PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 1, 1980

ARCHIMEDEAN LATTICE-ORDERED FIELDS THAT ARE ALGEBRAIC OVER THEIR o-SUBFIELDS

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Several properties of archimedean lattice-ordered fields which are algebraic over their o-subfield will be shown to be equivalent. Among these properties are the following: Two geometric descriptions of the positive cone. A sufficient condition for an intermediate field of the lattice-ordered field and its o-subfield to be lattice-ordered. A description of the additive structure of the lattice-ordered field. Two statements on the extendibility of lattice orders to total orders. A statement on the extendibility of a given lattice order to a lattice order on a real closure.

Introduction. It has been shown in [4], Kap. 2 that each archimedean l-field (=lattice-ordered field) K with positive cone P_K has a largest subfield L which admits a total order P_L with $P_L \cdot P_K \subseteq P_K$. L is called the o-subfield of K. In this paper archimedean l-fields that are algebraic over their o-subfield will be investigated. In §1 several geometric and structural properties of l-fields are considered. §2 contains a discussion of the extendibility of lattice orders to total orders. Finally, in §3 it is shown how Wilson's construction of lattice orders on the real field in [5] can be used to construct lattice orders on extension fields of l-fields.

All the proofs in this paper are based on the following representation of l-fields by continuous functions: By Hölder's theorem the archimedean totally ordered o-subfield L of the l-field K is isomorphic to a unique subfield of the reals. Identify L with this subfield. Since K is algebraic over L, the set E_K of embeddings of K over L into $\boldsymbol{C}$ can be topologized via infinite Galois theory to become a Boolean space. Let $C(E_K)$ be the Banach algebra of continuous functions of E_K into $\boldsymbol{C}$ with the norm given by $N(f) = \max(|f(\alpha)|; \alpha \in E_K)$. Define $\phi_K(x) = (\alpha(x))_{\alpha \in E_K}$ for all $x \in K$. Then ϕ_K embeds K into $C(E_K)$ by infinite Galois theory.

After defining $e_\alpha: C(E_K) \to \boldsymbol{C}$ to be the evaluation map at $\alpha \in E_K$ and $\bar{S}$ to be the closure of the subset S of a topological space, the main results of this paper can be summarized in the following

MAIN THEOREM. *For the archimedean l-field K which is algebraic over its o-subfield L, these are equivalent*:

(1) *There is an* $\alpha \in E_K$ *with* $\alpha(K) \subseteq \boldsymbol{R}$ *such that* $\overline{\phi_K(P_K)} \cap e_\alpha^{-1}(0) = 0$.

(2) *There is an* $\alpha \in E_K$ *with* $\alpha(K) \subseteq \boldsymbol{R}$ *such that* $\overline{\phi_K(P_K)} \cap e_\alpha^{-1}(1) \subseteq N^{-1}(1)$, i.e., $e_\alpha(x) = N(x)$ *for all* $x \in \overline{\phi_K(P_K)}$.

(3) $N\phi_K: P_K \to \boldsymbol{R}$ *is a semiring homomorphism.*

(4) $N\phi_K: P_K \to \boldsymbol{R}$ *is a semigroup homomorphism* wrt *addition.*

(5) *For all* $x \in P_K = P_K \backslash \{0\}$, $L(x)$ *is a convex l-subfield of* K.

(6) K *is equal to its own basis subgroup.*

(7) $\overline{\phi_K(P_K)}$ *is a partial order, and the quotient order* $Q(P_K)$ *of* P_K *is a total order.*

(8) $\overline{\phi_K(P_K)}$ *is a partial order, and* P_K *is uniquely extendible to a total order.*

(9) *There exists a real closure* R *of* K *such that the lattice order* P_K *of* K *can be extended to a lattice order* P_R *of* R *with* $\overline{\phi_R(P_R)}$ *a partial order.*

(Note that $\overline{\phi_K(P_K)}$ is always a pre-order on the algebra $C(E_K)$.)

The geometric property (2) of the positive cone has been considered by Wilson in [6] (see also [4], Satz 5.1). Conditions (1), (3), and (4) are evidently closely related to (2). (5)-(8) were investigated in [4] for finite field extensions $L \subseteq K$. (9) has been inspired by Wilson's construction of a nontrivial lattice order on the reals ([5]). Since (2) and (5)-(8) have been shown to hold for finite extensions $L \subseteq K$, the theorem essentially states that those conditions which are true for finite extensions turn out to be equivalent for algebraic extensions.

