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SWEEDLER'S TWO-COCYCLES AND HOCHSCHILD COHOMOLOGY

DAVE RIFFELMACHER

For any algebra C over a commutative ring k Sweedler defined a cohomology set which generalizes Amitsur's second cohomology group $H^2(C/k)$. Any Sweedler C-two-cocycle σ gives rise to a change of rings functor $(\ )^\sigma$ from the category of C-bimodules to the category of C^σ-bimodules, where C^σ is the k-algebra with multiplication altered by σ, which in turn induces a map $\phi^n(\sigma, M)\colon H^n(C, M) \to H^n(C^\sigma, M^\sigma)$ on Hochschild cohomology for any C-bimodule M and any positive integer n. In this paper, several properties of $\phi^n(\sigma, M)$ are derived, including: If C is a finite dimensional algebra over a field k, $\phi^1(\sigma, M)$ is an injection for all σ and M.

1. Introduction. In §2 we establish our notation conventions and review the basic definitions of Sweedler's two-cocycles and Hochschild cohomology. We also recall the change of rings functor $(\ )^\sigma$ associated with a Sweedler C-two-cycle σ from the category of C-bimodules to the category of C^σ-bimodules for any algebra C over a commutative ring k.

The map $\phi^n(\sigma, M)$ induced by $(\ )^\sigma$ from the nth Hochschild cohomology group $H^n(C, M)$ of C with coefficients in the C-bimodule M to $H^n(C^\sigma, M^\sigma)$ is studied in §3. This map links the multiplicative cohomology of Sweedler and Amitsur to the additive cohomology of Hochschild. We provide an example to show that this map need not be surjective but show that if σ is invertible in an appropriate sense $\phi^n(\sigma, M)$ is actually an isomorphism. In particular, if σ is an invertible (i.e., Amitsur) two-cocycle contained in a commutative subalgebra A of C, then $\phi^n(\sigma, M)$ is an isomorphism. The behavior of $\phi^n(\sigma, M)$ under base extension of k and two-cocycles equivalent to σ is considered, and several other results are derived which are useful in studying $\ker \phi^n(\sigma, M)$.

In §4 we prove that if C is a finite dimensional algebra over a field k, $\phi^1(\sigma, M)$ is injective for all σ and M; that is, σ induces an injection of the group of equivalence classes of k-derivations of C with values in M into the group of equivalence classes of k-derivations of C^σ with values in M^σ. This result is compared with Flanigan's work on Gerstenhaber's deformation theory.

2. Notation and preliminaries. Let C be an algebra over a commutative ring k and let unadorned $\otimes$ and Hom represent $\otimes_k$ and Hom_k, respectively. Denote the n-fold tensor product $C \otimes \cdots \otimes C$

by $C^{\otimes n}$. We denote the opposite k-algebra of C by C^0 and call a left $C \otimes C^0$-module a C-bimodule. In this section we provide a brief review of the pertinent features of Sweedler's theory of two-cocycles and Hochschild's cohomology theory.

Following Sweedler [7], we call an element $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ in $C \otimes C \otimes C$ a C-two-cocycle if

$$\sum_{i,j} a_i a_j \otimes b_j \otimes c_j b_i \otimes c_i = \sum_{i,j} a_i \otimes b_i a_j \otimes b_j \otimes c_j c_i$$

and there is an element e_σ in C with

$$\sum_i a_i e_\sigma b_i \otimes c_i = 1 \otimes 1 = \sum_i a_i \otimes b_i e_\sigma c_i \ .$$

Given a C-two-cocycle σ we may form a k-algebra C^σ as follows. As an abelian group C^σ is equal to C. For any x in C, we use the notation x^σ to indicate that we are considering x as an element of C^σ. The product $*$ of any two elements x^σ and y^σ in C^σ is defined by

$$x^\sigma * y^\sigma = \left(\sum_i a_i x b_i y c_i\right)^\sigma \ .$$

The C-two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is said to be cohomologous to the C-two-cocycle $\tau = \sum_i r_i \otimes s_i \otimes t_i$ via $\delta = \sum_i x_i \otimes y_i$ in $C \otimes C$ if

$$\sum_{i,j} x_i a_j \otimes b_j \otimes c_j y_i = \sum_{i,j,l} r_i x_j \otimes y_j s_i x_l \otimes y_l t_i$$

and

$$\sum_i x_i e_\sigma y_i = e_\tau \ .$$

If the element $\sum_i x_i \otimes y_i^0$ in $C \otimes C^0$ is invertible, δ is called vertible, and σ and τ are said to be equivalent, denoted $\sigma \sim {}^\delta\tau$. In this case the k-algebra map $R^\delta \colon C^\sigma \to C^\tau$ defined by $R^\delta(c^\sigma) = (\sum_i x_i c y_i)^\tau$ is an isomorphism.

