PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 2, 1980

Λ-HOMOLOGY COBORDISM BUNDLES

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Let K be a set of primes and Λ the localization of the integers away from K. In this paper we compute the homotopy types of $G(K)/H(K)$ and $H(K)/PL$, when $H(K)$ is the classifying monoid for Λ-homology cobordism bundles, with applications to the space $BH(K)$.

1. Introduction. Let Λ be a subring of $\boldsymbol{Q}$ with unit, and K the set of primes invertible in Λ. This paper is concerned with Λ-homology cobordism bundles, which are defined as in [13], using Λ-coefficients throughout.

In § 2, we define Λ-homology cobordism sphere, disc and cone bundles and discuss their basic properties, including representability, existence of normal bundles and transversality. Most of the results of this section are known in some form (from the bundle theories of [19], [28], straightforward generalizations of the $\Lambda = \boldsymbol{Z}$ case in [13], [15], [8] or special cases of [9], [22]).

In § 3, we consider rational surgery obstructions for simply connected manifolds. Our main result is a product formula for $\boldsymbol{Z}/m$-manifolds, which allows us to apply the Morgan-Sullivan construction]18].

In § 4 we compute the homotopy type of $G(K)/H(K)$. A similar construction has been briefly sketched by Quinn [19], following, as does the one given here, the construction of Sullivan [24] for $G/\widetilde{\mathrm{PL}}$. We show that $G(K)/H(K) \cong \Lambda^{\cdot +} \times K(\tilde{\psi}_3^K \otimes \Lambda, 4) \times Y$, where Y is given by the fiber diagram

$$\begin{array}{ccc} Y & \longrightarrow & \left(\prod_{i>0} K(L_i(1;\Lambda) \otimes \Lambda, i)\right)_{(2)} \\ \big\downarrow & & \big\downarrow \\ (BO_K)_2 & \longrightarrow & \prod_{i>0} K(\boldsymbol{Q}, 4i) \end{array} .$$

Here ψ_n^K denotes the group of PL Λ-homology n-spheres, modulo H_K-cobordism, and $\tilde{\psi}_3^K$ is the kernel of an invariant $\tilde{\psi}_3^K \to (\boldsymbol{Z}/8) \otimes \Lambda$.

In § 5, we compute the homotopy type of $H(K)/\widetilde{\mathrm{PL}}$. Our result is: $H(K)/\widetilde{\mathrm{PL}} \cong (\mathrm{BS}\widetilde{\mathrm{PL}})^{(K)} \times \prod_{i>0} K(\psi_i^K \otimes \Lambda, i)$, where $(\mathrm{BS}\widetilde{\mathrm{PL}})^{(K)}$ is the fiber of $\mathrm{BS}\widetilde{\mathrm{PL}} \to \mathrm{BS}\widetilde{\mathrm{PL}}^K$.

Finally, in § 6, we consider applications to Λ-homology cobordism bundles. The homotopy groups of $BH(K)$ are shown to be

$$\pi_i(BH(K)) \cong \begin{cases} \pi_i(B\widetilde{\mathrm{PL}}) \otimes \Lambda & i \not\equiv 0 \mod (4) \\ \pi_i(B\widetilde{\mathrm{PL}}) \otimes \Lambda \oplus \operatorname{tor} \bar{W}(\Lambda) \otimes \Lambda & i = 4j > 4 \\ \Lambda \oplus \psi_3^K \otimes \Lambda & i = 4 . \end{cases}$$

We also show that, unless $2 \in K$ or $K = \phi$, $BH(K)$ is not "computable" in terms of BTOP, showing that a conjecture of Quinn [19] is false.

2. Λ-homology manifolds and Λ-homology cobordism bundles. A polyhedron M is called a *Λ-homology manifold* of dimension n if M has a subdivision M' so that $\tilde{H}_*(LK(x, M'); \Lambda) \cong \tilde{H}_*(S^{n-1}; \Lambda)$ or 0. The *boundary* of M, $\partial M = \{x \in M' : \tilde{H}_*(\mathrm{LK}(x, M'); \Lambda) = 0\}$ is a Λ-homology manifold of dimension $n-1$.

A *Λ-homology n-sphere* is a Λ-homology n-manifold Σ so that $H_*(\Sigma; \Lambda) \cong H_*(S^n; \Lambda)$; a *$\Lambda$-homology n-disc* is a compact Λ-acyclic Λ-homology n-manifold Δ. The prefix "PL" indicates that Σ or Δ is a PL manifold. A *Λ n-cell* is the cone cM over a Λ-homology $(n-1)$-sphere or $(n-1)$-disc M; such a Λ n-cell is a Λ-homology n-manifold with boundary M or $M \cup c(\partial M)$. An *H_K-cobordism* is a Λ-homology manifold triad $(W; M_+, M_-)$ with $H_*(W; M_\pm; \Lambda) = 0$. (Again the prefix "PL" means that W is a PL-manifold.)

A *Λ-cell decomposition* of a simplicial complex X is a collection $\mathscr{A}$ of subpolyhedra of X so that

(i) each $\Delta \in \mathscr{A}$ is a Λ-cell,

(ii) X has a subdivision X' so that every simplex of X' lies in the interior of a unique element of $\mathscr{A}$,

and

(iii) if $\Delta \in \mathscr{A}$, $\partial\Delta$ is a union of elements of $\mathscr{A}$.

Let X be a simplicial complex with a Λ-cell decomposition $\mathscr{A}$. A *Λ-homology cobordism (n-sphere) bundle* ξ over X is a complex $E = E(\xi)$ over $\mathscr{A}$ (see [13], pg. 96) so that, for each $\nabla^m \in \mathscr{A}$,

(i) $E(\Delta)$ is a Λ-homology $(n+m)$-manifold with

$$\partial E(\Delta) = E(\partial\Delta) = \bigcup_{\substack{\Delta_0 \in \mathscr{A}|\Delta \\ \Delta_0 \neq \Delta}} E(\Delta_0) ,$$

and

(ii) there is a complex W over $\mathscr{A}|\Delta$ so that $W(\Delta_0)$ is an H_K-cobordism between $E(\Delta_0)$ and $\Delta_0 \times S^n$ for each $\Delta_0 \in \mathscr{A}|\Delta$.

Here $\mathscr{A}|\Delta$ denotes the Λ-cell decomposition of Δ consisting of those $\Delta_0 \in \mathscr{A}$ with $\Delta_0 \subset \Delta$.

If ξ^m, η^n are Λ-homology cobordism sphere bundles over X, Y, then we define their *product* $\xi \times \eta$ to be the space $E(\xi) \times E(\eta)$ over the induced Λ-cell decomposition of $X \times Y$. *Restrictions* are defined in the obvious way. If ξ is a Λ-homology cobordism sphere bundle over X, and $f: Y \to X$ is a simplicial map, then the *pull-back* $f^*\xi$ is defined to be $\varepsilon^0 \times \xi | G_f$, where ε^p denotes the trivial bundle $Y \times S^q$, and we identify Y with the graph G_f of f. If ξ, η are Λ-homology cobordism sphere bundles over X, their *Whitney sum* $\xi \oplus \eta$ is defined by $\Delta^*(\xi \times \eta)$, where $\Delta: X \to X \times X$ is the diagonal.

Two Λ-homology cobordism n-sphere bundles ξ, η over X are *isomorphic*, written $\xi \cong \eta$, if there is a complex G over $\mathscr{A}$ so that for each $\Delta \in \mathscr{A}$, $G(\Delta)$ is an H_K-cobordism between $E(\xi)(\Delta)$ and $E(\eta)(\Delta)$.

We similarly define a *Λ-homology n-disc bundle* ξ over X to be a complex $E = E(\xi)$ over $\mathscr{A}$ so that, for each $\Delta^m \in \mathscr{A}$,

(i) $E(\Delta)$ is a Λ-homology $(n+m)$-manifold, with $\partial E(\Delta)$ containing $E(\partial\Delta)$ as a codimension 0 submanifold,
and

(ii) there is a complex W over $\mathscr{A}|\Delta$ so that $W(\Delta_0)$ is an H_K-cobordism between $(E(\Delta_0); E(\partial\Delta_0), \overline{\partial E(\Delta_0) - E(\partial\Delta_0)})$ and $(\Delta_0 \times D^n; \partial\Delta_0 \times D^n, \Delta_0 \times S^{n-1})$ for each $\Delta_0 \in \mathscr{A}|\Delta$.

A *Λ-homology n-cone bundle* is a Λ-homology n-disc bundle ξ with $E(\Delta) = c(\overline{\partial E(\Delta) - E(\Delta)})$. The concepts given above generalize to disc and cone bundles in the obvious way.

PROPOSITION 2.1. ([13], *Prop.* 3.3). *There exist bijective correspondence between the isomorphism classes of Λ-homology cobordism $(n-1)$-sphere, n-cone, and n-disc bundles over X.*

Thus we may freely pass among the three types of bundles defined above. The term *Λ-homology cobordism n-bundles* shall refer either Λ-homology cobordism $(n-1)$-sphere, n-disc or n-cone bundles.

Let $k_n(X)$ denote the set of isomorphism classes of Λ-homology cobordism n-bundles over the simplicipal complex X.

THEOREM 2.2. *k_n^Δ is representable,* i.e., *there is an H-space $BH(K)_n$ so that $k_n^\Delta(X) \cong [X, BH(K)_n]$.*

The construction of $BH(K)_n$ follows from Martin and Maunder [13]: Let $H(K)_n$ be the Δ-monoid with i-simplexes given by isomorphisms of the trivial bundle $\Delta^i \times D^n$ over Δ^i. Then the classifying space $BH(K)_n$ of $H(K)_n$ represents k_n^Δ.

PROPOSITION 2.3. ([13], *Prop.* 3.4). *Let ξ be a Λ-homology cobordism bundle over a Λ-homology manifold M. Then $E(\xi)$ is a Λ-homology manifold.*

Let ξ be a Λ-homology n-sphere bundle over a Λ-cell decomposition $\mathscr{A}$ of X. Let $\Delta_0, \Delta_1, \Delta_2 \in \mathscr{A}$ with $\Delta_0 \subset \Delta_1 \cap \Delta_2$, and W_i complexes over $\mathscr{A}|\Delta_i$, $i = 1, 2$, so that $W_i(\Delta_0)$ is an H_K-cobordism between $E(\Delta_0)$ and $\Delta_0 \times S^n$. Attaching W_1 and W_2 along $E(\Delta_0)$, we get an H_K-cobordism of $\Delta_0 \times S^n$ with itself, and so an automorphism of $H_n(\Delta_0 \times S^n; \Lambda) \cong \Lambda$. We say that ξ is *orientable* if we may always choose W_1, W_2 so that this automorphism is the identity times a positive unit in Λ; a choice of these H_K-cobordisms is called an *orientation*. We may define a Λ-monoid $SH(K)_n$ so that $BSH(K)_n$ classifies oriented Λ-homology cobordism n-sphere bundles.

PROPOSITION 2.4. ([13], *Cor.* 3.9). *$\pi_1(BH(K)_n) \cong \boldsymbol{Z}/2$ and $BSH(K)_n$ is the universal cover of $BH(K)_n$.*

The same holds for $BH(K)$, $BSH(K)$, where $BH(K) = \varinjlim BH(K)$, and $BH(K)_n \to BH(K)_{n+1}$ is defined by block-by-block suspension, etc.