Quite naturally, the question arises whether (1)-(9) do not, in fact, hold for arbitrary algebraic extensions. I have been unable to determine the answer to this problem.

The terminology concerning l-groups is that of [2].

1. Geometric and structural properties. This section is devoted entirely to the proof of the equivalence of conditions (1)-(6) of the Main Theorem. The proof will be arranged to reflect the obvious close relationship of the members of each of the pairs (1) and (2), (3) and (4), (5) and (6).

Proof of (1) $\leftrightarrow$ (2). (1) $\to$ (2): Suppose that (1) holds for $\alpha \in E_K$. Then it can be readily verified that $e_\alpha(\overline{\phi_K(P_K)}) \subseteq \boldsymbol{R}^+$. Now assume (by way of contradiction) that (2) does not hold for α, i.e., there is some $\beta \in E_K$, $\beta \neq \alpha$, and some $a \in \overline{\phi_K(P_K)}$ such that $0 \leqq e_\alpha(a) < |e_\beta(a)| = N(a)$. Since a is not contained in the closed set $C = \{x \in C(E_K); N(x) = |e_\alpha(x)|\}$, there exists a neighborhood U of a with $U \cap C = \phi$. Therefore there is some $b \in \phi_K(P_K)$ with $b \notin C$. By finiteness of $b(E_K)$, there is a partition $E_K = E_1 \cup \cdots \cup E_n$ such that b is constant on each E_i. Obviously, $b/N(b)$ lies in the unit ball of the finite dimensional subspace

$C(b) = \{f \in C(E_K); f$ is constant on all $E_i\}$ of $C(E_K)$. The sequence $(b/N(b))^n$, $n \in N$ has a convergent subsequence with limit $d \in \overline{\phi_K(P_K)}$ by compactness of the unit ball in $C(b)$. From $|e_\alpha(b/N(b))| < 1$ and $|e_\beta(b/N(b))| = 1$ it follows that $e_\alpha(d) = 0$ and $|e_\beta(d)| = 1$. In particular, $d \neq 0$. But then $e_\alpha(d) = 0$ contradicts (1).

(2) $\rightarrow$ (1): Suppose that (2) holds for $\alpha \in E_K$. Then $e_\alpha(x) = N(x)$ for all $x \in \overline{\phi_K(P_K)}$. Hence, $e_\alpha(x) = 0$ implies $x = 0$, i.e., (1) holds for α.

The following corollaries follow immediately from the proof of (1) $\leftrightarrow$ (2).

COROLLARY 1. *Properties* (1) *and* (2) *hold for exactly the same* $\alpha \in E_K$.

COROLLARY 2. (1) *and* (2) *hold for at most one* $\alpha \in E_K$.

Proof. If (2) holds for $\alpha, \beta \in E_K$, then $\alpha(x) = N\phi_K(x) = \beta(x)$ for all $x \in P_K$. Now $K = P_K - P_K$ implies $\alpha = \beta$.

Proof of (1), (2) $\leftrightarrow$ (3) $\leftrightarrow$ (4). (2) $\rightarrow$ (4): Suppose that (2) holds for $\alpha \in E_K$. Then $N\phi_K(x) = \alpha(x)$ for all $x \in P_K$. Clearly, the restriction of α to P_K is a semigroup homomorphism wrt addition.

(4) $\rightarrow$ (3): It must only be shown that the restriction of $N\phi_K$ to $P_K\backslash\{0\}$ is a multiplicative homomorphism. Pick $x, y \in P_K\backslash\{0\}$. There is some $\alpha \in E_K$ such that $N\phi_K(x+y) = |\alpha(x+y)|$. Since $N\phi_K$ is an additive homomorphism, an elementary computation shows that $N\phi_K(x) = |\alpha(x)|$, $N\phi_K(y) = |\alpha(y)|$. But then it follows immediately that $N\phi_K(xy) = |\alpha(xy)| = |\alpha(x)|\,|\alpha(y)| = N\phi_K(x)N\phi_K(y)$.

