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THE ENDOMORPHISMS OF A DIRICHLET ALGEBRA

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Let $A(\Delta)$ be the disc algebra. A Dirichlet subalgebra of $A(\Delta)$ is obtained by forming $A(p) = A(\Delta) \cap \{f \circ p: f \in A(\Delta)\}$ where $p(e^{it})$ in a singular homeomorphism of the unit circle. In this paper the endomorphisms of $A(p)$ are described and for a restricted class of these endomorphisms the spectra is computed.

1. **Introduction.** The disc algebra $A(\Delta)$ is the uniform algebra of functions analytic on the open unit disc $\Delta = \{z: |z| < 1\}$ and continuous on the closure $\bar{\Delta}$. In [3] Browder and Wermer gave a method for the construction of algebras of $A(\Delta)$ which are Dirichlet algebras on the unit circle $\Gamma = \{z: |z| = 1\}$. Their method goes as follows: Let $p(e^{it})$ be a homeomorphism of Γ where $p'(e^{it}) = 0$ for almost all t. Such homeomorphisms will be called *singular*. Let $A(\Delta)_p = \{f \circ p: f \in A(\Delta)\}$ and set $A(p) = A(\Delta) \cap A(\Delta)_p$. Then $A(p)$ is a Dirichlet algebra on Γ. Properties of $A(p)$ are given in [1], [2], [8].

In §2 we show that two singular homeomorphisms p_1 and p_2 may lead to nonisomorphic algebras $A(p_1)$ and $A(p_2)$. Nonetheless, some general characteristics of the space of endomorphisms of an algebra $A(p)$ are given in §3. In §4 the techniques of Kamowitz [4], [5] are applied to describe the spectra of certain of the endomorphisms of $A(p)$.

We collect some information about the algebra $A(p)$ and refer the reader to [7], Section 8.2 for further discussion. To describe the maximal ideal space $M_{A(p)}$ of $A(p)$, we first make the convention that if $f \in A(p)$, then $\widetilde{f \circ p}$ will stand for the analytic extension of $f \circ p(e^{it})$ to Δ. Also, we let $f(y)$ denote the value of $f \in A(p)$ evaluated at the maximal ideal y. Then $M_{A(p)}$ consists of Γ and two parts lying off Γ. One of these parts is identified with Δ and the other, to be denoted by Δ_p, consists of points of the form $\Psi(z)$ where $z \in \Delta$ defined by $f(\Psi(z)) = \widetilde{f \circ p}(z)$ for $f \in A(p)$.

When the maximal ideal space is given the Gelfand topology, we have $\bar{\Delta} \backslash \Delta = \Gamma$ and $\bar{\Delta}_p \backslash \Delta_p = \Gamma$. The Gelfand topology coincides with the usual topology on $\bar{\Delta}$ and the map $\Psi: \Delta \to \Delta_p$ is a homeomorphism. Moreover, Ψ extends to Γ to map $\bar{\Delta}$ homeomorphically onto $\bar{\Delta}_p$ by setting $\Psi(e^{it}) = p(e^{it})$.

Given $z_0 \in \Delta$ we claim there are functions $f \in A(p)$ such that $f'(z_0) \neq 0$. Likewise, there are functions $f \in A(p)$ such that

$\widetilde{f \circ p}'(z_0) \neq 0$. We verify this claim for $z_0 = 0$ as an illustration of the general case. Suppose instead that every $f \in A(p)$ satisfied $f'(0) = 0$. Then $\mu = [(1 + \cos(t))/2\pi]dt$ is a representing measure for the origin which is supported on Γ. But normalized Lebesque measure on Γ also is a representing for the origin contradicting the uniqueness of representing measures for a Dirichlet algebra.

As a result of these observations we conclude:

(1) Given $z_0 \in \Delta$ and a positive integer k there is $f \in A(p)$ such that $f(z) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$. Similarly, there is $f \in A(p)$ such that $\widetilde{f \circ p}(z) = (z - z_0)^k h(z)$ where $h(z_0) \neq 0$.