Let $B\widetilde{\mathrm{PL}}$ penote the classifying space for PL block bundles. By [3], there is a natural map $B\widetilde{\mathrm{PL}} \to BH(K)$; let $H(K)/\widetilde{\mathrm{PL}}$ denote its homotopy fiber.

THEOREM 2.5. ([3], *Theorem* 3.6). *$\pi_n(H(K)/\widetilde{\mathrm{PL}}) \otimes \Lambda \cong \psi_n^K \otimes \Lambda$, where ψ_n^K is the group of* PL *Λ-homology n-spheres modulo* PL *H_K-cobordism.*

Let $\bar{W}(\Lambda)$ denote the Witt group of even quadratic forms over Λ, and $\bar{W}(\Lambda, \boldsymbol{Z}) = \operatorname{coker}(\bar{W}(\boldsymbol{Z}) \to \bar{W}(\Lambda))$; $\bar{W}(\Lambda, \boldsymbol{Z})$ is a torsion group, all elements having order dividing 8, and is not finitely generated in general (cf. § 3). We have the following calculation of ψ_n^K for $n \geqq 4$.

PROPOSITION 2.6. *$\psi_4^K = 0$ and* $\psi_n^K \otimes \Lambda \cong \begin{cases} 0 & n \not\equiv 3 \bmod (4) \\ \bar{W}(\Lambda, \boldsymbol{Z}) & n = 4k-1 > 3. \end{cases}$

Proof. Using the notation of [5], we have

$$\begin{aligned} \psi_4^K &\cong \psi_4^Q \otimes \boldsymbol{Z}_{(K)} \oplus \psi_4 \otimes \boldsymbol{Z}_K \\ &\cong \operatorname{tor}(A_4^Q) \otimes \boldsymbol{Z}_{(K)} = 0 . \end{aligned}$$

The result for $n > 4$ follows from [3], Theorem 3.5.

THEOREM 2.7. *If $n \geqq 3$, then there is a map $\phi_n: BH(K)_n \to BG(K)_n$ so that*

$$\begin{array}{ccc} B\widetilde{\mathrm{PL}}_n & \longrightarrow & BG_n \\ \downarrow & & \downarrow \\ BH(K)_n & \longrightarrow & BG(K)_n \end{array}$$

commutes (up to homotopy); the maps ϕ_n *are compatible with stabilization.*

Proof. The argument is basically the same as that in [8], § 6. Let $\bar{G}(K)_n$ denote the Λ-set with i-simplexes PL Λ S^{n-1}-block fibrations over $\Delta^i \times I$, trivial over $\Delta^i \times \{0, 1\}$; note that $B\bar{G}(K)_n \cong BG(K)_n$.

Define a map $H(K)_n \to \bar{G}(K)_n$ inductively on cells as follows: Since the 2-skeletons of H_n and $H(K)_n$ coincide, we may use the construction of [8] for cells of dimension 1 and 2. Assume an i-cell, $i \geqq 3$, is represented by a Λ-homology cobordism $(n-1)$-sphere bundle ξ over $\Delta^i \times I$, trivial over $\Delta^i \times \{0, 1\}$, and, inductively, a Λ S^{n-1}-block fibration over $\Delta^i \times I$. Since $\dim E(\xi) \geqq 5$ and $E(\xi)$ is a smooth manifold in a neighborhood of its dual 3-skeleton, it follows from Corollary 3.3 of [5] that we may do surgery on $E(\xi)$ rel $\partial E(\xi)$ to get a new Λ-homology cobordism bundle ξ' with $\pi_1(E(\xi')) = 0$. The remainder of the proof follows as in [8].

The next theorem shows the existence of normal bundles.

THEOREM 2.8. *Let* M^n, N^{n+k} *be compact* Λ*-homology manifolds with* M *embedded as a full subcomplex of* N. *Then* M *has a* Λ*-homology cobordism* k*-cone bundle neighborhood in* N.

Proof. We use the notation of Stone [22]. Let $\{X_1, \cdots, X_m\}$ be the intrinsic variety of M. By Theorem 2.1 of [22], $\{X_0, \cdots, X_m\}$ has a regular neighborhood stratification in N, and so has a cone block bundle neighborhood ξ in N. By construction, ξ is the desired Λ-homology cobordism bundle.

We now turn to transversality. A bundle theory with the proper transversality theorems has been developed by Quinn [19], and we show that this theory coincides, stably, with ours. This is sufficient for the applications in § 3.

Let X be a finite complex. A $Q(K)$*-bundle* over X is a pair (E, B) where B is a regular neighborhood of X in some Λ-homology manifold and E is a relative regular neighborhood of $(B, \partial B)$ is some PL-manifold. We also assume B is stratified ([22]) in E. (Quinn calls these PL_K-bundles, which is ambiguous, since these bundles are not equivalent to PL-block bundles if $K = \phi$). Let $BQ(K)$ denote the classifying space for stable $Q(K)$-bundle (cf. [19]).

Let $Y \subset X$. A *$Q(K)$-bundle structure* on a neighborhood of Y in X is a $Q(K)$-bundle (E, B) over Y, a regular neighborhood N of Y in X, and an embedding $(N, Y) \to (E, Y)$, transverse to B.

THEOREM 2.9. ([19], *Theorem* 3.4). *Let Y be a subcomplex of X with a $Q(K)$-bundle structure, and $f: M \to X$, where M is a Λ-homology manifold. Then f is homotopic to a map g transverse to Y, in the sense that $g^{-1}(Y)$ is a Λ-homology manifold with canonical $Q(K)$-bundle structure.*

We now construct a natural equivalence between the sets of stable classes of Λ-homology cobordism bundles and $Q(K)$-bundles. Let $\mathscr{M}_n$ denote the category of compact PL-manifolds and isotopy classes of embeddings (cf. [26]), and $\mathscr{M} = \varinjlim \mathscr{M}_n$, where stabilization is defined by cartesian product with I. Let $\mathscr{C}$, $\mathscr{A}b$ denote the categories of finite simplicial complexes and abelian groups. Let $R: \mathscr{C} \to \mathscr{M}$ denote the "regular neighborhood" functor of [26].

Define $H, Q: \mathscr{M}_n \to \mathscr{A}b$ to be the contravariant functors sending M to the set of stable isomorphism classes of Λ-homology cobordism bundles, $Q(K)$-bundles over M. Clearly, H and Q induce functors $\mathscr{M} \to \mathscr{A}b$.

We construct natural transformations $T: Q \to H$, $S: H \to Q$ of follows: let ξ be a $Q(K)$-bundle over a PL-manifold M. As in the proof of Lemma 4.2 of [19], we may assume that $E(\xi)$ is a regular neighborhood of $B(\xi)$ in some $\boldsymbol{R}^s$, s large. By Theorem 2.8, $B(\xi)$ is a Λ-homology cobordism bundle over M, and we let $T(M)(\xi) = B(\xi)$.

If ξ is a Λ-homology cobordism cone bunele over M, then $E(\xi)$ is a Λ-homology manifold by Proposition 2.3, and we define $S(M)\xi = (E(\nu_{E(\xi)}), E(\xi))$ where $\nu_{E(\xi)}$ is the stable normal bundle of $E(\xi)$.

It is easy to see that T and S are natural transformations and that $T \circ S = 1, S \circ T = 1$. We can define a natural equivalence $[\,, BQ(K)] \to [\,, BH(K)]$ (as functors $\mathscr{C} \to \mathscr{A}b$) by

$$\begin{aligned}[X, BQ(K)] &\cong [R(X), BQ(K)] \\ &= Q(R(X)) \\ &\xrightarrow{T} H(R(X)) \\ &= [R(X), BH(K)] \\ &\cong [X, BH(K)]\,.\end{aligned}$$

This natural transformation is induced by the map $BQ(K) \to BH(K)$ that forgets the top space E (of a pair (E, B) as above), and so we have:

THEOREM 2.10. $BH(K) \cong BQ(K)$.

This implies the following by [19], Lemma 4.2. (See also [28].)

COROLLARY 2.11. *$BSH(K)$ is K-local.*

Let $w^{PL} \in H^*(B\widetilde{PL}; Z/2)$, $p^{PL} \in H^*(B\widetilde{PL}; Q)$ denote the universal Stiefel-Whitney, Pontrjagin classes, and let $\phi: B\widetilde{PL} \to BH(K)$ be the natural map.

PROPOSITION 2.12. (i) *If $2 \notin K$, then there is a universal Stiefel-Whitney class $w \in H^*(BH(K); Z/2)$ so that $\phi^* w = w^{PL}$.*

(ii) *There is a universal Pontrjagin class $p \in H^*(BH(K); Q)$ so that $\phi^* p = p^{PL}$.*

Proof. (i) follows from Theorem 2.7, Theorem 4.2 of [25] and the construction of Stiefel-Whitney classes for spherical fibrations. (ii) is proven exactly as in [15] using the construction of Pontrjagin class for rational homology manifolds of [27].

3. **Rational surgery obstructions.** This section is devoted to proving the product formula for rational surgery obstruction necessary to apply the Morgan-Sullivan construction [18]. This is the crucial step in the computation of $G(K)/H(K)$ (cf. § 4).

Let $L_n(\Lambda)$ be the functor of Wall [30] applied to the ring Λ. By [1],

$$L_n(\Lambda) \cong \begin{cases} 0 & n \ \text{odd} \\ Z/2 \otimes \Lambda & n \equiv 2 \bmod(4) \\ \bar{W}(\Lambda) & n \equiv 0 \bmod(4) . \end{cases}$$

These groups have the following geometric significance: Let $f: M^n \to X^n$ be a normal map between a compact manifold M and a simply-connected Λ-Poincare space X so that $\deg(f) \in \Lambda^{\cdot}$, $f|\partial M: \partial M \to \partial X$ is a Λ-homology equivalence, and $n \geqq 5$.

THEOREM 3.1. ([1]). *There is an obstruction $s(f) \in L_n(\Lambda)$ so that $s(f) = 0$ if and only if f is normaly cobordant to a Λ-homology equivalence.*

If $2 \notin K$, the obstruction $s: L_{2k+2}(\Lambda) \to Z/2$ is the Kervaire invariant, and the usual constructions (e.g., [7]) apply. We will need the following existence theorem from [1].

THEOREM 3.2. *Let $k \geqq 1$ and $x \in \bar{W}(\Lambda)$. Then there exists a degree 1 normal map $f: M \to D^{4k}$ so that $H_*(\partial M; \Lambda) \cong H_*(S^{4k-1}; \Lambda)$ and*

$s(f) = x$.

We now compute $\bar{W}(\Lambda)$. Recall the following results concerning Witt groups from Lam [12] or Milnor-Husemoller [17]: Let p be an odd prime. The first and second residue homomorphisms $\bar{\alpha}_p$, $\bar{\beta}_p$: $W(\boldsymbol{Q}_p) \to W(\boldsymbol{F}_p)$ define an isomorphism of $W(\boldsymbol{Q}_p)$ with $W(\boldsymbol{F}_p) \oplus W(\boldsymbol{F}_p)$, where

$$W(\boldsymbol{F}_p) \cong \begin{cases} \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 & p \equiv 1 \mod(4) \\ \boldsymbol{Z}/4 & p \equiv 3 \mod(4) . \end{cases}$$

Define α_p, β_p: $W(\boldsymbol{Q}) \to W(\boldsymbol{F}_p)$ to be the compositions of $\bar{\alpha}_p$, $\bar{\beta}_p$ with the functorial map $W(\boldsymbol{Q}) \to W(\boldsymbol{Q}_p)$. This can be extended to $p = 2$ by letting α_2 be the signature mod(2) and β_2 the 2-adic valuation of the determinant. (Note that $W(\boldsymbol{F}_2) \cong \boldsymbol{Z}/2$.)