(3) $\rightarrow$ (2): Define a map $\alpha: K \rightarrow C$ by $x \mapsto N\phi_K(x^+) - N\phi_K(x^-)$. Since P_K generates K and since $N\phi_K$ is a semiring homomorphism, α is a field homomorphism into the reals. Obviously, $\alpha | L$ is the identity. Thus $\alpha \in E_K$, and $\alpha = e_\alpha\phi_K$. Therefore $e_\alpha\phi_K = \alpha = N\phi_K$ on P_K, or equivalently, e_α and N agree on $\phi_K(P_K)$. By continuity they also agree on $\overline{\phi_K(P_K)}$, i.e., (2) holds for α.

For the next step in the proof of the Main Toeorem it is useful to recall the following fact from [4] (part (i), proof of Satz 5.3):

LEMMA 1. *If the archimedean l-field K which is algebraic over its o-subfield has property* (2), *then the l-ideal* $I(x)$ *generated by* $x \in P_K\backslash\{0\}$ *is contained in* $L(x)$.

Proof of (1)-(4) $\leftrightarrow$ (5) $\leftrightarrow$ (6). (2) $\rightarrow$ (5): For all $x \in P_K\backslash\{0\}$ and all

$a \in P_K\backslash\{0\} \cap L(x)$, $I(a) \subseteqq L(x)$ by Lemma 1. Since $L(x)$ has a strong order unit, this implies that $L(x)$ is an l-ideal of K, hence a convex l-subfield.

(5) $\rightarrow$ (6): Since any disjoint subset of K is linearly independent over L ([4], Lemma 3.1), it follows from (5) that K has property (F) of [1]. Now K is equal to its basis subgroup by Theorem 7.3 of [1], since K is archimedean.

(6) $\rightarrow$ (5): For any $x \in P_K\backslash\{0\}$, the partially ordered field $L(x) \subseteqq K$ has a strong order unit u. Thus, $L(x)$ is contained in the l-ideal $I(u)$ generated by u. Since K is its own basis subgroup by hypothesis and since the maximal o-subgroups of K are one-dimensional over L ([4], Satz 2.3), $I(u)$ is of finite dimension over L. Moreover, an easy computation shows that $I(u)$ is multiplicatively closed. Thus, $I(u)$ is a convex l-subfield of K which is of finite dimension over its o-subfield. By [4], Satz 5.3, $L(x)$ is a convex l-subfield of $I(u)$, hence also of K.

(5) $\rightarrow$ (3): For any $x \in P_K\backslash\{0\}$, $L(x)$ is an l-field which is finite over its o-subfield L. By [6] (see also [4], Satz 5.1), $L(x)$ has property (2), hence also property (3). Clearly, this implies that $N\phi_K\colon P_K = \bigcup_{x \in P_K\backslash\{0\}} P_{L(x)} \rightarrow \boldsymbol{R}$ is a semiring homomorphism.

2. Extendibility of lattice orders to total orders.

In the next step of the Main Theorem, conditions (7) and (8) will be dealt with. The implications (1)-(6) $\rightarrow$ (7) and (1)-(6) $\rightarrow$ (8) are contained in the following corollaries, which are immediate consequences of the considerations in the preceding section.

COROLLARY 3. *If the l-field K has properties* (1)-(6), *then* $\overline{\phi_K(P_K)}$ *is a partial order.*

Proof. Let $\alpha \in E_K$ be the unique element for which (1) holds. From $-\overline{\phi_K(P_K)} \subseteqq e_\alpha^{-1}(\boldsymbol{R}^-)$ and $\overline{\phi_K(P_K)} \subseteqq e_\alpha^{-1}(\boldsymbol{R}^+)$ it follows that $-\overline{\phi_K(P_K)} \cap \overline{\phi_K(P_K)} \subseteqq e_\alpha^{-1}(0) \cap \overline{\phi_K(P_K)} = 0$, whence the pre-order $\overline{\phi_K(P_K)}$ is a partial order.