(2) If $g: \bar{\Delta} \to \bar{\Delta}$ is continuous and $f \circ g \in A(\Delta)$ for all $f \in A(p)$, then $g \in A(\Delta)$. Similarly, if $f \circ \Psi \circ g \in A(\Delta)$ for all $f \in A(p)$, then $g \in A(\Delta)$.

2. **Isomorphisms.** Let $A(p)$ and $A(q)$ be two Dirichlet algebras as described in §1. If T is a homomorphism (linear and multiplicative) from $A(p)$ to $A(q)$, then T induces a continuous mapping $\Phi: M_{A(q)} \to M_{A(p)}$ defined as follows: If $y \in M_{A(q)}$, then $\Phi(y)$ is the maximal ideal which satisfies $f(\Phi(y)) = T(f)(y)$ for all $f \in A(p)$. We call Φ the *mapping induced by* T. The mapping Φ has the following properties:

(a) $\Phi^{-1}(\Gamma) \subset \Gamma$.

(b) Either $\Phi(\Delta) \subset \Delta$ or $\Phi(\Delta) \subset \Delta_p$.

(c) Either $\Phi(\Delta_q) \subset \Delta$ or $\Phi(\Delta_q) \subset \Delta_p$.

To establish (a) we suppose there exists $x \in \Gamma$ and $y \in \Phi^{-1}(x)$ where $y \notin \Gamma$. Each point of Γ is a peak point for $A(q)$. Let $f \in A(q)$ peak at x. Then $T(f)(y) = f \circ \Phi(y) = f(x)$ and $T(f)$ peaks at y. However, this contradicts the maximum principle when $y \notin \Gamma$. Properties (b) and (c) follow from (a) and the fact that Δ, Δ_p, and Δ_q are open and connected.

In order to state the following theorem we let $\Psi: \bar{\Delta} \to \bar{\Delta}_p$ be defined by the equation $f(\Psi(z)) = \widetilde{f \circ p}(z)$ for $f \in A(p)$ and let $\chi: \bar{\Delta} \to \bar{\Delta}_q$ be defined by $f(\chi(z)) = \widetilde{f \circ q}(z)$ for $f \in A(q)$.

THEOREM 1. *Let* $T: A(p) \to A(q)$ *be an isomorphism and let* $\Phi: M_{A(q)} \to M_{A(p)}$ *be the induced map. Then* Φ *is a homeomorphism with* $\Phi(\Gamma) = \Gamma$ *and either* (i) $\Phi(\Delta) = \Delta$, $\Phi(\Delta_q) = \Delta_q$ *and* $\Phi \in A(\Delta)$, $\Psi^{-1} \circ \Phi \circ \chi \in A(\Delta)$. *(Hence,* Φ *and* $\Psi^{-1} \circ \Phi \circ \chi$ *are conformal maps of* Δ *onto itself.) Or* (ii) $\Phi(\Delta) = \Delta_p$, $\Phi(\Delta_q) = \Delta$ *and* $\Psi^{-1} \circ \Phi \in A(\Delta)$, $\Phi \circ \chi \in A(\Delta)$. *(Hence,* $\Psi^{-1} \circ \Phi$ *and* $\Phi \circ \chi$ *are conformal maps of* Δ *onto itself.)*

Proof. It is easy to check that Φ is a homeomorphism with $\Phi(\Gamma) = \Gamma$. Either $\Phi(\Delta) \subset \Delta$ or $\Phi(\Delta) \subset \Delta_p$. Suppose $\Phi(\Delta) \subset \Delta$. Then

$\Phi(\Delta_q) \subset \Delta_p$ or else Δ_p would be omitted from the range of Φ altogether. Again, since Φ is onto we conclude $\Phi(\Delta) = \Delta$ and $\Phi(\Delta_q) = \Delta_p$. Using Property 2) listed in §1 we conclude that Φ and $\Psi^{-1} \circ \Phi \circ \chi \in A(\Delta)$. Consideration of the case $\Phi(\Delta) \subset \Delta_p$ leads to alternative (ii). This completes the proof.