We have $\alpha_p(q) = \beta_p(q \otimes \langle p \rangle)$ if $p \neq 2$, and

$$\alpha_p(q \otimes q') = \alpha_p(q)\alpha_p(q') + \beta_p(q)\beta_p(q')(p \neq 2)$$
$$\beta_p(q \otimes q') = \beta_p(q)\alpha_p(q') + \alpha_p(q)\beta_p(q') .$$

Let σ: $W(\boldsymbol{Q}) \to \boldsymbol{Z}$ be the signature homomorphism. The natural map $W(\Lambda) \to W(\boldsymbol{Q})$ is injective, and

$$\sigma \oplus \bigoplus_{p \in K} \beta_p \colon W(\Lambda) \xrightarrow{\cong} \boldsymbol{Z} \oplus \bigoplus_{p \in K} W(\boldsymbol{F}_p) .$$

This computes $\bar{W}(\Lambda)$ if $2 \in K$, and by [17], $\sigma/8$: $W(\Lambda) \xrightarrow{\cong} \boldsymbol{Z}$ if $K = \phi$. If $2 \notin K$, $K \neq \phi$, choose $p_0 \in K$, so that $p_0 \equiv 3 \mod(4)$ if such a p_0 exists, and to be arbitrary if all primes in K are 1 mod(4). Let $K_0 = K - \{p_0\}$.

PROPOSITION 3.3. *Let* $2 \notin K$, $K \neq \phi$.
(i) *If all primes in* K *are* 1 mod(4), *then*

$$\bar{W}(\Lambda) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2 \oplus \bigoplus_{p \in K_0} W(\boldsymbol{F}_p) .$$

(ii) *Otherwise*, $\bar{W}(\Lambda) \cong \boldsymbol{Z} \oplus \bigoplus_{p \in K_0} W(\boldsymbol{F}_p)$.

Proof. Let $\mathscr{U} \subset \boldsymbol{C}^{\cdot}$ be the group of roots of unity and γ_p: $W(\boldsymbol{Q}_p) \to \mathscr{U}$ the Gauss sum character of [17]. Define Φ_K: $W(\Lambda) \to \boldsymbol{Z}/8$ by exp $(2\pi i \Phi_K(q)/8) = \exp(2\pi i \sigma(q)/8)$. $\prod_{p \in K} \gamma_p(q \otimes Q_p)^{-1}$. By Theorem 1.1 of [2], $\bar{W}(\Lambda) \cong \ker(\Phi_K)$.

If $p \equiv 1 \mod(4)$, let π_1, π_2: $W(\boldsymbol{F}_p) \to \boldsymbol{Z}/2$ be the projections. Note that $\pi_1\beta_p(q) = \pi_2\beta_p(q \otimes \langle s_p \rangle)$, where s_p is some quadratic nonresidue mod(p). By [2],

$$\gamma_p(q \otimes Q_p) = \begin{cases} ((-1)^{(p+1)/4} \cdot i)^{\beta_p(q)} & p \equiv 3 \bmod(4) \\ (-1)^{\pi_1 \beta_p(q)} & p \equiv 5 \bmod(8) \\ (-1)^{\pi_2 \beta_p(q)} & p \equiv 1 \bmod(8) . \end{cases}$$

Let $(n; x_1(p_1), \cdots, x_k(p_k))$ denote the element $x \in W(\Lambda)$ with $\sigma(x) = n$, $\beta_{p_i}(x) = x_i$ and $\beta_p(x) = 0$, $p \neq p_1, \cdots, p_k$. If follows easily that $x \in \bar{W}(\Lambda)$ if and only if

$$n + \sum_{p_i \equiv 3(4)} (-1)^{(p_i-3)/4} 2x_i + \sum_{p_i \equiv 5(8)} 4\pi_1(x_i) + \sum_{p_i \equiv 1(8)} 4\pi_2(x_i) \equiv 0 \bmod(8) .$$

Thus $\bar{W}(\Lambda)$ is generated by $(2; (-1)^{(p+1)/4}(p))$ if $p \equiv 3 \bmod(4)$, $(4; (1, 0)(p))$ if $p \equiv 5 \bmod(8)$, and $(4; (0, 1)(p))$ if $p \equiv 1 \bmod(8)$ $(p \in K)$.

It is now easy to check that the isomorphisms above are given by $\sigma/2 \oplus \bigoplus_{p \in K_0} \beta_p$ if $p_0 \equiv 3 \bmod(4)$, $\sigma/4 \oplus \pi_1\beta_{p_0} \oplus \bigoplus_{p \in K_0} \beta_p$ if $p \equiv 5 \bmod(8)$ and $\sigma/4 \oplus \pi_2\beta_{p_0} \oplus \bigoplus_{p \in K_0} \beta_p$ if $p \equiv 1 \bmod(8)$.

Let $a_K = \gcd\{|\sigma(q)|: q \in \bar{W}(\Lambda)\}$. By the proof of Proposition 3.3,

$$a_K = \begin{cases} 1 & 2 \in K \\ 2 & 2 \notin K, \text{ some } p \in K \text{ is } 3 \bmod(4) \\ 4 & K \neq \phi, \text{ all } p \in K \text{ are } 1 \bmod(4) \\ 8 & K = \phi . \end{cases}$$

Let Σ be a $\boldsymbol{Z}/2$-homology 3-sphere and $\mu(\Sigma) \in \boldsymbol{Z}/16$ the invariant of [10]. It is easily checked that μ defines a homomorphism $\mu_K: \psi_3^K \to \boldsymbol{Z}/16$ if $2 \notin K$. By the result above, the image of μ_K is contained in $\boldsymbol{Z}/(16/a_K)$. Combining this with Theorem 3.2, we have

Proposition 3.4. *There is a surjective homomorphism* $\mu_K: \psi_3^K \to \boldsymbol{Z}/(16/a_K) \otimes \Lambda$. *We let* $\tilde{\psi}_3^K = \ker(\mu_K)$.

Let X be a closed, oriented $\boldsymbol{Q}$-Poincare complex of dimension n. Define $\alpha_p(X), \beta_p(X) \in W(\boldsymbol{F}_p)$ to be 0 if $n \not\equiv 0 \bmod(4)$, and the corresponding invariants of the cup product pairing on $H^{n/2}(X; \boldsymbol{Q})$ if $n \equiv 0 \bmod(4)$. This definition can be extended to $\boldsymbol{Q}$-Poincare pairs in the usual way.

Lemma 3.5. *If X is a compact oriented $\boldsymbol{Q}$-Poincare complex, then* $\alpha_p(\partial X) = \beta_p(\partial X) = 0$.

Proof. Assume X is of dimension $4k + 1$. We have a commutative ladder

$$\begin{array}{ccccc} H^{2k}(X; \boldsymbol{Q}) & \xrightarrow{j^*} & H^{2k}(\partial X; \boldsymbol{Q}) & \longrightarrow & H^{2k+1}(X, \partial X; \boldsymbol{Q}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{2k+1}(X, \partial X; \boldsymbol{Q}) & \longrightarrow & H_{2k}(\partial X; \boldsymbol{Q}) & \xrightarrow[j_*]{} & H_{2k}(X; \boldsymbol{Q}) , \end{array}$$

where the vertical isomorphisms are given by Poincare duality, and so $H_{2k}(\partial X; \boldsymbol{Q})/\ker(j_*) \cong \mathrm{Im}(j_*) \cong \mathrm{Im}(j^*) \cong \ker(j_*)$. Therefore, dim $(\mathrm{Im}(j^*)) = \dim(H_{2k}(\partial X; \boldsymbol{Q}))/2$, and for $x = j^*y \in \mathrm{Im}(j)^*$,

$$\begin{aligned}\langle x \cup x, [\partial X]\rangle &= \langle j^*y \cup j^*y, [\partial X]\rangle \\ &= \langle y \cup y, j_*[\partial X]\rangle \\ &= 0 .\end{aligned}$$

By [17], the cup product form on $H^{2k}(\partial X; \boldsymbol{Q})$ is split, and so is split over $\boldsymbol{Q}_p$. The result now is an easy consequence of Sylvester's theorem [12].

LEMMA 3.6. *If X is a closed oriented Poincare space, then* $\beta_p(X) = 0$, $\alpha_p(X) = \sigma(X)\cdot 1$.

Proof. Assume $\dim(X) = 4k$, and let $q: H^{2k}(X) \to \boldsymbol{Z}$ be the cup product pairing. By [17], $\gamma_p(q \otimes \boldsymbol{Q}_p) = 0$ for p odd, and so $\beta_p(q) = 0$ by Lemma 2.1 of [2] (cf. the proof of Proposition 3.3). Since $|\det(q)| = 1$, $\beta_2(q) = 0$.

Let $q' = q \oplus \sigma(X)\cdot\langle -1\rangle$. Then $\sigma(q') = 0$, $\beta_p(q') = 0$ for all p, and so $q' = 0$ in $W(\boldsymbol{Q})$. Therefore $0 = \alpha_p(q') = \alpha_p(q) - \sigma(X)\cdot 1$.

The following "Novikov additivity" result is proved in the same way as the signature case [4].

PROPOSITION 3.7. *Let X, Y be compact, oriented $\boldsymbol{Q}$-Poincare complexes and $f: \partial X \to \partial Y$ an orientation-reversing* PL *homeomorphism. Then*

$$\alpha_p(X \cup_f Y) = \alpha_p(X) + \alpha_p(Y)$$
$$\beta_p(X \cup_f Y) = \beta_p(X) + \beta_p(Y) .$$

We now turn to computing the obstruction $s: L_{4k}(\Lambda) \to \bar{W}(\Lambda)$. Let $f: M^{4k} \to X$ be a normal map as before with $\deg(f) = 1$. Doing surgery on M, rel(∂M) we may assume that $f_*: H_i(M) \to H_i(X)$ is an isomorphism for $i < 2k$ and $A = \ker(f_*: H_{2k}(M; \Lambda) \to H_{2k}(X; \Lambda))$ is a free Λ-module. Self-intersections of $2k$-spheres in M that are null-homotopic in X define a nonsingular even quadratic form q over Λ; $s(f)$ is defined to be the Witt class of q. We let $\sigma(f) = \sigma(q)$, $\beta_p(f) = \beta_p(q)$ and $\alpha_p(f) = \alpha_p(q)$. By [7], $\sigma(f) = \sigma(M) - \sigma(X)$.

PROPOSITION 3.8. $\beta_p(f) = \beta_p(M) - \beta_p(X)$; $\alpha_p(f) = \alpha_p(M) - \alpha_p(X)$.