COROLLARY 4. *If the l-field K has properties* (1)-(6), *then the quotient order* $Q(P_K)$ *of* P_K *is a total order.*

Proof. By [4], Satz 4.14, for all $x \in P_K\backslash\{0\}$ the quotient order $Q(P_{L(x)})$ on the convex l-subfield $L(x)$ of K is total. Thus, $Q(P_K) = \bigcup_{x \in P_K\backslash\{0\}} Q(P_{L(x)})$ is a total order for K.

COROLLARY 5. *If the l-field K has properties* (1)-(6), *then the lattice order of K is uniquely extendible to a total order.*

Clearly, Corollary 5 follows from the totality of the quotient order without any assumptions on the validity of properties (1)-(6). Next it will be shown that the converse of this is also true, i.e., the quotient order is total if the lattice order has a unique extension to a total order. In fact, the following result is even stronger than that.

THEOREM 1. *For the archimedean l-field K which is algebraic over its o-subfield, the quotient order $Q(P_K)$ is an intersection of total orders.*

Proof. Clearly, $Q(P_K)$ induces the natural total order on the o-subfield L, i.e., the unique total order which makes K a partially ordered L-vector space. With this total order L is archimedean, or (AS) in the terminology of [3]. Since K is an algebraic extension field of L and since $K = Q(P_K) - Q(P_K)$ is clearly primitive ([3], p. 919), Theorem 3.1 of [3] shows that K is an (AS)-field. Now, by Theorem 3.2 of [3], $Q(P_K)$ is an intersection of total orders on K.

COROLLARY 6. *The lattice order of the l-field K can be extended to a total order.*

This immediate corollary of Theorem 1 is exactly Satz 4.10 of [4]. Thus, an alternative proof (which is much shorter than the original one) of this important theorem has been found. The next corollary essentially states that (7) and (8) are equivalent:

COROLLARY 7. *The lattice order of K is uniquely extendible to a total order* iff *the quotient order is a total order.*

To establish the equivalence of (1)-(8) of the Main Theorem, it only remains to be shown that (1)-(6) follow from (7) and (8). This proof requires the following two rather technical lemmas.

LEMMA 2. *Suppose that K is an archimedean l-field which is algebraic over its o-subfield. If $\overline{\phi_K(P_K)}$ is a partial order, then for all $a \in P_K \backslash \{0\}$ there is some $\alpha \in E_K$ such that $\alpha(a) = N\phi_K(a)$.*

Proof. Pick $a \in P_K$, $a \neq 0$. Then $\phi_K(a)(E_K)$ is finite. Thereby E_K is partitioned into finitely many subsets E_i, $i \in I$, on each of which $\phi_K(a)$ is constant. $C(a) = \{x \in C(E_K);\ x \text{ is constant on } E_i,\ i \in I\}$ is a finite dimensional subalgebra of $C(E_K)$, which is canonically isomorphic to $\boldsymbol{C}^I$ by $\varphi\colon C(a) \to \boldsymbol{C}^I\colon x \mapsto (x(E_i))_{i \in I}$. Since $\phi_K(L(a)) \subseteq C(a)$ by infinite Galois theory, the restriction ψ of $\varphi\phi_K$ to $L(a)$ is well-defined and embeds $L(a)$ into C^I. Let $\pi_i: C^I \to C$ be the canonical projections, and write $\psi_i = \pi_i \psi$.

Define $b_0 = \psi(a)/N\phi_K(a)$. Then every $i \in I$ is in exactly one of the following subsets of I:

$$\begin{aligned}
I_1 &= \{i \in I;\ \pi_i(b_0) = 1\}\ ,\\
I_2 &= \{i \in I;\ \pi_i(b_0) \neq 1 \text{ is a root of unity}\}\ ,\\
I_3 &= \{i \in I;\ |\pi_i(b_0)| = 1, \pi_i(b_0) \text{ is no root of unity}\}\ ,\\
I_4 &= \{i \in I;\ |\pi_i(b_0)| < 1\}\ .
\end{aligned}$$

In this notation, the claim of the lemma is that I_1 is nonempty.

Obviously, $I_1 \cup I_2 \cup I_3 \neq \phi$ by the definition of N. One can easily define a sequence in $\overline{\psi(P_{L(a)})}$ with components 1 on $I_1 \cup I_2$, limit 1 on I_3, and limit 0 on I_4. The limit d_1 of this sequence is in $\overline{\psi(P_{L(a)})}$ by closure.