COROLLARY. *If $p(e^{it})$ and $q(e^{it})$ give different orientations to Γ, then $A(p)$ and $A(q)$ are not isomorphic.*

Proof. Assume alternative (i) of the theorem. Then $\Psi^{-1} \circ \Phi \circ \chi$ is analytic on $\bar{\Delta}$ and maps Γ homeomorphically onto itself. However, the orientation of Γ is reversed by this map which gives us a contradiction. A similar argument applies to alternative (ii).

The next example shows that $A(p)$ and $A(q)$ need not be isomorphic when p and q give the same orientation to Γ.

EXAMPLE 1. In [10] a strictly increasing, continuous, singular function $h(x)$ is constructed which maps the interval $[0, 1]$ onto itself. Furthermore, $h(x)$ has the property that if $h'(x_0)$ exists for some $x_0 \in (0, 1]$, then $h'(x_0) = 0$. Also, if $\rho = 3$ where ρ is a constant appearing in [10], then it follows that $h'(0) = 0$. (Derivatives at 0 and 1 are one-sided.) For such an $h(x)$ we define $p(e^{it}) = e^{i2\pi h(t)}$.

Next, we indicate how to construct a strictly increasing, continuous, singular function $k(x)$ taking $[0, 1]$ onto itself which satisfies $k'(0) = 1$ and $k'(1) = 1$. To obtain $k'(0) = 1$ we set $\alpha(x) = e^x - 1$ and $\beta(x) = \alpha^{-1}(x)$ and construct $k(x)$ to lie between $\alpha(x)$ and $\beta(x)$. A similar approach applies to obtain $k'(1) = 1$. Now define $q(e^{it}) = e^{i2\pi k(t)}$.

Suppose T is an isomorphism of $A(p)$ to $A(q)$ with induced map Φ. Assume alternative (i) of Theorem 1 holds. Let g be given by $g = p^{-1} \circ \Phi \circ q$. Then g and Φ are differentiable on Γ with nonvanishing derivative. Consider $\Phi \circ q = p \circ g$. At $t = 0$, $\Phi \circ q$ is differentiable with nonzero derivative. This implies that p is differentiable at $g(0)$ with nonzero derivative which is a contradiction. A similar argument will eliminate the possibility of alternative (ii). Hence, $A(p)$ and $A(q)$ are not isomorphic.

REMARK. It is easy to construct algebras $A(p)$ and $A(q)$ which are isomorphic. For example, given p we let $q = g^{-1} \circ p \circ g$ where g is a conformal map of the disc onto itself. Then $A(p)$ and $A(q)$ are isomorphic by the map $T: A(p) \to A(q)$ where $T(f) = f \circ g$.

3. Endomorphisms of $A(p)$. Let T be endomorphism of $A(p)$ with induced map Φ. There are four cases to be considered.

Case (i). $\Phi(\Delta) \subset \Delta$ and $\Phi(\Delta_p) \subset \Delta$.
In this case $T(f)(z) = f(\Phi(z)) \in A(\Delta)$ for all $f \in A(p)$ and thus $\Phi \in A(\Delta)$. Likewise, $T(f) \circ \Psi(z) = f(\Phi \circ \Psi(z)) \in A(\Delta)$ for all $f \in A(p)$ and thus $\Phi \circ \Psi \in A(\Delta)$. We conclude that $\Phi \in A(p)$ and $\|\Phi\| \leqq 1$. Such endomorphisms exist since to each $\Phi \in A(p)$ where $\|\Phi\| \leqq 1$ we can associate an endomorphism T_Φ where $T_\Phi(f) = f \circ \Phi$.