Proof. By Lemma 4.5 and [7], Theorem V.1.3, $\beta_p(a) = \beta_p(q^*)$, $\alpha_p(q) = \alpha_p(q^*)$, where q^* is defined by the cup product pairing on $A^* = \mathrm{coker}(f^*: H^{2k}(X, \partial X; \boldsymbol{Q}) \to H^{2k}(M, \partial M; \boldsymbol{Q}))$. We have $H^{2k}(M,$

$\partial M; \boldsymbol{Q}) = A^* \oplus f^*H^{2k}(X, \partial X; \boldsymbol{Q})$, and for $x \in A^*$, $y \in H^{2k}(X, \partial X; \boldsymbol{Q})$,

$$\begin{aligned}\langle x \cup f^*y, [M, \partial M]\rangle &= f^*y \cap (x \cap [M, \partial M]) \\ &= y \cap f_*(x \cap [M, \partial M]) \\ &= y \cap (f_*x \cap [X, \partial X]) \\ &= 0 .\end{aligned}$$

Since f is of degree 1, the cup product pairing on $f^*H^{2k}(X, \partial X; \boldsymbol{Q})$ is equivalent to that on $H^{2k}(X, \partial X; \boldsymbol{Q})$, and so $\beta_p(M) = \beta_p(q^*) + \beta_p(X)$, $\alpha_p(M) = \alpha_p(q^*) + \alpha_p(X)$.

Now assume f is of degree $n \in \Lambda^{\cdot+}$. Let $v_p(n)$ be the p-adic valuation of n and $e_p(n) = p^{-v_p(n)} \cdot n$.

COROLLARY 3.9. (i) *If* $v_p(n)$ *is even, then* $\beta_p(f) = \beta_p(M) - \langle e_p(n)\rangle \beta_p(X)$ *and* $\alpha_p(f) = \alpha_p(M) - \langle e_p(n)\rangle \alpha_p(X)$.

(ii) *If* $v_p(n)$ *is odd, then* $\beta_p(f) = \beta_p(M) - \langle e_p(n)\rangle \alpha_p(X)$,

$$\begin{aligned}\alpha_p(f) = \alpha_p(M) - \langle e_p(n)\rangle \beta_p(X),\ p \neq 2,\ \beta_2(f) \\ = \beta_2(M) + \alpha_2(X) + \beta_2(X),\ \alpha_2(f) = \alpha_2(M) + \alpha_2(X) .\end{aligned}$$

Proof. Let $(Y, \partial Y)$ be the Λ-Poincare pair with underlying space X and fundamental class $n[X, \partial X]$. Then f induces a degree 1 map $f'\colon (M, \partial M) \to (Y, \partial Y)$ with $s(f) = s(f')$. If q, q' are the quadratic forms corresponding to X, Y, then $nq(x) = q'(x)$, and the result follows from the definition of the first and second residues.

COROLLARY 3.10. *If* $f\colon M \to N$ *is a normal map between closed, oriented manifolds of degree* n, *then for* $p \neq 2$

$$\beta_p(f) = \begin{cases} 0 & v_p(n) \text{ even} \\ -\sigma(N) \cdot \langle e_p(n)\rangle & v_p(n) \text{ odd} \end{cases}$$

$$\alpha_p(f) = \begin{cases} \sigma(M) \cdot 1 - \sigma(N) \cdot \langle e_p(n)\rangle & v_p(n) \text{ even} \\ \sigma(M) \cdot 1 & v_p(n) \text{ odd} . \end{cases}$$

We have the following product formulas for the invariants α_p, β_p.

THEOREM 3.11. *Let* $f\colon M \to X$, $g\colon N \to Y$ *be degree* 1 *normal maps as above. Then*

$$\begin{aligned}\alpha_2(f \times g) &= \alpha_2(f)\alpha_2(f) + \alpha_2(f)\alpha_2(Y) + \alpha_2(g)\alpha_2(X) \\ \alpha_p(f \times g) &= \alpha_p(f)\alpha_p(g) + \beta_p(f)\beta_p(g) + \alpha_p(f)\alpha_p(Y) + \beta_p(f)\beta_p(Y) \\ &\quad + \alpha_p(g)\alpha_p(X) + \beta_p(g)\beta_p(X) \quad (p \neq 2) \\ \beta_p(f \times g) &= \beta_p(f)\alpha_p(g) + \beta_p(g)\alpha_p(f) + \beta_p(f)\alpha_p(Y) + \beta_p(Y)\alpha_p(f) \\ &\quad + \beta_p(g)\alpha_p(X) + \beta_p(X)\alpha_p(g) .\end{aligned}$$

Proof. First assume, $\dim M$, $\dim N \equiv 0 \bmod(4)$. By Proposition 3.8,

$$\begin{aligned}\alpha_p(f \times g) &= \alpha_p(M \times N) - \alpha_p(X \times Y)\\ &= (\alpha_p(M)\alpha_p(N) + \beta_p(M)\beta_p(N)) - (\alpha_p(X)\alpha_p(Y)\\ &\quad + \beta_p(X)\beta_p(Y))\ (p \neq 2)\end{aligned}$$

$$\begin{aligned}\beta_p(f \times g) &= \beta_p(M \times N) - \beta_p(X \times Y)\\ &= (\beta_p(M)\alpha_p(N) + \beta_p(N)\alpha_p(M)) - (\beta_p(X)\alpha_p(Y) + \beta_p(Y)\alpha_p(X))\end{aligned}$$

and the result follows from Proposition 3.8 by eliminating $\alpha_p(M)$, $\alpha_p(N)$, $\beta_p(M)$, $\beta_p(N)$ from these equations. (The α_2 case follows from the corresponding signature result.)

If $\dim(M \times N) \not\equiv 0 \bmod(4)$, then the result is trivial, and we consider the case $\dim M = 4l + 1$, $\dim N = 4h - 1$. Let $k = l + h$ and assume $\partial M = \phi = \partial N$ (the bounded case being similar). We may write $H^{2k}(M \times N; \boldsymbol{Q}) = A \oplus B$, where

$$\begin{aligned}A &= \bigoplus_{i=2l+1}^{2k} H^i(M; \boldsymbol{Q}) \otimes H^{2k-i}(N; \boldsymbol{Q})\\ &\cong \bigoplus_{j=2h+1}^{2k} H^{2k-j}(M; \boldsymbol{Q}) \otimes H^j(N; \boldsymbol{Q})\\ &= B\,.\end{aligned}$$

Furthermore, for $z \in A$, $\langle z \cup z, [M \times N]\rangle = 0$ since for $x \in H^i(M; \boldsymbol{Q})$, $i \geqq 2l + 1$, $x \cup x = 0$. Therefore, the form on $H^{2k}(M \times N; \boldsymbol{Q})$ is split. Similarly, the form on $H^{2k}(w \times Y; \boldsymbol{Q})$ is split, and so $\alpha_p(f \times g) = 0 = \beta_p(f \times g)$. Since the right sides of the equations above are zero by definition, we have equality.

Finally, assume $\dim M = 4l + 2$, $\dim N = 4l - 2$. Let $x_1, \cdots, x_r, y_1, \cdots, y_r$ be a symplectic basis for $H^{2l+1}(M; \boldsymbol{Q})$ and A the subspace spanned by $x_1, \cdots, x_r$. Then $A \otimes H^{2h-1}(N; \boldsymbol{Q})$ is a subkernel of $H^{2l+1}(M; \boldsymbol{Q}) \otimes H^{2h-1}(N; \boldsymbol{Q})$, and $\bigoplus_{i \neq 2l+1} H^i(M; \boldsymbol{Q}) \otimes H^{2k-i}(N; \boldsymbol{Q})$ ($k = l + h$) is split by the argument above. Therefore $H^{2k}(M \times N; \boldsymbol{Q})$ is split and the result holds as before.

Together with Corollary 3.9, this implies

COROLLARY 3.12. *If $f: M \to X$ is a degree n normal map and N a compact manifold, then*

$$\begin{aligned}\alpha_2(f \times 1_N) &= \alpha_2(f)\alpha_2(N)\\ \alpha_p(f \times 1_N) &= \alpha_p(f)\alpha_p(N) + \beta_p(f)\beta_p(N)\ (p \neq 2)\\ \beta_p(f \times 1_N) &= \beta_p(f)\alpha_p(N) + \alpha_p(f)\beta_p(N)\,.\end{aligned}$$

By Lemma 3.6, we have

COROLLARY 3.13. *If $f: M \to X$ is a degree n normal map and N a closed manifold, then*

$$\alpha_p(f \times 1_N) = \alpha_p(f)\sigma(N)$$
$$\beta_p(f \times 1_N) = \beta_p(f)\sigma(N) .$$

In order to apply the results of [18], we must extend Corollary 3.13 to $\boldsymbol{Z}/m$-manifolds. For a $\boldsymbol{Z}/m$-manifold M, we let $\bar{M}$ denote the manifold M is obtained from by identifying the m isomorphic copies of $\delta M \subset \partial M$ (cf. [16] or [18]).

Let M^n, N^n be $\boldsymbol{Z}/m$-manifolds, $\pi_1(\bar{N}) = \pi_1(\delta N) = 0$, and $f: M \to N$ a normal map of degree $r \in \Lambda^{\cdot +}$.

THEOREM 3.14. *Let $n \geqq 6$. Then f is normally cobordant to a Λ-homology equivalence if and only if an obstruction $s(f)$ in*

$$\begin{cases} (\mathrm{tor}\ \bar{W}(\Lambda)) \otimes \boldsymbol{Z}/m & n \equiv 1 \bmod(4) \\ (\boldsymbol{Z}/2) \otimes (\boldsymbol{Z}/m) \otimes \Lambda & n \equiv 2, 3 \bmod(4) \\ \bar{W}(\Lambda) \otimes \boldsymbol{Z}/m & n \equiv 0 \bmod(4) \end{cases}$$

vanishes.

Proof. (i) $n \equiv 1 \bmod(4)$: By Theorem 3.1 and Corollary 3.9, the obstructions to completing surgery on $f|\delta M: \delta M \to \delta N$ are $\sigma(\delta M) - \sigma(\delta N)$ and $\beta_p(\delta M) - \langle e_p(r)\rangle\beta_p(\delta N)$ $(v_p(r) \equiv 0 \bmod(2))$, $\beta_p(\delta M) - \langle e_p(r)\rangle \alpha_p(\delta N)$ $(v_p(r) \equiv 1 \bmod (2))$. Since $m\delta M$, $m\delta N$ are boundaries, $\sigma(\delta M) = \sigma(\delta N) = 0$, $m\beta_p(\delta M) = m\beta_p(\delta N) = m\alpha_p(\delta N) = 0$, and so the obstruction lies in $(\mathrm{tor}\ \bar{W}(\Lambda)) \otimes \boldsymbol{Z}/m$. By Theorem 3.1, there are no further obstructions.

(ii) $n \equiv 2, 3 \bmod(4)$: The arguments are identical to Theorem 3.4 of [16].

(iii) $n \equiv 0 \bmod(4)$: By Theorem 3.1, we may assume that $f|\delta M$ is a Λ-homology equivalence. By Theorem 3.2 and Proposition 3.7, the surgery obstruction of $f: \bar{M} \to \bar{N}$ may be changed by any element of $m\bar{W}(\Lambda)$ and the result follows.

To compute the obstruction in dimensions $0 \bmod(4)$, we introduce a generalization of Milgram's semi-index [16]. Let K be a set of odd primes and define $I^K: W(\boldsymbol{Q}) \to \boldsymbol{Z}/8$ by

$$I^K(q) = \sigma(q) + \sum_{\substack{p \equiv K \\ p - 3(4)}} (-1)^{(p+1)/4} 2\beta_p(q) + \sum_{\substack{p \equiv K \\ p\,|\,5(8)}} 4\pi_1\beta_p(q) + \sum_{\substack{p \equiv K \\ p\,|\,1(8)}} 4\pi_2\beta_p(q) .$$

LEMMA 3.15. *I^K defines an isomorphism of* $\mathrm{Im}(W(\boldsymbol{Z}) \to W(\boldsymbol{Q})/\bar{W}(\Lambda))$ *with* $\boldsymbol{Z}/8$.