Next it will be shown that $I_1 \cup I_2 \neq \phi$. Clearly, this is true if $I_3 = \phi$. Therefore suppose that $I_3 \neq \phi$. Now consider $b_1 = b_0 d_1$. The support of b_1 is contained in $J_1 = I_1 \cup I_2 \cup I_3$. Again, define a sequence in $\overline{\psi(P_{L(a)})}$ with components 1 on $I_1 \cup I_2$, limit -1 on a nonempty subset $J_2' \subseteqq I_3$, and limit 1 on $I_3 \backslash J_2'$. Again by closure, the limit d_2 of this sequence is contained in $\overline{\psi(P_{L(a)})}$. Also, $b_2 = (b_1 + b_1 d_2)/2 \in \overline{\psi(P_{L(a)})}$. The support of b_2 is contained in $J_2 = J_1 \backslash J_2' \not\supseteqq J_1$. By iteration, there is a smallest $r \in N$ such that the support of $b_r = (b_{r-1} + b_{r-1} d_r)/2$ is contained in $I_1 \cup I_2$. Now assume (by way of contradiction) that $I_1 \cup I_2 = \phi$. But then $0 = b_r = b_{r-1}/2 + b_{r-1} d_r/2$ is a nontrivial representation of 0 in $\overline{\psi(P_{L(a)})}$. This is a contradiction, since $\overline{\psi(P_{L(a)})}$ is a partial order, being the isomorphic image of $\overline{\phi_K(P_{L(a)})} \subseteqq \overline{\phi_K(P_K)}$.

The final step of this proof is to show that $I_1 \neq \phi$. This is obviously true if $I_2 = \phi$. If $I_2 \neq \phi$, let $k \in N$ be a common multiple of the exponents of all the roots of unity involved. Define $c = \sum_{l=1}^{k} b_r^l$. Then the components of c are k on I_1, 0 elsewhere. If I_1 were empty, $0 = c = \sum_{l=1}^{k} b_r^l$ would again be a nontrivial representation of 0 in the partial order $\overline{\psi(P_{L(a)})}$: contradiction. Thus, $I_1 \neq \phi$ as claimed.

LEMMA 3. *Suppose that $\overline{\phi_K(P_K)}$ is a partial order for the archimedean l-field K which is algebraic over its o-subfield L. Then* $L(a) = L(a + a^2)$, *and* $\{\alpha \in E_K;\ |\alpha(a + a^2)| = N\phi_K(a + a^2)\} = \{\alpha \in E_K;\ \alpha(a + a^2) = N\phi_K(a + a^2)\} = \{\alpha \in E_K;\ \alpha(a) = \phi_K(a)\}$.

Proof. This is an easy computation.

The next lemma shows how the subfields $L(a)$, $a \in P_K \backslash \{0\}$ can be embedded order preservingly into the reals. In the proof of (7), (8) $\to$ (1)-(6) these embeddings will be put together to give an embedding of K into the reals.

LEMMA 4. *Suppose that $\overline{\phi_K(P_K)}$ is a partial order for the archimedean l-field K which is algebraic over its o-subfield L. Then $\alpha: L(a) \to \boldsymbol{R}$ is order preserving for all $a \in P_K \backslash \{0\}$ and all α such that $\alpha(a) = N\phi_K(a)$.*

Proof. $\alpha(a) = N\phi_K(a)$ clearly shows that α embeds $L(a)$ into $\boldsymbol{R}$. So it is only left to show that α is order preserving: Assume (without loss of generality by Lemma 3) that $\{\alpha \in E_K;\ \alpha(a) = N\phi_K(a)\} = \{\alpha \in E_K;\ |\alpha(a)| = N\phi_K(a)\}$. Now, suppose (by way of contradiction) that there is some $x \in P_{L(a)}$ for which $\alpha(x) < 0$. Note that both $\phi_K(a)(E_K)$ and $\phi_K(x)(E_K)$ are finite and that $\phi_K(x)$ is constant on those subsets of E_K on which $\phi_K(a)$ is constant. By $|\alpha(a)/\beta(a)| > 1$ for all $\beta \in E_K$ with $\beta(a) \neq N\phi_K(a)$, there is some $n \in \boldsymbol{N}$ such that $|\alpha(a)/\beta(a)|^n > |\beta(x)/\alpha(x)|$ for all $\beta \in E_K$, $\beta(a) \neq N\phi_K(a)$. From this it follows that $\{\alpha \in E_K;\ |\alpha(a^n x)| = N\phi_K(a^n x)\} = \{\alpha \in E_K;\ \alpha(a) = N\phi_K(a)\}$. But then $\{\alpha \in E_K;\ \alpha(a^n x) = N\phi_K(a^n x)\} = \phi$ by $\alpha(a^n x) = \alpha(a)^n \alpha(x) = N\phi_K(a)^n \alpha(x) < 0$. This contradicts Lemma 2.