Case (ii). $\Phi(\Delta) \subset \Delta_p$ and $\Phi(\Delta_p) \subset \Delta_p$.
In this case $T(f)(z) = f(\Phi(z)) = f \circ \Psi(\Psi^{-1} \circ \Phi(z)) \in A(\Delta)$ for all $f \in A(p)$ and thus $\Psi^{-1} \circ \Phi \in A(\Delta)$. A similar argument shows $\Psi^{-1} \circ \Phi \circ \Psi \in A(\Delta)$. We conclude that $\Psi^{-1} \circ \Phi \in A(p)$ and $\|\Psi^{-1} \circ \Phi\| \leqq 1$. Such endomorphisms exist since to each $g \in A(p)$ where $\|g\| \leqq 1$ we can associate an endomorphism T^g where $T^g(f) = f \circ \Psi \circ g$. If Φ is the map induced by T^g, then $\Phi(z) = \Psi(g(z))$ and $\Phi(\Psi(z)) = \Psi \circ g(\Psi(z))$.

Case (iii). $\Phi(\Delta) \subset \Delta$ and $\Phi(\Delta_p) \subset \Delta_p$.
In this case $\Phi(\Gamma) = \Gamma$. By arguments similar to those in cases (i) and (ii) we have Φ and $\Psi^{-1} \circ \Phi \circ \Psi$ in $A(\Delta)$. Hence, Φ and $\Psi^{-1} \circ \Phi \circ \Psi$ are finite Blaschke products and $\Phi(\Delta) = \Delta$, $\Phi(\Delta_p) = \Delta_p$.

Examples of such endomorphisms may be given for particular $p(e^{it})$. Example 2 was suggested by W. Knight.

EXAMPLE 2. Let $\alpha(x)$ be a strictly increasing, continuous, singular function mapping $[0, \pi]$ onto $[\pi, 2\pi]$. Extend α to $(\pi, 2\pi]$ by defining $\alpha(x) = \alpha(x - \pi) - \pi$ and extend α to $\boldsymbol{R}$ by periodicity. Set $p(e^{it}) = e^{i\alpha(t)}$ and notice that p is a singular homeomorphism of Γ. Let $\Phi(e^{it}) = -e^{it}$. It can be readily verified that $p \circ \Phi(e^{it}) = \Phi \circ p(e^{it})$. Consider $A(p)$ and let $T(f) = f \circ \Phi$. Using the equation $p \circ \Phi = \Phi \circ p$ we see that T is an automorphism of $A(p)$ which falls into case (iii).

Case (iv). $\Phi(\Delta) \subset \Delta_p$ and $\Phi(\Delta_p) \subset \Delta$.
By arguments similar to those in the previously considered cases we see that $\Phi(\Gamma) = \Gamma$, $\Phi(\Delta) = \Delta_p$, $\Phi(\Delta_p) = \Delta$ and that $\Psi^{-1} \circ \Phi$ and $\Phi \circ \Psi$ belong to $A(\Delta)$.

Examples of such endomorphisms may be given for particular $p(e^{it})$.

EXAMPLE 3. Let $p(e^{it})$ be a singular homeomorphism of Γ which satisfies $p \circ p(e^{it}) = e^{it}$. See [7] p. 207 for a construction. Set $T(f) = f \circ p$ for $f \in A(p)$. Then T is an automorphism of $A(p)$ which falls into case (iv).

The next result gives a property of certain automorphisms of $A(p)$. We adopt the following notation. If T is an automorphism, then $T^n = T \circ \cdots \circ T$ (n times) and I will denote the identity automorphism.