The proof is immediate from the proof of Proposition 3.3.

Let $g: P \to Q$ be a degree $r \in \Lambda^{\cdot +}$ normal map between closed, simply connected $(4n-1)$-manifolds. Let $G: W \to Q \times I$ be a normal cobordism from g to a Λ-homology equivalence. Assume G is $(2n)$-connected, and let q_G denote the intersection pairing on $K_{2n}(W; \boldsymbol{Q})$. Define the *K-index* of g by $I^K(g) = I^K(q_G) \in \boldsymbol{Z}/8$. Note that $I^K(g) \equiv \sigma(q_G) \bmod(a_K)$.

LEMMA 3.16. *$I^K(g)$ is independent of G.*

Proof. Let G' be another such normal cobordism. Then $G + G'$: $W \bigcup_P W' \to Q \times I$ is a degree r normal map which is a Λ-homology equivalence when restricted to the boundary. Thus $q_{(G+G')} \cong q_G - q_{G'}$ is an even quadratic form over Λ and so

$$0 = I^X(q_{(G+G')}) = I^K(q_G) - I^K(q_{G'})$$

by Lemma 3.15.

Let $f: M \to N$ be a map as in Theorem 3.14 with $n \equiv 0 \bmod(4)$. Define $\sigma_\infty(f) = (1/a_K)\sigma(s(f)) \in \boldsymbol{Z}/m$, $\sigma_p(f) = \beta_p(s(f)) \in W(\boldsymbol{F}_p) \otimes \boldsymbol{Z}/m$.

PROPOSITION 3.17. $\sigma_\infty(f) = 1/a_K(\sigma(\bar{M}) - \sigma(\bar{N}) + mI^K(f|\delta M))$, $\sigma_p(f) = \beta_p(f)$.

Proof. Clear from the proof of Theorem 4.14.

Let M, N be $\boldsymbol{Z}/m$-manifolds. Define $M \otimes N$ to be the $\boldsymbol{Z}/m$-manifold obtained from $(\bar{M} \times \bar{N} - (\delta M \times \mathring{c}(m) \cup \delta N \times \mathring{c}(m))) \cup \delta M \times \delta N \times W$, where W is a $\boldsymbol{Z}/m$-manifold with $\delta W = m * m$. By [18], $\otimes$ is well-defined and associative up to cobordism.

Define $\sigma(M) = \sigma(\bar{M})$, $\beta_p(M) = \beta_p(\bar{M})$. By the proof of Lemma 3.5, σ and β_p are cobordism invariant mod(m).

PROPOSITION 3.18. $\sigma(M \otimes N) = \sigma(M)\sigma(N)$, $\beta_p(M \otimes N) = \beta_p(M) \alpha_p(N) + \beta_p(N)\alpha_p(M)$ $(\text{mod}(m))$.

Proof. Choose W above so that $H_*(W, \partial W; Q) = 0$. By Mayer-Vietoris, $H^*(\overline{M \otimes N}; \boldsymbol{Q}) \cong H^*(\bar{M}; Q) \otimes H^*(\bar{N}; \boldsymbol{Q})$. By the usual argument (e.g., [23]), the equations above hold for this choice of W, and hence any choice since we are working mod(m).

THEOREM 3.19. *Let $m = 2^k$ and N a closed oriented smooth $\boldsymbol{Z}/m$-manifold. Then $\sigma_v(f \times 1_N) = \sigma_v(f)\sigma(N)$ for $v = 2, 3, 5, \cdots, \infty$.*

Proof. The $v = \infty$ case follows exactly as in [18] (and does not require N to be smooth). Assume v is a finite prime $p \in \Lambda^{\cdot}$. By

Propositions 3.13 and 3.18, we need only show that $\beta_p\colon \Omega_*^{SO}(\boldsymbol{Z}/m) \to W(\boldsymbol{F}_p) \otimes \boldsymbol{Z}/m$ is 0.

First assume $k \geqq 3$. By [18], there is an exact sequence

$$\cdots \longrightarrow \Omega_*^{SO} \xrightarrow{\times m} \Omega_*^{SO} \xrightarrow{r_m} \Omega_*^{SO}(\boldsymbol{Z}/m) \xrightarrow{\delta} \Omega_{*-1}^{SO} \cdots .$$

By Lemma 3.6, β_p vanishes on $\mathrm{Im}(r_m)$. Let $[V] \in \Omega_{4n}^{SO}(\boldsymbol{Z}/m)$. By [29], $\delta V + \delta V = \partial W$ for some smooth manifold W. Then $\delta(\sharp_{m/2} W \cup (-V)) = 0$, so that $[\sharp_{m/2} W \cup (-V)] = r_m[U]$ for some $[U] \in \Omega_{4n}^{SO}$. Therefore

$$\begin{aligned} 0 &= \beta_p(r_m[U]) \\ &= \beta_p(\sharp_{m/2} W) - \beta_p(V) \text{ by Proposition 4.7} \\ &= (m/2)\beta_p(W) - \beta_p(V) \\ &= -\beta_p(V) \text{ since } m/2 \equiv 0 \bmod (4) . \end{aligned}$$

For $m = 2$ or 4, we have a commutative ladder

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & \Omega_*^{SO} & \xrightarrow{\times m} & \Omega_*^{SO} & \xrightarrow{r_m} & \Omega_*^{SO}(\boldsymbol{Z}/m) & \xrightarrow{\delta_m} & \Omega_{*-1}^{SO} & \longrightarrow \cdots \\ & \downarrow{=} & & \downarrow{xz} & & \downarrow{R} & & \downarrow{=} & \\ \cdots \longrightarrow & \Omega_*^{SO} & \xrightarrow{\times z_m} & \Omega_*^{SO} & \xrightarrow{r_{2m}} & \Omega_*^{SO}(\boldsymbol{Z}/2m) & \xrightarrow{\delta_{2m}} & \Omega_{*-1}^{SO} & \longrightarrow \cdots \end{array}$$

and so β_p vanishes on $\delta_m^{-1}(\Omega_{*-1}^{SO})$ by the argument above and [29].

4. The homotopy type of $G(K)/H(K)$. In this section, we use the methods of Sullivan [24] to compute the homotopy type of $G(K)/H(K)$.

LEMMA 4.1. *$G(K)/H(K) \simeq \Lambda^{\cdot +} \times SG(K)/SH(K)$.*

Proof. We have the following homotopy commutative diagram of fibrations

$$\begin{array}{ccccc} BSG(K) & \longrightarrow & BG(K) & \longrightarrow & K(\Lambda^{\cdot}, 1) \\ \uparrow & & \uparrow & & \uparrow \\ BSH(K) & \longrightarrow & BH(K) & \longrightarrow & K(\{\pm 1\}, 1) \end{array}$$

by [25] and Proposition 2.6. Since the fiber of $K(\{\pm 1\}, 1) \to K(\Lambda^{\cdot}, 1)$ is $K(\Lambda^{\cdot}/\{\pm 1\}, 0) = \Lambda^{\cdot +}$, $G(K)/H(K)$ is given as stated.

LEMMA 4.2. *$SG(K)/SH(K)$ is K-local.*

Proof. Both $BSG(K)$ and $BSH(K)$ are K-local by [25] and Corollary 2.11.

In order to construct odd primary characteristic classes for $SG(K)/SH(K)$, we must introduce variants of the homomorphisms σ_v defined in § 3.

Assume $2 \notin K$ and let M^{4n} be a smooth, closed, oriented manifold, $\phi: M \to SG(K)/SH(K)$. By Theorem 2.10, ϕ determines a $Q(K)$-bundle ξ over M that is fiber Λ-homotopy equivalent to ν_M^t. It follows that $T(\xi)_K$ and $T(\nu_M)_K$ are homotopy equivalent. Let $\alpha \in H_{4n+t}(T(\nu_M))$ correspond to the natural collapse $S^{4n+t} \to T(\nu_M)$; α determines an element $\alpha' \in H_{4n+t}(T(\xi); \Lambda)$, and there exists a $k \in \Lambda^{\cdot +} \cap \boldsymbol{Z}$ so that $k\alpha'$ is represented by $f: S^{4n+t} \to T(\xi)$. Making f transverse to M, we get a normal map $f_0: N \to M$ (of degree k). Define

$$\hat{\sigma}_\infty(\phi) = \frac{1}{a_K}\Big(\frac{1}{k}\,\sigma(N) - \sigma(M)\Big) \in \Lambda \otimes \boldsymbol{Z}[1/2]\,,$$

$$\hat{\sigma}_p(\phi) = \frac{1}{k}\,\beta_p(N) - \beta_p(M) \in W(\boldsymbol{F}_p)\,.$$

PROPOSITION 4.3. *The invariants $\hat{\sigma}_\infty$, $\hat{\sigma}_p$ determine homomorphisms*

$$\hat{\sigma}_\infty: \Omega_{4n}^{SO}(SG(K)/SH(K)) \longrightarrow \Lambda,\ \hat{\sigma}_p: \Omega_{4n}^{SO}(SG(K)/SH(K)) \longrightarrow W(\boldsymbol{F}_p)\,.$$

Proof. We verify this for $\hat{\sigma}_\infty$; the proof for $\hat{\sigma}_p$ is similar. Since σ is a cobordism invariant, we need only show that $\hat{\sigma}_\infty(\phi)$ is independent of k and has odd denominator (i.e., $1/k\ \sigma(N) - \sigma(M) \equiv 0 \bmod(a_K)$).

Suppose $f_1, f_2: S^{4n+t} \to T(\xi)$ represent $k_1\alpha', k_2\alpha'$ respectfully. It follows easily that k_2N_1 (the disjoint union of k_2 copies of N_1) is cobordant to k_1N_2, so that $k_2\sigma(N_1) = k_1\sigma(N_2)$.

To see that $1/k\sigma(N) - \sigma(M) \equiv 0 \bmod(a_K)$, notice that $\sigma(N) - \sigma(M) \equiv 0 \bmod(a_K)$, since this is the signature invariant of the normal map f. We have

$$\frac{1}{k}\,\sigma(N) - \sigma(M) = \frac{1}{k}(\sigma(N) - \sigma(M)) + \Big(\frac{1-k}{k}\Big)\sigma(M)$$

and need to consider 3 cases: (i) $a_K = 8$: $K = \phi$, so that $k = 1$; (ii) $a_K = 4$: Then all primes in K are $1 \bmod(4)$ and so $k \equiv 1 \bmod(4)$; (iii) $a_K = 2$: $2 \notin K$, so k is odd. In all cases, $1 - k \equiv 0 \bmod(a_K)$ and the result follows.

PROPOSITION 4.4. *$\hat{\sigma}_\infty$, $\hat{\sigma}_p$ are multiplicative with respect to the signature.*

Proof. This is clear for $\hat{\sigma}_\infty$, and if $[P] \in \Omega_*^{SO}$,

$$\begin{aligned}\hat{\sigma}_p(\pi_1\circ f\colon M\times P \longrightarrow SG(K)/SH(K)) &= \frac{1}{k}\beta_p(N\times P) - \beta_p(M\times P)\\ &= \frac{1}{k}\beta_p(N)\sigma(P) - \beta_p(M)\sigma(P)\\ &= \hat{\sigma}_p(\phi)\sigma(P)\end{aligned}$$

by the methods of § 3.