Proof of (1)-(6) $\leftrightarrow$ (7) $\leftrightarrow$ (8). (8) $\to$ (2): (By Corollaries 3, 4, 5, 7 this completes the proof.) P_K is uniquely extendible to a total order by hypothesis. Thus, by Hölder's theorem, there is a unique order preserving $\alpha \in E_K$. Assume (by way of contradiction) that (2) does not hold for this α, i.e., there is some $a \in P_K$ such that $N\phi_K(a) > \alpha(a)$. Then $\alpha \notin \{\beta \in E_K;\ \beta(a) = N\phi_K(a)\}$. Again, one can assume without loss of generality (by Lemma 3) that $\{\beta \in E_K;\ \beta(a) = N\phi_K(a)\} = \{\beta \in E_K;\ |\beta(a)| = N\phi_K(a)\}$. By Lemma 2, this is a nonempty, compact subset of E_K. Now define for all $b \in P_K \backslash \{0\}$, $E(b) = \{\beta \in E_K;\ \beta(a) = N\phi_K(a)$, $\beta: L(a, b) \to \boldsymbol{R}$ preserves the order$\}$. For $b \in P_K \backslash \{0\}$ pick $c \in P_K$ such that $L(a, b) = L(c)$ (this is possible by part (i) of the proof of Satz 4.10 in [4]). As in the proof of Lemma 4, one sees that there is some $n \in \boldsymbol{N}$ with $L(c) = L(a^n c)$ and $\phi \neq \{\beta \in E_K;\ \beta(a^n c) = N\phi_K(a^n c)\} \subseteqq \{\beta \in E_K;\ \beta(a) = N\phi_K(a)\}$. Now it follows from Lemma 4 that $\{\beta \in E_K;\ \beta(a^n c) = N\phi_K(a^n c)\} \subseteqq E(b)$. In particular, $E(b) \neq \phi$.

So far, it has been shown that each $E(b)$ is a nonempty, closed subset of $\{\beta \in E_K;\ \beta(a) = N\phi_K(a)\}$. Now, if $b_1, \cdots, b_k \in P_K \backslash \{0\}$, then there is some $c \in P_K$ such that $L(a, b_1, \cdots, b_k) = L(a, c)$ ([4], Satz 4.10, part (i) of the proof). Therefore $E(c) \subseteqq \bigcap_{i=1}^k E(b_i)$, and the intersection is nonempty. But then also $\bigcap_{b \in P_K \backslash 0} E(b) \neq \phi$ by compactness of $\{\beta \in E_K;\ \beta(a) = N\phi_K(a)\}$. This gives a contradiction to the uniqueness of α, since any $\gamma \in \bigcap_{b \in P_K \backslash \{0\}} E(b)$ is another order preserving embedding of K into $\boldsymbol{R}$.

For a certain class of fields, in particular for the reals, the following stronger version of the equivalence of (1)-(8) holds:

COROLLARY 8. *If the total order of the o-subfield admits only a unique extension to a total order of the l-field K, then* (1)-(6) *are equivalent to:* $\overline{\phi_K(P_K)}$ *is a partial order. Moreover, the quotient order is always total.*

3. **Extendibility of lattice orders to over-fields.** The direction (9) $\rightarrow$ (1)-(8) of the last remaining equivalence has already been established by Corollary 8. The other direction of this equivalence is a corollary of the next result, which will be stated after a few notations have been introduced: For the l-field K the quotient order is an intersection of total orders by Theorem 1. Identify this set of total orders with the subset $T_K \subseteqq E_K$ of order preserving embeddings of K into the reals. By infinite Galois theory, T_K is a compact subspace of E_K. If $K \subseteqq M$ is an algebraic field extension, let $T_M \subseteqq E_M$ be those embeddings of M into the reals which extend some $\alpha \in T_K$. Let $\varphi: T_M \rightarrow T_K$ be the restriction of the canonical surjection $E_M \rightarrow E_K$.