THEOREM 2. *Let T be an automorphism of $A(p)$ such that the induced map Φ satisfies $\Phi(\Delta) \subset \Delta$ and $\Phi(\Delta_p) \subset \Delta_p$. If either Φ or $\Psi^{-1} \circ \Phi \circ \Psi$ has a fixed point in Δ, then T is a nilpotent element in the group of automorphisms of $A(p)$.*

Proof. We assume first that $\Phi(0) = 0$. Then $\Phi(e^{it}) = ce^{it}$ where $|c| = 1$. Assume there is no positive integer n for which $c^n = 1$. Then $\{c, c^2, c^3, \cdots\}$ is dense in Γ. Let $\alpha_0 \in \Gamma$ be given. We will show that the map U_0 on $A(p)$ defined by $U_0(f)(e^{it}) = f(\alpha_0 e^{it})$ defines an automorphism of $A(p)$. Let α_n be a sequences taken from $\{c, c^2, c^3, \cdots\}$ which converses to α_0. To see that U_0 maps $A(p)$ to $A(p)$, we let $f \in A(p)$ and note that $f(\alpha_n e^{it}) \in A(p)$ for each n and that $f(\alpha_n e^{it})$ converges to $f(\alpha_0 e^{it})$ uniformly on Γ. It is evident that U_0 is an endomorphism and it is not difficult to verify that it is actually an automorphism with induced map satisfying case (iii). In particular, $p^{-1}(\alpha_0 p(e^{it}))$ has nonvanishing derivative since this function gives the boundary values of a conformal map of the disc.

Let e^{it_0} be a point where the derivative of $p(e^{it})$ vanishes. Then select $\alpha_0 \in \Gamma$ so that $\alpha_0 p(e^{it_0})$ is a point where the derivative of $p^{-1}(e^{it})$ vanishes. In this event the derivative of $p^{-1}(\alpha_0 p(e^{it}))$ vanishes at e^{it_0} which gives a contradiction. Consequently, there must be an integer $n > 0$ for which $c^n = 1$. But then $T^n = I$.

If $\Phi(z_0) = z_0 \in \Delta$ and $z_0 \neq 0$, then let $g(z)$ be a conformal map of Δ taking 0 to z_0. Set $q = g^{-1} \circ p \circ g$ and $\Theta = g^{-1} \circ \Phi \circ g$. An automorphism of $A(q)$ is given by T' where $T'(f) = f \circ \Theta$. Since $\Theta(0) = 0$, the previous results apply to show T' nilpotent. It follows that T is nilpotent.

Now assume $\Psi^{-1} \circ \Phi \circ \Psi(z_0) = z_0 \in \Delta$. An automorphism of $A(p^{-1})$ is defined by T_1 where $T_1(f) = f(\Psi^{-1} \circ \Phi \circ \Psi)$. The previous result applies to T_1 to prove that T_1 is nilpotent. It follows that T is nilpotent. This completes the proof.

REMARK. If T is an automorphism satisfying case (iv), then $T \circ T$ falls into case (iii). Let Φ be induced by T. If $\Phi \circ \Phi$ has a fixed point in Δ or Δ_p, then Theorem 2 applies and shows that $T^{2n} = I$ for some integer $n > 0$.

The next example shows that for particular $A(p)$ there exists automorphisms which are not nilpotent.

EXAMPLE 4. This construction employs the methods of [9], Lemma 5. Let α be a conformal map of Δ which has two fixed points on Γ. Let $L(z)$ be a linear fractional map taking Δ to the upper half plane and taking the two fixed points of α to 0 and ∞. Then $g = L \circ \alpha \circ L^{-1}$ has fixed points 0 and ∞ and $\text{Im}\,[g(i)] > 0$. Hence, $g(z) = az$ with $a > 0$ and $a \neq 1$.

We will construct a strictly increasing, continuous, singular function $h(x)$ on $\boldsymbol{R}$ such that $h \circ g = g \circ h$. We assume $a > 1$, the case $a < 1$ being similar. Let $h_0(x)$ be a strictly increasing, continuous, singular function mapping $[1, a]$ onto itself. Define $h(x)$ on $(0, \infty)$ by setting

$$h(x) = a^n h_0(a^{-n}x) \quad \text{for} \quad x \in [a^n, a^{n+1}]$$

where n takes integer values. Extended $h(x)$ to $\boldsymbol{R}$ by setting $h(0) = 0$ and $h(x) = -h(-x)$ if $x < 0$. It can be checked that $h(x)$ satisfies the specified relationship with $g(x)$.