We may similarly define $\hat{\sigma}_\infty^m\colon \Omega_{4n}^{SO}(SG(K)/SH(K); \boldsymbol{Z}/2^m) \to \boldsymbol{Z}/2^m$, $\hat{\sigma}_p^m\colon \Omega_{4n}^{SO}(SG(K)/SH(K); \boldsymbol{Z}/2^m) \to W(\boldsymbol{F}_p)\otimes \boldsymbol{Z}/2^m$ by

$$\hat{\sigma}_\infty^m(\phi) = \frac{1}{a_K}\left(\frac{1}{k}\left(\sigma(N) + 2^m I^K(f_0|\delta N)\right) - \sigma(M)\right),$$

$$\hat{\sigma}_p^m(\phi) = \frac{1}{k}\beta_p(N) - \beta_p(M)$$

(with notation as above and in § 3). Again by the methods of § 3, we have

PROPOSITION 4.5. *$\hat{\sigma}_\infty^m$, $\hat{\sigma}_p^m$ are multiplicative with respect to the signature.*

THEOREM 4.6. *Let $2\notin K$. Then there exist unique classes $\mathscr{L}_*\in H^{4*}(SG(K)/SH(K); \boldsymbol{Z}_{(2)})$, $\beta_p^*\in H^{4*}(SG(K)/SH(K); W(\boldsymbol{F}_p))$ so that if $[\phi, M]\in \Omega_*^{SO}(SG(K)/SH(K))$ or $\Omega_*^{SO}(SG(K)/SH(K); \boldsymbol{Z}/2^m)$, then*

$$\sigma_\infty(\phi) = \langle \mathscr{L}_M \cup \phi^*\mathscr{L}_*, [M]\rangle$$
$$\sigma_p(\phi) = \langle \mathscr{L}_M \cup \phi^*\beta_p^*, [M]\rangle .$$

Proof. The existence and uniqueness of $\mathscr{L}_*$ follows exactly as in [18]. For β_p^*, let T_p be the Moore space $M(\boldsymbol{Z}/b_p, 1)$ where $b_p = 4(2)$ if $p\equiv 3 \bmod(4)$ $(p\equiv 1\bmod(4))$. Let $X = SG(K)/SH(K)$, and define $\sigma'\colon \Omega_{4*+1}^{SO}(X^+\wedge T_p; \boldsymbol{Z}_{(2)})$ to be 0 and $\sigma_m'\colon \Omega_{4*+1}^{SO}(X^+\wedge T_p; \boldsymbol{Z}/2^m)\to \boldsymbol{Z}/2^m$ to be the composition

$$\begin{aligned}\Omega_{4*+1}^{SO}(X^+\wedge T_p; \boldsymbol{Z}/2^m) &\cong \tilde{\Omega}_{4*+2}^{SO}(X^+\wedge T_p\wedge M(\boldsymbol{Z}/2^m, 1))\\ &\longrightarrow \tilde{\Omega}_{4*+2}^{SO}(X^+\wedge M(\boldsymbol{Z}/b_p, 2)) \quad (m\geqq 2 \text{ if } b_p = 4)\\ &\longrightarrow \Omega_{4*}^{SO}(X; \boldsymbol{Z}/b_p)\\ &\longrightarrow \boldsymbol{Z}/b_p \subset \boldsymbol{Z}/2^m ,\end{aligned}$$

where the final map is $\hat{\sigma}_p^m$ is $b_p = 4$ and one of $\pi_i\circ\hat{\sigma}_p^m$ if $b_p = 2$. If $m = 1$, $b_p = 4$, define $\hat{\sigma}_m$ similarly, replacing $\boldsymbol{Z}/b_p$ with $\boldsymbol{Z}/2$, and $\hat{\sigma}_p^m$ with $2\hat{\sigma}_p^m$. It is easily checked that σ', σ_m' are compatible, so by Corollary 4.3 of [18] (and the remarks following), there exist unique classes $\beta_p^*\in H^{4*+1}(X^+\wedge T_p; \boldsymbol{Z}_{(2)}) \cong H^{4*}(X; \boldsymbol{Z}/4)$ $(p\equiv 3 \bmod(4))$,

$\pi_i\beta_p^* \in H^{4*+1}(X^+ \wedge T_p; Z_{(2)}) \cong H^{4*}(X; Z/2)$ ($p \equiv 1 \bmod(4)$) so that (letting $\beta_p^* = (\pi_1\beta_p^*, \pi_2\beta_p^*) \in H^{4*}(X; Z/2 \oplus Z/2)$), $\hat{\sigma}_p^m(\phi) = \langle \mathscr{L}_M \cup \phi^*\beta_p^*, [M]\rangle$ for $[\phi, M] \in \Omega_*^{SO}(X; Z/2^m)$.

We may similarly define $\hat{\sigma}_p'\colon \Omega_{4*}^{SO}(SG(K)/SH(K)) \to W(F_p)$ by $\hat{\sigma}_p'(\phi) = 1/k\alpha_p(N) - \alpha_p(M)$ (notation as before). It follows from the proofs of Lemma 3.6 and Proposition 3.17 that $\hat{\sigma}_p'$ is multiplicative with respect to the signature, and so we may construct a class $\alpha_p^* \in H^{4*}(SG(K)/SH(K);\ W(F_p))$ as before.

There is also a Kervaire class $\kappa_* \in H^{4*+2}(SG(K)/SH(K); Z/2)$, as in [18] or [20], classifying the homorphism $\hat{c}\colon \Omega_{4*+2}^{SO}(SG(K)/SH(K)) \to Z/2$, $\hat{c}[\phi, M]$ = the Kervaire invariant of the normal map determined by ϕ.

The classes $\mathscr{L}_*$, κ_*, β_p^*, α_p^* satisfy the following product formulas: Let m be the H-multiplication on $SG(K)/SH(K)$ defined by Whitney sum. Then

(i) $m^*(\mathscr{L}_*) = \mathscr{L}_* \otimes 1 + 1 \otimes \mathscr{L}_* + a_K(\mathscr{L}_* \otimes \mathscr{L}_*)$

(ii) $m^*(\kappa_*) = \kappa_* \otimes 1 + 1 \otimes \kappa_*$

(iii) $m^*(\beta_p^*) = \beta_p^* \otimes 1 + 1 \otimes \beta_p^* + \alpha_p^* \otimes \beta_p^* + \beta_p^* \otimes \alpha_p^*$

(iv) $m^*(\alpha_p^*) = \alpha_p^* \otimes 1 + 1 \otimes \alpha_p^* + \alpha_p^* \otimes \alpha_p^* + \beta_p^* \otimes \beta_p^*$.

The proofs of (i), (ii) follow as in [18], [20]. For (iii), let $[\phi, M]$, $[\psi, N] \in \Omega_*^{SO}(SG(K)/SH(K))$. Then

$$\hat{\sigma}_p(m\circ(\phi \times \psi)) = \hat{\sigma}_p(\phi)\hat{\sigma}_p'(\psi) + \hat{\sigma}_p'(\phi)\hat{\sigma}_p(\psi) + \hat{\sigma}_p(\phi)\sigma(N) + \hat{\sigma}_p'(\psi)\sigma(M)$$

and so

$$\begin{aligned}\langle m\circ(\phi \times \psi)^*\beta_p^* \cup \mathscr{L}_{M\times N}, [M \times N]\rangle &\\ &= \hat{\sigma}_p(m\circ(\phi \times \psi))\\ &= \langle(\phi^*\beta_p^* \otimes 1 + 1 \otimes \psi^*\beta_p^* + \phi^*\beta_p^* \otimes \psi^*\alpha_p^*\\ &\quad + \phi^*\alpha_p^* \otimes \psi^*\beta_p^* \cup \mathscr{L}_{M\times N}, [M \times N]\rangle\ .\end{aligned}$$

A similar formula holds for $Z/2^m$-manifolds and the result follows by uniqueness; (iv) is similar.

THEOREM 4.7.

$$\pi_{4n}(SG(K)/SH(K)) \cong \begin{cases} 0 & n \text{ odd} \\ Z/2 \otimes \Lambda & n \equiv 2 \bmod(4) \\ \bar{W}(\Lambda) \otimes \Lambda & n \equiv 0 \bmod(4),\ n > 4 \\ \Lambda \oplus \tilde{\Psi}_3^K \otimes \Lambda & n = 4\ . \end{cases}$$

Proof. Consider the long exact homotopy sequence of the fibration

$$(SH(K)/S\widetilde{\mathrm{PL}})_K \longrightarrow (SG(K)/S\widetilde{\mathrm{PL}})_K \longrightarrow SG(K)/SH(K)\ .$$

By Theorem 2.5, $\pi_n(SH(K)/S\widetilde{\mathrm{PL}})_K \cong \pi_n(H(K)/\widetilde{\mathrm{PL}}) \otimes \Lambda \cong \psi_n^K \otimes \Lambda$, and by [1], $\pi_n(SG(K)/S\widetilde{\mathrm{PL}})_K \cong \pi_n(G/\widetilde{\mathrm{PL}}) \otimes \Lambda$. Since $\psi_n^K \otimes \Lambda = 0$ for n even and $\pi_n(G/\widetilde{\mathrm{PL}}) \otimes \Lambda = 0$ for n odd, the homotopy sequence of the fibration above reduces to short exact sequences

$$0 \longrightarrow \pi_n(G/\widetilde{\mathrm{PL}}) \otimes \Lambda \longrightarrow \pi_n(SG(K)/SH(K)) \longrightarrow \psi_{n-1}^K \otimes \Lambda \longrightarrow 0\ .$$

The cases when n is odd or $2 \bmod(4)$ are now immediate.

Define $\sigma_K\colon \pi_{4k}(SG(K)/SH(K)) \to \bar{W}(\Lambda) \otimes \Lambda \cong \Lambda \oplus \mathrm{tor}(\bar{W}(\Lambda)) \otimes \Lambda$ by Propositions 3.3 and 4.3. (Note that $\hat{\sigma}_\infty$ is defined even when $2 \in K$.) We have the following diagram, for $k > 1$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{4k}(G/\mathrm{PL}) \otimes \Lambda & \xrightarrow{i_\#} & \pi_{4k}(SG(K)/SH(K)) & \xrightarrow{\partial} & \psi_{4k-1}^K \otimes \Lambda & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \sigma/8} & & \big\downarrow{\scriptstyle \sigma_K} & & \big\downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & \Lambda & \xrightarrow[j]{} & \bar{W}(\Lambda) \otimes \Lambda & \xrightarrow[\pi]{} & \bar{W}(\Lambda, Z) \otimes \Lambda & \longrightarrow & 0
\end{array}$$

with exact rows, and $\sigma_K \circ i_\# = j \circ (\sigma/8)$ by construction. (s is the isomorphism of § 2.) Since σ is also an isomorphism, σ_K is an isomorphism by the 5-lemma provided the right square above commutes.

