpha-\beta)^{-1}(z-\alpha)(z-\beta)$.

THEOREM 5.11. *Let $\{T_t\}$, $t \in \boldsymbol{R}^+$, be a strongly continuous one-parameter semigroup of isometries of H^p into H^p, $1 \leqq p < \infty$, $p \neq 2$, and let $\{T_t\}$ have conformal group $\{\phi_t\}$ of type* (iii). *If $\{\phi_t\}$ has the representation* (5.10), *then there are unique nonnegative constants μ and ν and a unique real constant δ such that*

$$\begin{aligned} T_t f = \{\exp [i\delta t - \mu(e^{ct} - 1)Q_{\alpha,\beta} \\ + (\nu(e^{-ct} - 1)/Q_{\alpha,\beta})]\}(\phi_t')^{1/p} f(\phi_t), \ \textit{for} \ t \in \boldsymbol{R}^+, \ f \in H^p . \end{aligned} \tag{5.12}$$

Conversely, if a positive number c and unimodular complex numbers α and β are given, and if the type (iii) *group $\{\phi_t\}$ is defined by* (5.10), *then* (5.12) *defines a strongly continuous semigroup of isometries of H^p into H^p for $\delta \in \boldsymbol{R}$, $\mu \geqq 0$, $\nu \geqq 0$.*

Proof. Let $\{\psi_t\}$ be the inner coefficient of $\{T_t\}$, and let $L(\cdot, \cdot)$ and W be as in Proposition 3.7. Let G be a primitive of W/q on D. As previously, (5.3) holds in the present situation. Let $U = \operatorname{Re} G$. We remark that once the proper conformal mapping for transforming U is introduced, there will be some similarity with the proof of Theorem 5.1. We shall omit details which are obvious modifications of the proof of Theorem 5.1, and emphasize aspects which are special to the case at hand. We observe that $Q_{\alpha,\beta}$ maps D onto the right half plane, and hence $M = \operatorname{Log}(Q_{\alpha,\beta})$ maps D onto the strip Y given by $|\operatorname{Im} w| < \pi/2$. Define F on Y to be $G(M^{-1}(w))$, and put $u = \operatorname{Re} F = U(M^{-1}(w))$. By direct calculation we get that $dF/dw = W(M^{-1}(w))/c$. Hence, writting $w = x + iy$, $\partial u/\partial x \leqq 0$. The relevant form of the Herglotz theorem for Y gives:

$$\begin{aligned} -\partial u/\partial x = \rho_1 e^x \cos y + \rho_2 e^{-x} \cos y \\ + \int_{\boldsymbol{R}\setminus\{0\}} \{e^x \cos y/[e^{2x}\cos^2 y + (e^x \sin y - s)^2]\} d\gamma(s) , \end{aligned} \tag{5.13}$$

where ρ_1, ρ_2 are nonnegative constants, and γ is a measure on $\boldsymbol{R}$, $\gamma \geqq 0$, and γ is finite on bounded Borel sets. Since G' belongs to Hardy spaces of index less than 1/2, G belongs to some Hardy space. In particular, for $t \in \boldsymbol{R}^+$, (5.5) holds m-a.e. on K (in the present setting). After observing that if $z \in K\setminus\{\alpha, \beta\}$, $\{\phi_t(z): t \in \boldsymbol{R}\}$ is the component of z in $K\setminus\{\alpha, \beta\}$ [3, Theorem (2.5)], we find that there are real constants A and B such that for almost all $x \in \boldsymbol{R}$ (with respect to linear Lebesgue measure) $u(x, y) \to A$ as $y \to \pi/2$, and

$u(x, y) \to B$ as $y \to -\pi/2$. If x_1 and x_2 are values of x for which these limits (with respect to y) hold, and $x_1 < x_2$, then integration of (5.13) with respect to x from x_1 to x_2, followed by interchanging the order of integration, shows that

$$\begin{aligned} &u(x_1, y) - u(x_2, y) \\ (5.14)\qquad &\geqq \int_{\boldsymbol{R}\setminus\{0\}} s^{-1} \arctan \left[(e^x - (\sin y)s)/(s \cos y)\right]\Big|_{x=x_1}^{x=x_2} d\gamma(s) \;. \end{aligned}$$

If we replace $\boldsymbol{R}\setminus\{0\}$ in (5.14) by the interval $e^{x_1} < s < e^{x_2}$, and let $y \to \pi/2$, we see that this interval is a γ-null set. Similarly, if we replace $\boldsymbol{R}\setminus\{0\}$ by the interval $-e^{x_2} < s < -e^{x_1}$, and let $y \to -\pi/2$, we see that the latter interval is a γ-null set. Thus $\gamma(\boldsymbol{R}\setminus\{0\}) = 0$. Using this in (5.13) we find that there are real constants a and b such that $u(x, y) = -\rho_1 \operatorname{Re}(e^w) + \rho_2 \operatorname{Re}(e^{-w}) + a \operatorname{Im}(w) + b$. This yields the following equation for G on D

$$(5.15)\qquad G = -\rho_1 Q_{\alpha,\beta} + (\rho_2/Q_{\alpha,\beta}) - ai \operatorname{Log}(Q_{\alpha,\beta}) + C \;,$$

where C is a complex constant (which can be disregarded in using (5.3)). In view of (5.10) $Q_{\alpha,\beta}(\phi_t) = e^{ct} Q_{\alpha,\beta}$. Using this fact, we get (5.12) from (5.15) and (5.3).

To obtain uniqueness, let us note that if $\delta_j, \mu_j, \nu_j (j = 1, 2)$ are appropriate triples of constants such that for the given semigroup $\{T_t\}$ (and $\{\phi_t\}, \alpha, \beta, c$) (5.12) holds with each triple inserted, then by uniqueness of inner coefficient, the corresponding expressions inside the "exp" sign in (5.12) must differ on $\boldsymbol{R}^+ \times D$ by $2\pi ni$, where n is a constant (integer). After transposing this gives

$$(5.16)\qquad i(\delta_1 - \delta_2)t + [(\nu_1 - \nu_2)(e^{-ct} - 1)/Q] = (\mu_1 - \mu_2)(e^{ct} - 1)Q + 2\pi ni \;,$$

where we have deleted subscripts in $Q_{\alpha,\beta}$. If we fix t at a positive value in (5.16) and let $z \to \alpha$, the right-hand side approaches $2\pi ni$, whereas if $\nu_1 \neq \nu_2$, the left-hand side approaches ∞. So $\nu_1 = \nu_2$. Similarly, let $z \to \beta$ to get $\mu_1 = \mu_2$, and then $\delta_1 = \delta_2$ follows.

Under the hypotheses of the converse it is straightforward to see that the exponential expression in (5.12) is a family $\{\psi_t\}$ of inner functions, and that (1.4) and (1.5) hold.

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