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A NOTE ON GAP-FREQUENCY PARTITIONS

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George Andrews has introduced gap-frequency partitions in order to interpret the Rogers-Selberg q-series identities related to the modulus seven. In this paper, we give a direct derivation of the generating function for such partitions. Our approach makes it much easier to extend and generalize the notion of gap-frequency partitions.

L. J. Rogers is known today primarily for his discovery of the Rogers-Ramanujan identities:

$$\prod_{\substack{n=1 \\ n\neq 0\ \pm 2\ (\mathrm{mod}\ 5)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m}, \tag{1}$$

$$\prod_{\substack{n=1 \\ n\neq 0\ \pm 1\ (\mathrm{mod}\ 5)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m}, \tag{2}$$

where $(a)_\infty = (a; q)_\infty = \prod_{i=0}^{\infty} (1-aq^i)$,

$$(a)_m = \frac{(a)_\infty}{(aq^m)_\infty}.$$

These analytic identities came to prominence largely because of P. A. MacMahon's combinatorial interpretation of them:

(3) For $r = 1$ or 2, and any positive integer n, the partitions of n into parts not congruent to 0, $\pm r$ mod 5 are equinumerous with the partitions of n into parts with difference at least two between parts, and in which one appears as a part at most $r-1$ times.

Statement (3) can be proved from equations (1) and (2) by viewing each side of the equations as a generating function (see [3], § 19.13).

It is less well known that Rogers also discovered similar identities for the modulus 7:

$$\prod_{\substack{n=1 \\ n\neq 0,\ \pm 1\ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2+2m}}{(q^2; q^2)_m}(-q^{2m+2})_\infty \tag{4}$$

$$\prod_{\substack{n=1 \\ n\neq 0,\ \pm 2\ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2+2m}}{(q^2; q^2)_m}(-q^{2m+1})_\infty \tag{5}$$

$$\prod_{\substack{n=1 \\ n\neq 0,\ \pm 3\ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q^2; q^2)_m}(-q^{2m+1})_\infty . \tag{6}$$

Equations (4) and (6) first appeared in [4]. All three are proved by Rogers in [5]. A. Selberg rediscovered them in [6].

There is also a *combinatorial theorem for the modulus* seven. It is a special case of a combinatorial theorem by B. Gordon, [2], which was stated for all odd moduli greater than or equal to five.

(7) For $r = 1, 2$ or 3, and any positive integer n, the partitions of n into parts not congruent to $0, \pm r \bmod 7$ are equinumerous with the partitions of n in which each part appears at most twice, the difference between nonidentical parts is at least two if either appears twice, and one appears as a part at most $r-1$ times.

While many proofs of statement (7) exist, until recently there was no proof which showed (7) as a direct consequence of equations (4)-(6). It was to supply such a proof that George Andrews introduced the notion of *gap-frequency partitions* (abbreviated g-f partitions) in [1]. The purpose of this paper is to provide a simpler derivation of the generating function for g-f partitions. This yields a more direct proof that equations (4)-(6) imply statement (7), and also leads to certain natural generalizations of g-f partitions.

The generating function for g-f partitions.

DEFINITION. A partition π is said to be a *gap-frequency* (or g-f) partition if whenever a summand s appears exactly t times, the next larger part is at least $s + t$, and if it is exactly $s + t$ it can appear at most t times.

EXAMPLE. $1 + 4 + 4 + 4 + 7 + 7 + 7$ is a g-f partition. Neither $1 + 3 + 3 + 3 + 6 + 6 + 6 + 6$ nor $2 + 3 + 5 + 5 + 5 + 7 + 7$ is a g-f partition.

DEFINITION. For positive integers r, x and n, let $S_{r,x}(n)$ denote the number of g-f partitions of n in which no part appears more than x times and one appears at most $r-1$ times.