THEOREM 2. *Suppope that the l-field K is archimedean and algebraic over its o-subfield L. Let* $K \subseteqq M$ *be an algebraic field extension. Then the following are equivalent*:

(a) *There exists a basis B for M over K such that* $P_K(B) = \{\sum_{b \in B} \alpha_b b;\ \alpha_b \in P_K,\ \alpha_b = 0$ *for almost all* $b\}$ *is a lattice order on the field M.*

(b) φ *is surjective, and has a continuous section* $\sigma: T_K \rightarrow T_M$.

REMARK. Condition (a) evidently means that Wilson's construction of a nontrivial lattice order on the reals in [5] is applicable to obtain a lattice order on M. Therefore Wilson's results will be used extensively in the proof of (b) $\rightarrow$ (a).

Proof. (a) $\rightarrow$ (b): For each total order T on K extending P_K, $T(B) = \{\sum_{b \in B} \alpha_b b;\ \alpha_b = 0$ for almost all b, $\alpha_b \in T\}$ is a lattice order on the field M. The quotient order of $(M, T(B))$ is total, since (6) obviously holds for $(M, T(B))$. This shows that T is extendible to a total order on M, i.e., φ is surjective. This argument even shows a bit more: Let $T'_M \subseteqq T_M$ be the order preserving embeddings of $(M, P_K(B))$ into $\boldsymbol{R}$, φ' the restriction of φ to T'_M. Then φ' is surjective. But φ' is also injective. For, let $\alpha, \beta \in T'_M$ be such that $\alpha | K =$

$\beta|K$. If T is the total order on K corresponding to $\alpha|K = \beta|K$. then $T(B)$ is contained in each of the total orders of M corresponding to α and β. But since the quotient order of $T(B)$ is total, the total orders corresponding to α and β must be equal, hence $\alpha = \beta$. Thus φ' is a continuous bijection of compact spaces, hence a homeomorphism. Define $\sigma = \varphi'^{-1}$.

(b) $\to$ (a): Define $\underline{N} = \{(N, B_N); K \subseteqq N \subseteqq M$ intermediate field, B_N basis of N over K such that $P_K(B_N)$ is a lattice order on N, the quotient order of $P_K(B_N)$ is the intersection of the restrictions of the elements of $\sigma(T_K)\}$. If $\underline{N}$ is partially ordered by $(N, B_N) \leqq (L, B_L)$ if $N \subseteqq L$, $B_N \subseteqq B_L$, then there exists a maximal $(N, B_N) \in \underline{N}$. Assume (by way of contradiction) that $N \neq M$. Pick $a \in M \backslash N$, and define $L = N(a)$.

By identifying T_K and $\sigma(T_K)$ and by infinite Galois theory, K, N, and L can be represented as subfields of the real Banach algebra $R(T_K) = C_R(T_K)$. Note that each of these subfields contains the rationals and separates points, so that they are dense in $R(T_K)$ by the Stone-Weierstraß theorem. Since the quotient order of $P_K(B_N)$ is the intersection of the total orders of N belonging to the elements of $\sigma(T_K)$, $x \in N$ is positive in the quotient order iff its image $\psi(x) \in R(T_K)$ is positive (wrt the pointwise order).

Now the decisive step in this proof is to find a primitive element b for the field extension $N \subseteqq L$ such that b's minimal polynomial is $X^n + b_{n-1}X^{n-1} + \cdots + b_0$ with the $b_i < 0$ in the quotient order of $P_K(B_N)$ and such that $\psi(b) > 0$. This is almost exactly Step 1 of the proof of the Main Lemma in [5]. The essential difference is that in this case the inequalities $b_i < 0$, $b > 0$ have to be established not only one total order at a time, but for all the total orders of $\sigma(T_K)$ simultaneously.