Definition 2.1. For any C-two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$, let the linear map $\phi(\sigma) \colon C^\sigma \otimes C^{\sigma 0} \to C \otimes C^0$ be defined by

$$x^\sigma \otimes y^{\sigma^0} \longrightarrow \sum_{i,j} a_i a_j x b_j \otimes (c_j b_i y c_i)^0 \ .$$

$\phi(\sigma)$ is a k-algebra map and hence induces a change of rings functor from the category $M(C)$ of C-bimodules to the category $M(C^\sigma)$ of C^σ-bimodules which we will denote by $(\ \ )^\sigma$. The interested reader is referred to [5, 6, 7] for more of the theory of Sweedler two-cocycles and the maps $(\ \ )^\sigma$.

Now we review the basic definition of Hochschild cohomology.

For any integer $n \geqq 1$ and any C-bimodule M define δ^n: Hom $(C^{\otimes n}, M) \to$ Hom $(C^{\otimes(n+1)}, M)$ by

$$\begin{aligned}(\delta f)(x_1 \otimes \cdots \otimes x_{n+1}) &= x_1 f(x_2 \otimes \cdots \otimes x_{n+1}) \\ &+ \sum_{0<i<n+1} (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\ &+ (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) x_{n+1} \,.\end{aligned}$$

The nth Hochschild cohomology group of C with values in M is then defined as $H^n(C, M) = \ker \delta^n / \operatorname{Im} \delta^{n-1}$. If C is a projective k-module, it may be shown [1, Chap. IX] that $H^n(C, M) = \operatorname{Ext}^n_{C \otimes C^0}(C, M)$. Further discussion of Hochschild cohomology may be found in [1, 4].

3. Map induced on Hochschild cohomology by a two-cocycle. The change of rings map $\phi(\sigma)$ associated with a Sweedler two-cocycle σ introduced in Definition 2.1 induces a map $\phi^n(\sigma, M)$: $H^n(C, M) \to H^n(C^\sigma, M^\sigma)$ on Hochschild cohomology which is the focus of this paper. In general, this map is not surjective, as the following example illustrates.

EXAMPLE 3.1. Let k be a field and C be a central separable k-algebra. Then $H^n(C, M) = \{0\}$ for all $n \geqq 1$ and all M. However, given any k-algebra D with k-dimension of D equal to the k-dimension of C, there is a C-two-cocycle τ with $D \simeq C^\tau$ [7, Theorem 6.1]. Choose τ so that $C^\tau = k[x]/\langle x^m \rangle$ with $m = k$-dimension of C. Then $H^1(C^\tau, C^\tau) \neq \{0\}$ since d/dx is a nontrivial Hochschild 1-cocycle. Thus $\phi^1(\tau, C)$ is not surjective.

In certain cases, however, $\phi^n(\sigma, M)$ is actually an isomorphism.

DEFINITION 3.2. A Sweedler C-two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is vertible if $\sum_{i,j} a_i a_j \otimes b_j^0 \otimes c_j b_i \otimes c_i^0$ is invertible in $C \otimes C^0 \otimes C \otimes C^0$.

LEMMA 3.3. *If a Sweedler C-two-cocycle σ is vertible, $\phi^n(\sigma, M)$ is an isomorphism for all n and M.*

Proof. Let $\sum_l u_l \otimes v_l^0 \otimes w_l \otimes z_l^0$ be the inverse of

$$\sum_{i,j} a_i a_j \otimes b_j^0 \otimes c_j b_i \otimes c_i^0 \quad \text{and}$$

define $\bar{\phi}(\sigma)$: $C \otimes C^0 \to C^\sigma \otimes C^{\sigma 0}$ by

$$\bar{\phi}(\sigma)\left\{\sum_i x_i \otimes y_i^0\right\} = \sum_{i,l} (u_l x_i v_l)^\sigma \otimes (w_l y_i z_l)^{\sigma 0} \,.$$

Then it may be verified directly that $\bar{\phi}(\sigma) = \phi(\sigma)^{-1}$. Therefore $\phi(\sigma)$

is an isomorphism and hence the induced maps $\phi^n(\sigma, M)$ on Hochschild cohomology are isomorphisms.