Let $\alpha \in \pi_{4k}(SG(K)/SH(K))$ and choose $x \in W(\Lambda)$ so that $\pi(x \otimes 1) = s \circ \partial(\alpha)$. Let $\alpha_x \in \pi_{4k}(SG(K)/SH(K))$ be the element corresponding to the normal map f of Theorem 3.2 with $s(f) = x$. By [3], $s \circ \partial(\alpha_x) = \pi(x)$, and so $\partial(\alpha \cdot \alpha_x^{-1}) = 1$. It follows that $\pi \circ \sigma_K(\alpha \cdot \alpha_x^{-1}) = 1$, and so $s \circ \partial(\alpha) = \pi(x \otimes 1) = \pi \circ \sigma_K(\alpha_x) = \pi \circ \sigma_K(\alpha)$.

For $k = 1$, we have

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & \longrightarrow & A & \overset{\cong}{\dashrightarrow} & \tilde{\psi}_3^K \otimes \Lambda & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \pi_4(G/\widetilde{\mathrm{PL}}) \otimes \Lambda & \longrightarrow & \pi_4(SG(K)/SH(K)) & \longrightarrow & \psi_3^K \otimes \Lambda & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \Lambda & \xrightarrow[(\times 16/a_K)]{} & \Lambda & \longrightarrow & Z/(16/a_K) \otimes \Lambda & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $A = \ker(\hat{\sigma}_\infty)$. By the argument above, $\hat{\sigma}_\infty$ is onto and both squares commute. Therefore, the dotted arrow above may be filled in, and $\pi_4(SG(K)/SH(K))$ is given as stated.

Let $K_2 = K \cup \{2\}$, $K_{\text{odd}} = K \cup \{\text{odd primes}\}$.

THEOREM 4.8. *$G(K)/H(K) \cong \Lambda^{\cdot+} \times K(\tilde{\psi}_3^K \otimes \Lambda, 4) \times Y$, where Y is given by the fiber diagram*

$$\begin{array}{ccc} Y & \longrightarrow & \prod_{i>0} K(L_i(1; \Lambda), i)_{K_{\text{odd}}} \\ \downarrow & & \downarrow \\ BO_{K_2} & \xrightarrow[p]{} & \prod_{i>0} K(Q; 4i) \, . \end{array}$$

Here p denotes the Pontrjagin class.

Proof. By Lemma 4.1, it suffices to compute the homotopy type of $SG(K)/SH(K)$. First consider $(SG(K)/SH(K))_2$. By Theorem 4.7,

$$\pi_n(SG(K)/SH(K))_2 \cong \begin{cases} 0 & n \neq 4k \\ \boldsymbol{Z}_{K_2} & n = 4k > 4 \\ \boldsymbol{Z}_{K_2} \oplus \tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_2} & n = 4 \, , \end{cases}$$

so there is a map $\phi_2: (SG(K)/SH(K))_2 \Rightarrow K(\tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_2}, 4)$ inducing the projection on π_4.

Write $SG(K)/SH(K) = \varinjlim X_j$, the direct limit of its finite subcomplexes. Since $X_j \subset SG(K)/SH(K)$, there are compatible signature homomorphisms, as in § 3,

$$\begin{array}{ccc} \Omega_*(X_j; \boldsymbol{Q}) & \longrightarrow & \boldsymbol{Q} \\ \downarrow & & \downarrow \\ \Omega_*(X_j; \boldsymbol{Q}/\boldsymbol{Z}) & \longrightarrow & \boldsymbol{Q}/\boldsymbol{Z} \end{array}$$

which determine an orientation class $\Delta'_j \in KO(X_j) \otimes \boldsymbol{Z}_2$ by [25], Theorem 6.3. Represent the K_2-localization of Δ'_j by a map $\Delta_j: (X_j)_{K_2} \to (BO)_{K_2}$. By naturality, the Δ'_js induce a map $\Delta: (SGH(K)/SH(K))_2 \to (BO)_{K_2}$.

By construction, $\Delta|(SG/SPL)_{K_2} \cong \Delta_S$, where $\Delta_S: (SG/S\widetilde{PL})_{K_2} \to (BO)_{K_2}$ is the localization of the orientation class of [25]. By the proof of Theorem 4.7, $(SG/S\widetilde{PL})_{K_2} \subset (SG(K)/SH(K))_2$ induces an isomorphism on π_n, $n \neq 4$, and the direct summand inclusion $\boldsymbol{Z}_{K_0} \to \boldsymbol{Z}_{K_0} \oplus \tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_2}$ if $n = 4$. Therefore,

$$(\phi_2, \Delta): (SG(K)/SH(K))_2 \longrightarrow K(\tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_2}, 4) \times (BO)_{K_2}$$

is a homotopy equivalence.

Let $k_1 \in H^5(K(\boldsymbol{Z}/2, 2), \boldsymbol{Z}_{K_{\text{odd}}} \oplus \tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_{\text{odd}}})$ be the first k-invariant of $(SG(K)/SH(K))_{(2)}$. If $f: (SG/S\widetilde{PL})_{K_{\text{odd}}} \subset (SG(K)/SH(K))_{(2)}$, then by

[11] and [24], $k_1 = f_\sharp(\delta Sq^2)$, where $f_\sharp$ is induced by the map on π_4, since f is 3-connected. But δSq^2 is of order 2, and $f_\sharp$ is multiplication by $16/a_K \equiv 0 \bmod(2)$, so $k_1 = 0$. Therefore there are maps $\phi_{(2)}$: $(SG(K)/SH(K))_{(2)} \to K(\boldsymbol{Z}/2 \otimes \Lambda, 2)$, $\phi'_{(2)}$: $(SG(K)/SH(K))_{(2)} \to K(\boldsymbol{Z}_{K_{\text{odd}}} \oplus \tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_{\text{odd}}}, 4)$ inducing isomorphisms on π_2, π_4 respectfully;

If $2 \notin K$, then by the remarks following Theorem 4.6, there is a class κ_{4i+2}: $(SG(K)/SH(K))^{(2)} \to K(\boldsymbol{Z}/2, 4i+2)$, $i \geqq 1$, and by Proposition 3.3 and Theorem 4.6, there is a class σ_{4i}: $(SG(K)/SH(K))_{(2)} \to K(\bar{W}(\Lambda) \otimes \boldsymbol{Z}_{(2)}, 4i)$, $i > 1$, inducing isomorphisms on π_{4i+42}, π_i, respectively. (For $2 \in K$, $(SG(K)\ SH(K))_{(2)} \cong (SG(K)/SH(K))$, so none of this is necessary.) Again,

$$(\phi_2, \phi_4, \prod \kappa_{4i+2}, \prod \sigma_{4i}) \colon (SG(K)/SH(K))_{(2)} \longrightarrow K(\tilde{\psi}_3^K \otimes \boldsymbol{Z}_{K_{\text{odd}}}, 4) \times \prod_{i \geqq 0} K(\boldsymbol{Z}/2 \otimes \Lambda, 4i+2) \times \prod_{i \geqq 1} K(\bar{W}(\Lambda) \otimes \boldsymbol{Z}_{K_{\text{odd}}}, 4i)$$

is a homotopy equivalence.

5. **The homotopy type of $H(K)/\widetilde{\mathrm{PL}}$.** In this section, we use the results of § 4 to compute the homotopy type of the classifying space $H(K)/\widetilde{\mathrm{PL}}$ of H_K-reductions of PL-block bundles. Let $(BS\widetilde{\mathrm{PL}})^{(K)}$ denote the fiber of $BS\widetilde{\mathrm{PL}} \to (BS\widetilde{\mathrm{PL}})_K$.

THEOREM 5.1. $H(K)/\widetilde{\mathrm{PL}} \cong (BS\widetilde{\mathrm{PL}})^{(K)} \times \prod_{i>0} K(\psi_i^K \otimes \Lambda, i)$.

Proof. We first compute $(H(K)/\widetilde{\mathrm{PL}})_K$. By [24] and Theorem 4.8, $\Omega^5(SG/S\widetilde{\mathrm{PL}})_{K_{\text{odd}}}$ and $\Omega^5(SG(K)/SH(K))_{K_{\text{odd}}}$ have no nonzero k-invariants, so all k-invariants of $\Omega^5(SH(K)/S\widetilde{\mathrm{PL}})_{K_{\text{odd}}} = \Omega^5(H(K)/\widetilde{\mathrm{PL}})_{K_{\text{odd}}} = \Omega^1(H(K)/\widetilde{\mathrm{PL}})_K$ vanish. Since the first (possibly) nonzero homotopy group of $H(K)/\widetilde{\mathrm{PL}}$ occurs in dimension 3 and the next in dimension 7, $(H(K)/\widetilde{\mathrm{PL}})_K \cong \prod_{i>0} K(\psi_i^K \otimes \Lambda, i)$.

For the localization at K, consider the diagram

$$\begin{array}{ccc} (SG(K)/S\widetilde{\mathrm{PL}})_{(K)} & \longrightarrow & (SG(K)/SH(K))_{(K)} \\ \phi \big\downarrow & & \big\downarrow \psi \\ (BS\widetilde{\mathrm{PL}})_{(K)} & \xrightarrow[(c,p)]{} & K(\psi_3^K \otimes \boldsymbol{Q}, 4) \times \prod_{i>0} K(\boldsymbol{Q}, 4i) \end{array}$$

where: ϕ is induced from $SG(K)/S\widetilde{\mathrm{PL}} \to BS\widetilde{\mathrm{PL}} \to BSG(K)$, ψ is the $\boldsymbol{Q}$-localization, c is the constant map and p is the Pontrjagin class.

Since $SG(K)/SH(K)$ is K-local, ψ is a homotopy equivalence by Theorem 4.8, and the diagram commutes up to homotopy by the proof of that theorem. We have $BSG(K)_{(K)} \cong *$, so that ϕ is a homotopy equivalence. Therefore,

$$(H(K)/\widetilde{\mathrm{PL}})_{(K)} \simeq \text{fiber of } (c, p)$$
$$\simeq (BS\widetilde{\mathrm{PL}})^{(K)} \times K(\psi_3^K \otimes \boldsymbol{Q}, 3)$$

since p is the $\boldsymbol{Q}$-localization, and $(BS\widetilde{\mathrm{PL}})^{(K)} \simeq$ fiber of $(BS\widetilde{\mathrm{PL}})_{(K)} \to (BS\widetilde{\mathrm{PL}})_{\boldsymbol{Q}}$.

Therefore we have a fiber diagram

$$\begin{array}{ccc} H(K)/\widetilde{\mathrm{PL}} & \longrightarrow & K(\psi_3^K \otimes \Lambda) \times \prod_{i=4}^{\infty} K(\psi_i^K \otimes \Lambda) \\ \downarrow & & \downarrow \\ K(\psi_3^K \otimes \boldsymbol{Q}, 3) \times (BS\widetilde{\mathrm{PL}})^{(K)} & \longrightarrow & K(\psi_3^K \otimes \boldsymbol{Q}, 3) \end{array}$$

and the result follows.

6. Application to Λ-homology cobordism bundles.

In this section, we use the results of §§4 and 5 to study the space $BH(K)$. We first commute its homotopy groups.