For $t, s \in N$ let $T(t, s)$ be the composition of the following (partially defined) transformations of $R(T_K)$:

$$x \longmapsto x/t\ ; \quad x \longmapsto x - s\ ; \quad x \longmapsto 1/x\ ; \quad x \longmapsto x - 1/n\ .$$

If $X^n + a_{n-1}(t, s)X^n + \cdots + a_0(t, s)$ is the minimal polynomial for $T(t, s)(a)$ over N, a mapping $\tau\colon N^* \times N^* \mapsto R(T_K)^{n+1}$ is defined by $\tau(t, s) = (T(t, s)(a), a_0(t, s), \cdots, a_{n-1}(t, s))$. τ is continuous wrt the topology induced by $R(T_K)$ on N^*.

Using [5], proof of the Main Lemma, Step 1, for each $\alpha \in T_K$. rationals t_α, s_α can be found such that $\alpha(T(t_\alpha, s_\alpha)(a)) > 0$ and $\alpha(a_i(t_\alpha, s_\alpha)) < 0$. The sets $T_K(\alpha) = \{\beta \in T_K;\ \beta(T(t_\alpha, s_\alpha)(a)) > 0$, $\beta(a_i(t_\alpha, s_\alpha)) < 0\} = \{\beta \in T_K;\ \beta(T(t_\alpha, s_\alpha)(a)) \geqq 0$, $\beta(a_i(t_\alpha, s_\alpha)) \leqq 0\}$ are clopen subsets of T_K by continuity of τ. Since T_K is compact, this leads to a finite partition $C_1, \cdots, C_m$ of T_K into clopen subsets such that each C_j is contained in some $T_K(\alpha)$. For each $j = 1, \cdots, m$, choose α_j such that $C_j \subseteqq T_K(\alpha_j)$. Now define $t', s' \in R(T_K)$ by: $t'(\alpha) = t_{\alpha_j}$ and $s'(\alpha) = s_{\alpha_j}$ for $\alpha \in C_j$, $j = 1, \cdots, m$. By denseness of $N \subseteqq R(T_K)$, and by continuity of the evaluation maps $R(T_K) \to \boldsymbol{R}$ at the points of T_K, there exist neighborhoods U of t', V of s' in $R(T_K)$ such that for all $t \in N^* \cap U$ and all $s \in N^* \cap V$ the desired inequalities hold: $\alpha(T(t, s)(a)) > 0$, $\alpha(a_i(t, s)) < 0$ for all $\alpha \in T_K$. Thus, $b = T(t, s)(a)$ is the desired primitive element of L over N. Next, an application of [5], proof of the Main Theorem, Step 2 leads to a primitive element c for L over N such that $c_0, \cdots, c_{n-1} < 0$ in the lattice order of N, where $X^n + c_{n-1}X^{n-1} + \cdots + c_0$ is the minimal polynomial of c over N, and such that $\alpha(c) > 0$ for all $\alpha \in T_K$. Define $B_L = B_N \cdot \{1, c, \cdots, c^{n-1}\}$. This is clearly a basis of L over K. By construction of this basis, $P_K(B_L) = P_K(B_N)(\{1, c, \cdots, c^{n-1}\})$ is a lattice order on L. Finally, each total order belonging to some element of $\sigma(T_K)$ extends $P_K(B_L)$. By the proof of (a) $\to$ (b), these are all the total orders extending $P_K(B_L)$. Therefore it follows from Theorem 1 that the quotient order of $P_K(B_L)$ is the intersection of the total orders belonging to the elements of $\sigma(T_K)$. Altogether, this shows that $(L, B_L) \in \underline{N}$, contradicting the maximality of N.

Proof of (1)-(8) $\leftrightarrow$ (9). The only remaining part of the proof is the implication (1)-(8) $\to$ (9). Because of (8), $|T_K| = 1$. Also, for any real closure R of K with the quotient order of P_K, there is only one total order on R extending the quotient order. Thus, (b) of Theorem 2 clearly holds. By (a) of Theorem 2, there is a basis B for R over K such that $P_K(B)$ is a lattice order for R. Since R with this lattice order is equal to its own basis subgroup, $\overline{\phi_R(P_R)}$ is a partial order.

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Received October 27, 1978 and in revised form June 22, 1979.

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