EXAMPLE 3.4. Let $A \subseteq C$ be k-algebras with A commutative. If σ in $A \otimes A \otimes A \subseteq C \otimes C \otimes C$ is an invertible (i.e., Amitsur) A-two-cocycle, clearly σ is a vertible C-two-cocycle and thus $\phi^n(\sigma, M)$ is an isomorphism for all n and M.

In the next section, we will show that $\ker\{\phi^1(\sigma, M)\} = \{0\}$ for any Sweedler two-cocycle σ if C is a finite dimensional algebra over a field k. First, however, we need several simplifying tools. Let σ and τ be C-two-cocycles such that $\sigma \sim {}^{\delta}\tau$. Then for any n and M, $\phi^n(\tau, M) = \underline{R}^{\delta} \circ \phi^n(\sigma, M)$ where $\underline{R}^{\delta}$ is induced by the algebra isomorphism $R^{\delta}: C^{\sigma} \to C^{\tau}$. Therefore we have

LEMMA 3.5. *If σ and τ are equivalent C-two-cocycles, M is a C-bimodule and n is a positive integer,*

$$\ker\{\phi^n(\sigma, M)\} = \{0\} \text{ iff } \ker\{\phi^n(\tau, M)\} = \{0\}\ .$$

LEMMA 3.6. *Let S_i, $i = 1, 2$, be algebras over the commutative ring k. Hence $C = S_1 \times S_2$ is a k-algebra with diagonal k-action and canonical projections $p_i: C \to S_i$, $i = 1, 2$. If*

$$\sigma = \sum_i (a_{i1}, a_{i2}) \otimes (b_{i1}, b_{i2}) \otimes (c_{i1}, c_{i2})$$

is a C-two-cocycle, then

(i) *$p_j(\sigma) = \sum_i a_{ij} \otimes b_{ij} \otimes c_{ij}$ is an S_j-two-cocycle.*

(ii) *If $\delta_j = \sum_i x_{ij} \otimes y_{ij}$ is a vertible element in $S_j \otimes S_j$, $j = 1, 2$, then*

$$\begin{aligned}\delta = (1, 0) \otimes (0, 1) &+ (0, 1) \otimes (1, 0)\\ &+ \sum_i (x_{i1}, 0) \otimes (y_{i1}, 0)\\ &+ \sum_i (0, x_{i2}) \otimes (0, y_{i2})\end{aligned}$$

is a vertible element in $C \otimes C$ and $\sigma \sim {}^{\delta}\tau$ defines a C-two-cocycle τ with $e_{\tau} = (\sum_i x_{i1} e_{\sigma_1} y_{i1}, \sum_i x_{i2} e_{\sigma_2} y_{i2})$ if $e_{\sigma} = (e_{\sigma_1}, e_{\sigma_2})$.

Proof.

(i) follows trivially since an algebra map clearly preserves the two-cocycle relations.

(ii) If $\bar{\delta}_j = \sum_i \bar{x}_{ij} \otimes \bar{y}_{ij}$ is the verse of δ_j, then one may show that

$$\begin{aligned}\bar{\delta} = (1, 0) \otimes (0, 1) &+ (0, 1) \otimes (1, 0)\\ &+ \sum_i (\bar{x}_{i1}, 0) \otimes (\bar{y}_{i1}, 0)\end{aligned}$$

$$+ \sum_i (0, \bar{x}_{i2}) \otimes (0, \bar{y}_{i2})$$

is the verse of δ. The form of e_τ is clear.