THEOREM 6.1. *The homotopy groups of $BH(K)$ are given as follows*:

$$\pi_i(BH(K)) \cong \begin{cases} \pi_i(B\widetilde{\mathrm{PL}}) \otimes \Lambda & i \not\equiv 0 \bmod(4) \\ \pi_i(B\widetilde{\mathrm{PL}}) \otimes \Lambda \oplus \operatorname{tor} \bar{W}(\Lambda) \otimes \Lambda & i = 4j,\ j > 1 \\ \Lambda \oplus \psi_3^K \otimes \Lambda & i = 4\ . \end{cases}$$

Proof. We have a homotopy commutative diagram

$$\begin{array}{ccccc} \Lambda^{\cdot+} \times (G/\widetilde{\mathrm{PL}})_K & \longrightarrow & B\widetilde{\mathrm{PL}}_K & \longrightarrow & BG(K) \\ \downarrow & & \downarrow & & \downarrow = \\ G(K)/H(K) & \longrightarrow & BH(K) & \longrightarrow & BG(K) \end{array}$$

of fibrations, which yields a commutative ladder

$$\begin{array}{ccccccccc} \cdots \to \pi_{i+1}(BG(K)) & \to & L_i(1) \otimes \Lambda & \to & \pi_i(B\widetilde{\mathrm{PL}}_K) & \to & \pi_i(BG(K)) & \to & L_{i-1}(1) \otimes \Lambda \cdots \\ \downarrow = & & \downarrow \phi_i & & \downarrow \psi_i & & \downarrow = & & \downarrow \phi_{i-1} \\ \cdots \to \pi_{i+1}(BG(K)) & \to & \pi_i(G(K)/H(K)) & \to & \pi_i(BH(K)) & \to & \pi_i(BG(K)) & \to & \pi_{i-1}(G(K)/H(K)) \to \cdots \end{array}$$

with exact rows, $i > 0$.

Case 1. $i \not\equiv 0 \bmod(4)$.

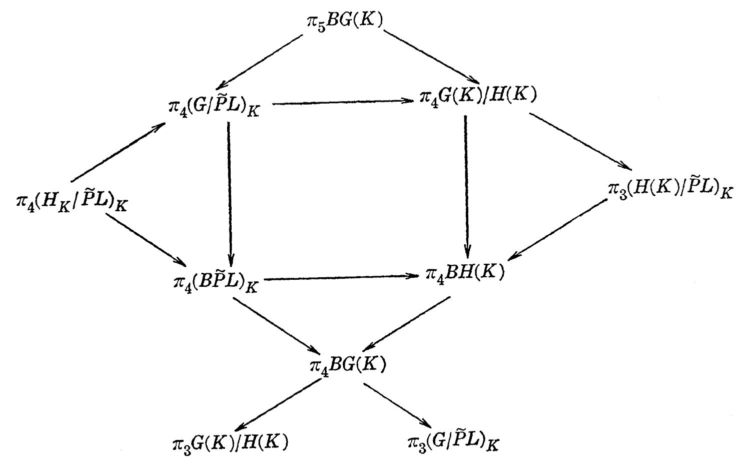
In this case, ϕ_i is an isomorphism by the results of § 4, and since ϕ_{4j}: $\bar{W}(\boldsymbol{Z}) \otimes \Lambda \to \bar{W}(\Lambda) \otimes \Lambda$, $j > 1$, ϕ_4: $\Lambda \to \Lambda \oplus \tilde{\psi}_3^K \otimes \Lambda$ are also injective, ψ_i is an isomorphism by the 5-lemma.

Case 2. $i = 4j, j > 1$.

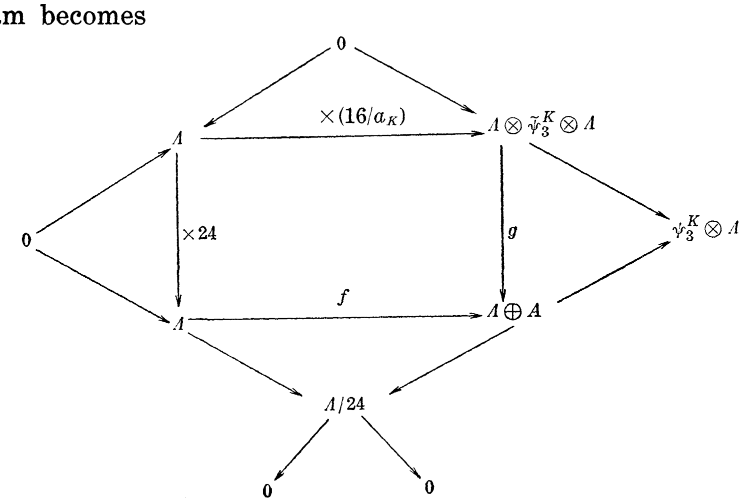
Again, ϕ_i is injective and ϕ_{i-1} is an isomorphism, so ψ_i is injective with $\operatorname{coker}(\psi_i) \cong \operatorname{coker}(\phi_i) \cong \bar{W}(\Lambda, \boldsymbol{Z}) \otimes \Lambda$. By Theorem 4.6, there is a map $K(\operatorname{tor} \bar{W}(\Lambda) \otimes \Lambda, i) \to G(K)/H(K) \to BH(K)$. This defines a section of $\pi_i(BH(K)) \to \bar{W}(\Lambda) \otimes \Lambda \subset \operatorname{tor}(\bar{W}(\Lambda, \boldsymbol{Z})) \otimes \Lambda$, and so $\operatorname{tor}(\bar{W}(\Lambda) \otimes \Lambda)$ is a direct summand of $\pi_i(BH(K))$. Since both $\pi_i(B\widetilde{\mathrm{PL}}) \otimes \Lambda$ and $\pi_i(BH(K))$ are rank 1 Λ-modules, $\pi_i(BH(K)) \cong \pi_i(B\widetilde{\mathrm{PL}}) \otimes \Lambda \oplus \operatorname{tor}(\bar{W}(\Lambda) \otimes \Lambda)$.

Case 3. $i = 4$:

Consider the following diagram



in which only $\pi_4 BH(K)$ is unknown. (Compare [21].) Since $\pi_4 BH(K)$ has rank at least 1, we may write $\pi_4 BH(K) \cong \Lambda \oplus A$, and the diagram becomes



Let $f|\Lambda, g|\Lambda: \Lambda \to \Lambda$ be multiplication by a, b respectfully. By commutativity of the square above, $24a = (16/a_K)b$ and by multiplication by a suitable unit in Λ, we may assume $a, b \in \boldsymbol{Z}_+$. We also have $a \cdot u \equiv 1 \bmod(24)$ for some $u \in \Lambda^\cdot$ and $b \cdot v$ divides 24 for some $v \in \Lambda^\cdot$. We consider 4 cases:

(1) $2, 3 \notin K$: Then we have $24a = (16/a_K)b$, b divides 24 and $a \equiv 1 \bmod(24)$. The first two of these imply that $b = 3, 6, 12$ or 24 and $a = 1, 2, 4$ or 8. Therefore $a = 1$ and $b = 3$ if $a_K = 2$, 6 if $a_K = 4$ and 12 if $a_K = 8$.

(2) $3 \in K, 2 \notin K$: We have $a_K = 2$, $24a = 8b$, bv divides 8 and $au \equiv 1 \bmod(8)$. As above $a = u^{-1}$, $b = 3u^{-1}$.

(3) $2 \in K, 3 \notin K$: As in (2), $a = u^{-1}$, $b = 3u^{-1}/2$.

(4) $2 \in K, 3 \in K$: In this case, $b \in \Lambda^\cdot$ and $a = 2b/3 \in \Lambda$.

In all cases, $a \in \Lambda^\cdot$, and a diagram chase shows that $A \cong \psi_3^K \otimes \Lambda$.

COROLLARY 6.2. *$BH(K)_2 \simeq K(\psi_3^K \otimes \Lambda, 4)_2 \times B\widetilde{\mathrm{PL}}_{K_2}$.*

Proof. Define $\phi: B\widetilde{\mathrm{PL}}_{K_2} \to BH(K)_2$ to be the natural map, and ψ to be the composition $K(\psi_3^K \otimes \Lambda, 4)_2 = K(\psi_3^K \otimes \Lambda, 4)_2 \to (G(K)/H(K))_2 \to BH(K)_2$, where the middle map is a splitting of the first factor of $(G(K)/H(K))_2$. Let $m: BH(K)_2 \times BH(K)_2 \to BH(K)_2$ be the H-multiplicatian induced by Whitney sum. Then $m \circ (\psi, \phi): K(\psi_3^K \otimes \Lambda, 4)_2 \times B\widetilde{\mathrm{PL}}_{K_2} \to BH(K)_2$ is a homotopy equivalence by the proof of the theorem.

For $2 \in K$, this gives the homotopy type of $BH(K)$.

COROLLARY 6.3. *If $2 \in K$, $BH(K) \simeq K(\psi_3^K \otimes \Lambda, 4) \times B\widetilde{\mathrm{PL}}_K$.*

Consider the diagram

$$(*) \qquad \begin{array}{ccc} B\widetilde{\mathrm{PL}}_K & \longrightarrow & B\widetilde{TOP}_K \\ \downarrow & & \\ BH(K) & & \end{array}$$

where all maps are the natural ones. Galewski and Stern [8] and Matumoto [14] have independently shown that when $K = \phi$, there is a map $BH(K) \to B\widetilde{TOP}_K$ making this diagram commute up to homotopy.

THEOREM 6.6. *If $K \neq \phi$ and $2 \notin K$, then there is no map $BH(K) \to B\widetilde{TOP}_K$ making (*) commute.*

Proof. Such a map induces a homotopy commutative diagram

$$\begin{array}{ccc} (G/\widetilde{\mathrm{PL}})_K & \longrightarrow & (G/\widetilde{\mathrm{TOP}})_K \\ \downarrow & \nearrow & \\ G(K)/H(K) & & \end{array}$$

and, applying π_4, we have

$$\begin{array}{ccc} \Lambda & \xrightarrow{\times 2} & \Lambda \\ {\scriptstyle \times(16a_K)}\downarrow & \nearrow{\scriptstyle f} & \\ \Lambda \oplus \psi_3^K \otimes \Lambda \, . & & \end{array}$$

The map f is then multiplication by $c \in \Lambda$ on Λ, and $(16/a_K)c = 2$. Therefore $2 \in K$ or $a_K = 8$, i.e., $K = \phi$.

Galewski and Stern [9] have shown that $BH \simeq B\widetilde{\mathrm{TOP}} \times K(\tilde{\psi}_3, 4)$ provided $\psi_3 \cong \boldsymbol{Z}/2 \oplus \tilde{\psi}_3$. Quinn [19] has conjectured a similar formula for $BH(K)$ in general, which we now show to be false if $2 \notin K$ and $K \neq \phi$.

THEOREM 6.5. *If $2 \notin K$ and $K \neq \phi$, then there is no homotopy equivalence*

$$BH(K) \longrightarrow B\widetilde{\mathrm{TOP}}_K \times K(\tilde{\psi}_3^K \otimes \Lambda, 4) \times \prod_{i>1} K(\operatorname{tor} \overline{W}(\Lambda) \otimes \Lambda, 4i) \, .$$

Proof. Assume ϕ is such a homotopy equivalence, and let $\phi'\colon BH(K) \to B\widetilde{\mathrm{TOP}}_K$ be ϕ followed by projection. By the proof of Theorem 6.1,

$$\begin{array}{ccc} B\mathrm{PL}_K & \longrightarrow & B\mathrm{TOP}_K \\ \downarrow & \nearrow{\scriptstyle \phi} & \\ BH(K) & & \end{array}$$

induces commutative diagrams after π_n is applied, and by the same argument as in Theorem 6.4, $2 \in K$ or $K = \phi$.

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Received March 20, 1978 and in revised form October 19, 1978.

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