THEOREM. *For positive integers r, x and n and for $|q| < 1$, let* $M(m_1, \cdots, m_x) = M = \sum_{j=1}^{x} j^2\binom{m_j}{2} + \sum_{1\leq i<j\leq x} ijm_im_j$. *Then*

$$\sum_{m_1,\cdots,m_x\geq 0} \frac{q^{M+m_1+2m_2+\cdots+(r-1)m_{r-1}+2(rm_r+(r+1)m_{r+1}+\cdots+xm_x)}}{(q;q)_{m_1}(q^2;q^2)_{m_2}\cdots(q^x;q^x)_{m_x}} = \sum_{n=0}^{\infty} S_{r,x}(n)q^n . \tag{8}$$

For fixed values of $m_2, \cdots, m_x$, the left side of (8) can be summed using Euler's formula:

$$(9) \qquad \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}} a^m}{(q;q)_m} = (-a;q)_\infty .$$

It is then a straightforward exercise to verify that when $x = 2$ the right-hand sides of equations (4)-(6) are obtained.

Note that $S_{r,2}(n)$ counts those partitions described in the second part of (7). The theorem is sufficient to prove that equations (4)-(6) imply statement (7).

Proof of the theorem.

DEFINITION. A *partition with attributes* is a partition in which parts of equal value may be distinguished by some attribute or characteristic. For example, parts may be colored red, blue, green, etc. A *partition with x attributes* is a partition in which *at most* x attributes or characteristics are used. In a partition with x attributes, each part will be denoted by an ordered pair, (d_i, a_i), where d_i is the value of the part and a_i is its attribute, $1 \leqq a_i \leqq x$.

The generating function for partitions into exactly m parts, each part greater than or equal to b is given by

$$q^{bm}(q;q)_m^{-1} .$$

It follows that

$$q^{b_1 m_1 + b_2 m_2 + \cdots + b_x m_x}(q;q)_{m_1}^{-1}(q;q)_{m_2}^{-1} \cdots (q;q)_{m_x}^{-1}$$

is the generating function for partitions with x attributes such that for $1 \leqq i \leqq x$, there are exactly m_i parts with attribute i, and each such part is greater than or equal to b_i.

DEFINITION. Let $R_r(m_1, \cdots, m_x; n)$ denote the number of partitions of n with x attributes such that for $1 \leqq i \leqq x$ there are exactly m_i parts with attribute i, each part with attribute i is divisible by i, and all parts with attribute $i \geqq r$ are greater than or equal to $2i$.

LEMMA 1.

$$\frac{q^{m_1 + 2m_2 + \cdots + (r-1)m_{r-1} + 2(rm_r + (r+1)m_{r+1} + \cdots + xm_x)}}{(q;q)_{m_1}(q^2;q^2)_{m_2} \cdots (q^x;q^x)_{m_x}} = \sum_{n=0}^{\infty} R_r(m_1, \cdots, m_x; n) q^n .$$

Proof. This lemma follows from the discussion given above and the definition of $R_r(m_1, \cdots, m_x; n)$.

DEFINITION. Let $S_r(m_1, \cdots, m_x; n)$ denote the number of g-f partitions of n in which no part appears more than x times, one appears at most $r-1$ times, and for $1 \leqq i \leqq x$, exactly m_i different integers appear i times.

LEMMA 2. *Let* $M(m_1, \cdots, m_x) = M = \sum_{j=1}^{x} j^2\binom{m_j}{2} + \sum_{1\leqq i<j\leqq x} ijm_im_j$. *Then* $R_r(m_1, \cdots, m_x; n-M) = S_r(m_1, \cdots, m_x; n)$.

Before proving Lemma 2, we note that it and Lemma 1 imply the theorem, since

$$\begin{aligned}\sum_{n=0}^{\infty} S_{r,x}(n)q^n &= \sum_{m_1 \cdots m_x \geqq 0} \sum_{n=M}^{\infty} S_r(m_1, \cdots, m_x; n)q^n \\ &= \sum_{m_1 \cdots m_x \geqq 0} \sum_{n=M}^{\infty} R_r(m_1, \cdots, m_x; n-M)q^n \\ &= \sum_{m_1 \cdots m_x \geqq 0} q^M \sum_{n=0}^{\infty} R_r(m_1, \cdots, m_x; n)q^n \\ &= \sum_{m_1, \cdots, m_x \geqq 0} \frac{q^{M+m_1+\cdots+(r-1)m_{r-1}+2(rm_r+\cdots+xm_x)}}{(q;q)_{m_1}\cdots(q;q)_{m_x}} .\end{aligned}$$

Proof of Lemma 2. We shall prove this lemma by establishing a one-to-one correspondence between partitions counted by $R_r(m_1, \cdots, m_x; n-M)$ and those counted by $S_r(m_1, \cdots, m_x; n)$.