We pull back to Γ by defining $p(e^{it}) = L^{-1} \circ h \circ L(e^{it})$. Then $p(e^{it})$ is singular and $p \circ \alpha(e^{it}) = \alpha \circ p(e^{it})$. It follows that the map T on $A(p)$ defined by $T(f) = f \circ \alpha$ is an automorphism of $A(p)$.

Let g_n be the composition of g with itself n times. We refer to such a composition as the nth iterate of g. Since $g_n(z)$ converges to 0 or ∞ for each z, the iterates α_n of α converge pointwise on Γ to the fixed points of α. In particular, $\alpha_n(z)$ is not the identity function for any n. Hence, $T^n \neq I$ for every n. This completes Example 4.

We conclude this section by suggesting two problems. Let $\text{Aut}\,(A(p))$ refer to the group of automorphisms of $A(p)$.

Problem 1. Is it possible to construct a $p(e^{it})$ so that $\text{Aut}(A(p))$ consists of only the identity?

Problem 2. Let the sets $N_{\varepsilon,f}(T) = \{U \in \text{Aut}\,(A(p)) \colon \| T(f) - U(f) \| < \varepsilon\}$ for $\varepsilon > 0$, $f \in A(p)$, and $T \in \text{Aut}\,(A(p))$ be a subbase for a topology on $\text{Aut}(A(p))$. With this topology $\text{Aut}(A(p))$ is a topological group. Is $\text{Aut}\,(A(p))$ discrete?

4. Spectra of endomorphisms of $A(p)$. We apply the methods of Kamowitz [4], [5] to obtain information about the spectra of endomorphisms of $A(p)$. Let T be an endomorphism. The *spectrum* of T, denoted by $\sigma(T)$, is given by $\{\lambda \in \boldsymbol{C} \colon \lambda - T \text{ is not invertible}\}$. Equivalently, $\lambda \in \sigma(T)$ if $\lambda - T$ is not 1-to-1 or not onto.

We consider the situation where an endomorphism has an induced map which has a fixed point in Δ. Thus, we consider a subset of the endomorphisms satisfying case (i) or (iii). The next lemma shows that we can assume the origin is the point fixed by the induced map. (Cf. [4], Lemma 1.)

LEMMA 1. *Let T be an endomorphism of $A(p)$ with induced map Φ where $\Phi(z_0) = z_0 \in \Delta$. Let $g(z)$ be a conformal map of Δ taking 0 to z_0 and set $q = g^{-1} \circ p \circ g$. If T_1 is the endomorphism of $A(q)$ defined by $T_1(f) = f(g^{-1} \circ \Phi \circ g)$, then the induced map Φ_1 satisfies $\Phi_1(0) = 0$, $\Phi_1'(0) = \Phi'(z_0)$, and furthermore, $\sigma(T) = \sigma(T_1)$.*

We omit the proof of Lemma 1. The approach developed by Kamowitz in [4] for describing the spectra of endomorphisms of $A(\Delta)$ can now be adapted with minor modifications to the endomorphisms of $A(p)$.

LEMMA 2. *Let T be an endomorphism of $A(p)$ with induced map Φ where $\Phi(0) = 0$. Then $\sigma(T) \supset \{(\Phi'(0))^n$: n is a positive integer$\}$.*

Proof. We have $\Phi(\Delta) \subset \Delta$ since Φ fixes the origin. The case where $\Phi'(0) = 0$ is obvious. Consider $0 < |\Phi'(0)| \leqq 1$. Given an integer $k > 0$ let $g(z) = z^k h(z) \in A(p)$ where $h(0) \neq 0$. Then following [4], Lemma 2 we may show that $(\Phi'(0))^k f - f \circ \Phi \neq g$ for any $f \in A(p)$. Hence, $(\Phi'(0))^k - T$ is not onto and $(\Phi'(0))^k \in \sigma(T)$.