Suppose now that k is a field and let L be an extension field of k. The natural injection $C \to C \otimes L$ induces a map $C \otimes C \otimes C \to (C \otimes L) \otimes_L (C \otimes L) \otimes_L (C \otimes L)$. Let $\sigma \otimes 1$ denote the image of the C-two-cocycle σ under this map. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccc}
\{0\} \to \ker \phi^n(\sigma, M) & \longrightarrow & H^n(C, M) & \longrightarrow & H^n(C^\sigma, M^\sigma) \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} \to \ker \phi^n(\sigma \otimes 1, M \otimes L) & \to & H^n_L(C \otimes L, M \otimes L) & \to & H^n_L(C^\sigma \otimes L, M^\sigma \otimes L)
\end{array}$$

where $H^n_L(-, -)$ denotes the Hochschild cohomology as an L-algebra rather than as a k-algebra. Since L is a field extension of k, it follows easily by linear algebra from the definition of Hochschild cohomology (cf. §2), noting $(C \otimes L) \otimes_L n \simeq (C^{\otimes n}) \otimes L$, that the map $H^n(C, M) \to H^n_L(C \otimes L, M \otimes L)$ is injective. Thus the above commutative diagram allows one to conclude that $\ker \phi^n(\sigma \otimes 1, M \otimes L) = \{0\}$ implies that $\ker \phi^n(\sigma, M) = \{0\}$.

LEMMA 3.7. *If C is an algebra over an algebraically closed field k which is Artinian as a ring and σ is a C-two-cocycle, there is a C-two-cocycle τ with $\sigma \sim {}^\delta\tau$ and $e_\tau = 1$.*

Proof. Since C is Artinian, the Jacobson radical $J(C)$ is nilpotent and thus by [5, Theorem 4.7] $\sigma \sim q(\tau)$ where q is the endomorphism of C induced by projection modulo $J(C)$. (Note that since k is algebraically closed, the semisimple algebra $C/J(C)$ is actually separable over k.) We may henceforth assume C is semisimple and thus by Wedderburn-Artin structure theory we have $C = \prod_{i=1}^m S_i$ where $S_i = M(n_i, k)$, the algebra of n_i by n_i matrices over k. Then by Lemma 3.6 it is sufficient to show that the S_i-two-cocycle $p_i(\sigma)$ is equivalent via δ_i to an S_i-two-cocycle τ_i with $e_{\tau_i} = 1$. If $n_i = 1$ and $S_i = k$, one may take $\delta_i = 1 \otimes e_{\sigma_i}$ (cf. [6, §1]). If $n_i > 1$, then the existence of $\delta_i = \sum_j x_{ji} \otimes y_{ji}$ with $e_{\sigma_i} \sim {}^{\delta_i}e_{\tau_i}$ and $\sum_j x_{ji} e_{\sigma_i} y_{ji} = 1$ is assured by [7, Theorem 6.1].

4. Injectivity of $\phi^1(\sigma, M)$. In this section we prove

THEOREM 4.1. *Let C be a finite dimensional algebra over a field k, σ be a Sweedler C-two-cocycle, and M be a C-bimodule. Then $\phi^1(\sigma, M)$: $H^1(C, M) \to H^1(C^\sigma, M^\sigma)$ is injective.*

In light of the results in §3, we need only to prove:

THEOREM 4.2. *If k is an algebraically closed field, C is a finite dimensional k-algebra, σ is a Sweedler C-two-cocycle, and M is a C-bimodule, then there is a Sweedler C-two-cocycle τ with σ equivalent to τ and $\phi^1(\tau, M)$ injective.*

To establish this theorem we need several preliminary results.

LEMMA 4.3. *If k is a field, C is a k-algebra, σ is a Sweedler C-two-cocycle, and M is a C-bimodule, the map $\phi^1(\sigma, M)$ is induced by the map*

$$d \longrightarrow \left(x^\sigma \longrightarrow \left(\sum_{i,j} a_i a_j e_\sigma d(b_j x b_i e_\sigma) c_i c_j\right)^\sigma\right)$$

from the group $\mathrm{Der}_k(C, M)$ of k-derivations of C with values in M to $\mathrm{Der}_k(C^\sigma, M^\sigma)$.

Proof. This lemma may be proved using the definition of $\phi(\sigma)$.