Consider a partition counted by $R_r(m_1, \cdots, m_x; n-M)$ with parts given by $(d_1, a_1), (d_2, a_2), \cdots$ and with the parts ordered from left to right such that if $d_\lambda/a_\lambda < d_\mu/a_\mu$, then (d_λ, a_λ) precedes (d_μ, a_μ) and if $d_\lambda/a_\lambda = d_\mu/a_\mu$ and $a_\lambda > a_\mu$, then (d_λ, a_λ) precedes (d_μ, a_μ). Clearly there is a unique such ordering of the ordered pairs.

This partition of $n-M$ with x attributes is transformed into a partition of n with x attributes if each ordered pair (d_λ, a_λ) is replaced by the pair (e_λ, a_λ) where

$$e_\lambda = d_\lambda + \sum_{k=1}^{\lambda-1} a_\lambda a_k .$$

We claim that our new partition is a partition of n. The total amount which has been added to our partition is

$$\sum_{\lambda=1}^{m_1+\cdots+m_x} \sum_{k=1}^{\lambda-1} a_\lambda a_k ,$$

which is the second elementary symmetric function of the numbers

$$\overbrace{1\cdots1}^{m_1}\ \overbrace{2\cdots2}^{m_2}\ \cdots\ \overbrace{x\cdots x}^{m_x} ,$$

which is equal to

$$\sum_{j=1}^{x} j^2 \binom{m_j}{2} + \sum_{1 \leqq i < j \leqq x} ijm_i m_j = M .$$

This proves our claim.

Observe that a_λ divides e_λ, and for each λ

$$\frac{e_\lambda}{a_\lambda} - \frac{e_{\lambda-1}}{a_{\lambda-1}} = \frac{d_\lambda}{a_\lambda} - \frac{d_{\lambda-1}}{a_{\lambda-1}} + a_{\lambda-1} \geqq a_{\lambda-1} .$$

Equality occurs when $d_\lambda/a_\lambda = d_{\lambda-1}/a_{\lambda-1}$, which implies that $a_{\lambda-1} \geqq a_\lambda$. Also, if $a_1 \geqq r$, then $e_1/a_1 = d_1/a_1 \geqq 2$. Thus if each part (e_λ, a_λ) is replaced by a_λ equal parts of value e_λ/a_λ, we have a partition counted by $S_r(m_1, \cdots, m_x; n)$. This procedure is uniquely reversible (equal parts are added together and the resulting part is given the attribute equal to the number of parts which were added, $\sum_{k=1}^{\lambda-1} a_\lambda a_k$ is then subtracted from the value of the λ'th part), and so the one-to-one correspondence is established.

This concludes the proof of Lemma 2, and so also the proof of the theorem.

A generalization.

DEFINITION. A partition π is said to be a *k-fold* g-f *partition* if whenever a summand s appears exactly t times, then the next larger part is at least $s + kt$, and if it is exactly $s + kt$, it can appear at most t times.

DEFINITION. For positive integers r, x, k and n, let $S_{r,x,k}(n)$ denote the number of k-fold g-f partitions of n in which no part appears more than x times and one appears at most $r - 1$ times.

By the method used above to find the generating function for $S_{r,x}(n) = S_{r,x,1}(n)$, it can be readily verified that

$$\sum_{n=0}^{\infty} S_{r,x,k}(n) q^n = \sum_{m_1, \cdots, m_x \geqq 0} \frac{q^{kM + m_1 + \cdots + (r-1)m_{r-1} + 2(rm_r + \cdots + xm_x)}}{(q; q)_{m_1} \cdots (q^x; q^x)_{m_x}} .$$

Since $S_{r,1,k}(n)$ counts the number of partitions of n into parts with minimal difference k, including at most $r - 1$ ones, we see that the right sides of the Rogers-Ramanujan identities are special cases of the generating function for $S_{r,x,k}(n)$.

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