In view of Theorem 2 and the spectral mapping theorem we conclude:

COROLLARY. *Let T be an automorphism of $A(p)$ with induced map Φ where $\Phi(0) = 0$. Let $c = \Phi'(0)$. Then there is an integer $n > 0$ such that $c^n = 1$ and $\sigma(T) = \{1, c, c^2, \cdots, c^{n-1}\}$.*

Let T be an endomorphism of $A(p)$ with induced map Φ with $\Phi(\Delta) \subset \Delta$. Let Φ_n denote the nth iterate of Φ. Then $\bigcap_{n=1}^{\infty} \Phi_n(\bar{\Delta})$ is called the *fixed set of* Φ. The fixed set is compact and connected.

THEOREM 3. *Let T be an endomorphism of $A(p)$ with induced map Φ where $\Phi(z_0) = z_0 \in \Delta$ and suppose the fixed set is infinite. If T is not an automorphism, then $\sigma(T) = \bar{\Delta}$.*

Proof. Apply the method of [4], Theorem 7 after replacing $g(z) = z^\nu$ by $g(z) = z^\nu h(z) \in A(p)$ where $||h|| \leqq 1$.

LEMMA 3. *Let T be an endomorphism of $A(p)$ with induced map Φ where $\Phi(\Delta) \subset \Delta$ and $\Phi(\Delta_p) \subset \Delta$. Suppose $\Phi(0) = 0$. Let ν be a positive integer and suppose every $f \in A(p)$ with a zero of order at least $(\nu + 1)$ at 0 is in the range of $\lambda - T$ where $\lambda \notin \{(\Phi'(0))^n$: n is a positive integer$\} \cup \{0, 1\}$. Then $\lambda - T$ is onto.*

Proof. Let $z^\nu q(z) \in A(p)$ where $q(0) = 1$. Define $g(z) = \Phi(z)^\nu -$

$\Phi'(0)^\nu z^\nu q(z)$. By assumption T falls into case (i) and so $\Phi \in A(p)$. Thus, $g \in A(p)$. Also, g vanishes at $z = 0$ to order at least $(\nu + 1)$. By hypothesis there is $h \in A(p)$ such that $(\lambda - T)h = g$. Set $f = h + z^\nu q$. Then

$$\begin{aligned}(\lambda - T)f &= (\lambda - T)h + (\lambda - T)z^\nu q \\ &= \Phi(z)^\nu - (\Phi'(0))^\nu z^\nu q(z) + \lambda z^\nu q(z) - \Phi(z)^\nu q(\Phi(z)) \\ &= \Phi(z)^\nu(1 - q(\Phi(z))) + (\lambda - \Phi'(0)^\nu)z^\nu q(z)\ .\end{aligned}$$

Since $(\lambda - T)f$ and $z^\nu q(z)$ belong to $A(p)$, so does $\Phi(z)^\nu(1 - q(\Phi(z)))$. Also, $\Phi(z)^\nu(1 - q(\Phi(z)))$ vanishes to order at least $(v + 1)$ at the origin and so $\Phi(z)^\nu(1 - q(\Phi(z)))$ belongs to the range of $\lambda - T$. Since $\lambda - \Phi'(0)^\nu \neq 0$, we conclude $z^\nu q(z)$ belongs the range of $\lambda - T$.

Let $f \in A(p)$ vanish to order at least ν at the origin. We will show that f belongs to the range of $\lambda - T$. We have $f(z) = az^\nu + r(z)$ and $az^\nu q(z) = az^\nu + s(z)$ where both $r(z)$ and $s(z)$ vanish to order at least $(\nu + 1)$ at the origin. Then $f(z) = az^\nu + s(z) - s(z) + r(z) = az^\nu q(z) - (s(z) - r(z))$. Since $f(z)$ and $az^\nu q(z)$ belong to $A(p)$, so does $s(z) - r(z)$. But then $s(z) - r(z)$ belongs to the range of $\lambda - T$. Also, $az^\nu q(z)$ belongs to the range of $\lambda - T$. Hence, $f(z)$ belongs to the range of $\lambda - T$. Now by use of induction we see that $\lambda - T$ is onto. This completes the proof.