LEMMA 4.4. *Let C be an algebra over the commutative ring k, M be a C-bimodule, and $d\colon C \to M$ be a k-derivation. Let $\alpha = \sum_i a_i \otimes b_i \otimes c_i$ and m be elements of $C \otimes C \otimes C$ and M, respectively, such that*

$$d(x) = \sum_i a_i x b_i m c_i - \sum_i a_i m b_i x c_i \equiv [x, m]_\alpha$$
$$d(b_i) = 0 \quad \textit{for all } i$$
$$\sum_i a_i b_i \otimes c_i = 1 \otimes 1 = \sum_i a_i \otimes b_i c_i \,.$$

Then $d(x) = xm - mx = [x, m]$ for all x in C.

Proof. Letting [,] denote the usual Lie bracket, we have

$$\begin{aligned} d(x) &= \sum_i a_i x b_i m c_i - \sum_i a_i m b_i x c_i \\ &= \sum_i a_i x [b_i, m] c_i + \sum_i a_i [b_i, m] x c_i + [x, m] \,. \end{aligned} \tag{4.5}$$

Since $d(b_i) = 0$ for all i,

$$[b_i, m] = -\sum_j a_j b_i [b_j, m] c_j - \sum_j a_j [b_j, m] b_i c_j \,.$$

Hence we may rewrite eqn. (4.5) as

$$d(x) = -\sum_{i\,j} a_i x a_j b_i [b_j, m] c_j c_i$$

$$\begin{aligned}
&- \sum_{i,j} a_i x a_j [b_j, m] b_i c_j c_i \\
&- \sum_{i,j} a_i a_j b_i [b_j, m] c_j x c_i \\
&- \sum_{i,j} a_i a_j [b_j, m] b_i c_j x c_i + [x, m] \\
&= -\sum_i a_i x d(b_i) c_i - \sum_i a_i d(b_i) x c_i + [x, m] \\
d(x) &= [x, m]
\end{aligned}$$

and we are done.

Now we are ready to prove Theorem 4.2:

Since we are assuming C is a finite dimensional algebra over an algebraically closed field, we may write $C = B \oplus J(C)$ with $J(C)$ the Jacobson radical of C and B a k-separable subalgebra of C. By [5, Theorem 4.7], σ is equivalent to its projection modulo $J(C)$, a C-two-cocycle σ_1 in $B \otimes B \otimes B$. Then using Lemma 4.3 and the fact that $H^1(B, M) = \{0\}$ [4, Theorem 4.1], the map $\mathrm{Der}_k(C, M) \to \mathrm{Der}_k(C^{\sigma_1}, M^{\sigma_1})$ which induces $\phi^1(\sigma_1, M)$ is given by $d \to (x^{\sigma_1} \to d(x)^{\sigma_1})$. Thus $\phi^1(\sigma_1, M)$ will be injective if for any m in M, $d(x^{\sigma_2}) = x^{\sigma_2} * m^{\sigma_2} - m^{\sigma_2} * x^{\sigma_2}$ for all x implies that $d(x) = xm - mx$ for some two-cocycle σ_2 equivalent to σ_1. This follows from Lemma 3.7 and Lemma 4.4.

REMARK 4.5. Theorem 4.2 may be paraphased as "multiplication alteration shrinks the separable part of C." Since $J(C)^\sigma \subseteqq J(C^\sigma)$ for $J(C)$ nilpotent [5, Lemma 2.1], multiplication alteration adds to the nilpotency of C. Hence the effect of multiplication alteration is in a sense opposite to the effect of Gerstenhaber's deformation theory [3] which adds to the separable part of C and shrinks the radical [2, Theorem 1].

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REFERENCES

1. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.

2. F. Flanigan, *Straightening-out and semirigidity in associative algebras*, Trans. Amer. Math. Soc., **138** (1969), 415-425.

3. M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math., **79** (1964), 59-103.

4. G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math., **46** (1945), 58-67.

5. D. Riffelmacher, *Multiplication alteration and related rigidity properties of algebras*, Pacific J. Math., **71** (1977), 139-157.

6. ———, *Sweedler's two-cocycles and generalizations of theorems on Amitsur*

cohomology, Trans. Amer. Math. Soc., **251** (1979), 255–265.
7. M. Sweedler, *Multiplication alteration by two-cocycles*, Illinois J. Math., **15** (1971), 302–323.

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BELL LABORATORIES
HOLMDEL, NJ 07733