THEOREM 4. *Let T be an endomorphism of $A(p)$ with induced map Φ where $\Phi(z_0) = z_0 \in \Delta$. Suppose the fixed set of Φ is $\{z_0\}$. Then $\sigma(T) = \{(\Phi'(z_0))^n$: n is a positive integer$\} \cup \{0, 1\}$.*

Proof. Since the fixed set is $\{z_0\}$, Φ must satisfy case (i) for under case (iii) Φ maps $\bar{\Delta}$ onto itself. In particular, Lemma 3 applies. The method of [4], Theorem 9 can now be applied with minor modifications.

REMARK. The spectra of endomorphisms whose induced map fixes a point of Δ_p can be described in a manner entirely parallel to the foregoing work. If T is an endomorphism of $A(p)$ and if the induced map Φ fixes a point of Δ_p, then Theorems 3 and 4 can be restated with $\Theta = \Psi^{-1} \circ \Phi \circ \Psi$ replacing Φ and with $\bigcap_{n=1}^{\infty} \Theta_n(\bar{\Delta})$ taking the role of the fixed set.

We give one result describing the spectra of automorphisms satisfying case (iv).

THEOREM 5. *Let T be an automorphism of $A(p)$ with induced map Φ where $\Phi(\Delta) \subset \Delta_p$ and $\Phi(\Delta_p) \subset \Delta$. Suppose $\Phi \circ \Phi(z_0) = z_0 \in \Delta$. If $T^2 \neq I$, then $\sigma(T) = \{\lambda\colon \lambda^2 \in \sigma(T^2)\}$.*

Proof. Using Lemma 1 and the corollary to Lemma 2 we conclude that there is an integer $n > 1$ where $c^n = 1$ and $\sigma(T^2) = \{1, c, c^2, \cdots, c^{n-1}\}$ for $c = (\Phi \circ \Phi)'(z_0)$. By the spectral mapping theorem $\sigma(T^2) = (\sigma(T))^2$. So, either $\sqrt{c}$ or $-\sqrt{c}$ belongs to $\sigma(T)$. According to [6], Theorem 1, $\sigma(T)$ is a finite subgroups of Γ. Since both $\sqrt{c}$ and $-\sqrt{c}$ generate the $2n$th roots of unity, we see that the $2n$th roots of unity comprise $\sigma(T)$. This completes the proof.

EXAMPLE 5. We construct a singular homeomorphism $p(e^{it})$ of Γ such that $p \circ p(e^{it}) = -e^{it}$. Let $h_1(x)$: $[0, \pi/2] \rightarrow [\pi/2, \pi]$ be strictly increasing, continuous and singular. Then for integer values of j define $h_j(x)$: $[\pi(j-1)/2, \pi j/2] \rightarrow [\pi j/2, \pi(j+1)/2]$ by

$$h_j(x) = h_{j-1}^{-1}(x) + \pi \quad \text{for} \quad j \geqq 2$$

and

$$h_j(x) = h_{j+1}^{-1}(x + \pi) \quad \text{for} \quad j \leqq 0 .$$

Now use the $h_j(x)$ to define $h(x)$: $\boldsymbol{R} \rightarrow \boldsymbol{R}$. It may be shown that $h \circ h(x) = x + \pi$. Set $p(e^{it}) = \exp(ih(t))$ to obtain the required map.

The map $p(e^{it})$ gives rise to an algebra $A(p)$ and automorphism $T(f) = f \circ p$. The induced map Φ satisfies case (iv) and $\Phi \circ \Phi(z) = -z$. Then $\sigma(T^2) = \{1, -1\}$ and by Theorem 5 $\sigma(T) = \{1, i, -1, -i\}$.

REMARK. It can be verified that the spectrum of the automorphism in Example 3 is given by $\{1, -1\}$.

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