PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 2, 1980

AN ANALOGUE OF THE WIENER-TAUBERIAN THEOREM FOR SPHERICAL TRANSFORMS ON SEMI-SIMPLE LIE GROUPS

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Let G be a semi-simple connected noncompact Lie group with finite center and K a fixed maximal compact subgroup of G. Fix a Haar measure dx on G and let $I_1(G)$ denote those functions in $L^1(G, dx)$ which are biinvariant under K. The purpose of this paper is to prove that if $f \in I_1(G)$ is such that its spherical Fourier transform (i.e., Gelfand transform) $\hat{f}$ is nowhere vanishing on the maximal ideal space of $I_1(G)$ and $\hat{f}$ "does not vanish too fast at ∞", then the ideal generated by f is dense in $I_1(G)$. This generalizes earlier results of Ehrenpreis-Mautner for $G = \mathrm{SL}(2, \boldsymbol{R})$ and R. Krier for G of real rank one.

1. Introduction. Let f be an L^1-function on $\boldsymbol{R}$ (or more generally on a locally compact abelian group). Then the celebrated Wiener-Tauberian theorem says that if the Fourier transform $\hat{f}$ is a nowhere vanishing function then the ideal generated by f is dense in $L^1(\boldsymbol{R})$. In [1] Ehrenpreis and Mautner observe that the corresponding result is not true if one considers the commutative Banach algebra of K-biinvariant functions on noncompact semi-simple Lie group G, where K is a maximal compact subgroup of G. More precisely, let $G = \mathrm{SL}(2, \boldsymbol{R})$ i.e., the group of 2×2 real matrices of determinant 1, and

$$K = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} ; \quad 0 \leqq \theta \leqq 2\pi \right\} \text{ and let}$$

$I_1(G)$ denote the commutative Banach algebra of K-biinvariant L^1-functions on G. For $f \in I_1(G)$, let $\hat{f}$ denote its spherical Fourier transform (see § 2). Then Ehrenpreis and Mautner observed that there exist functions $f \in I_1(G)$ such that $\hat{f}$ does not vanish anywhere on the maximal ideal space of $I_1(G)$ and yet the algebra generated by f is *not* dense in $I_1(G)$. However they were able to show that if $\hat{f}$ is non vanishing *and* $\hat{f}$ 'does not go to zero too fast at ∞' then the ideal generated by f is indeed dense in $I_1(G)$. (Theorems 6 and 7 of [1].) These results have been generalized by R. Krier [6] in his thesis when G is a noncompact connected semi-simple Lie group of real rank 1. (The author does not know whether Krier's results have been published.) The purpose of this note is to prove a theorem in the spirit of Theorem 7 of [1] without any restriction on the rank of G. While the basic technique we use is that of [1], we have to use the more recent results of Trombi-Varadarajan [7] and some observations of Gangolli-Warner [4] to prove our main theorem. Indeed in [3] Gangolli predicts that a theorem of the Trombi-Varadarajan type would yield a Tauberian type theorem.

2. **Notation and preliminaries.** (For any unexplained notation and terminology please see [5].) G will denote a connected non-compact semi-simple Lie group with finite center and K a fixed maximal compact subgroup of G. Fix an Iwasawa decomposition $G = KAN$ and let $\boldsymbol{a}$ be the Lie algebra of A. Let $\boldsymbol{a}^*$ be the real dual of $\boldsymbol{a}$ and $\boldsymbol{a}^*$ its complexification. Let ρ be the half-sum of the positive roots for the adjoint action of $\boldsymbol{a}$ on $\boldsymbol{g}$ (where $\boldsymbol{g}$ is the Lie algebra of G). The Killing form will induce a form $\langle\cdot,\cdot\rangle$ on $\boldsymbol{a}^*\times\boldsymbol{a}^*$. Then, as is well known, $\langle\cdot,\cdot\rangle$ is positive definite on $\boldsymbol{a}^*\times\boldsymbol{a}^*$. Extend the form $\langle\cdot,\cdot\rangle$ to a bilinear form on $\boldsymbol{a}_c^*\times\boldsymbol{a}_c^*$. This bilinear form also will be denoted by $\langle\cdot,\cdot\rangle$. Let W be the Weyl group of the symmetric space G/K. Then there is a natural action of W on $\boldsymbol{a}$, $\boldsymbol{a}^*$ and $\boldsymbol{a}_c^*$ and $\langle\cdot,\cdot\rangle$ is invariant under the action of W.

For each $\lambda\in\boldsymbol{a}_c^*$ let ϕ_λ be the elementary spherical function associated with λ. (Recall that ϕ_λ is given by the formula, $\phi_\lambda(x)=\int_K e^{(i\lambda-\rho)(H(xk))}dk$ – see [5] for details.) Then it is known that $\phi_\lambda = \phi'_\lambda$, iff $\exists s\in W$ with $s\lambda = \lambda'$. Let $F = \{\lambda; \phi_\lambda$ is a bounded function on $G\}$. Then it is known (a theorem of Helgason and Johnson) that:

$$F = \boldsymbol{a}^* + iC_\rho \text{ where } C_\rho = \text{convex hull of } \{s\rho\colon s\in W\} .$$

Let $P(\boldsymbol{a}_c^*)$ be the symmetric algebra over $\boldsymbol{a}_c^*$. Then each $u\in P(\boldsymbol{a}_c^*)$ gives rise to a differential operatior $\partial(u)$ on $\boldsymbol{a}_c^*$.

Let $I(G)$ be the set of all complex valued spherical functions on G, i.e., $I(G) = \{f; f(k_1xk_2) = f(x),\ k_1,\ k_2\in K,\ x\in G\}$. Fix a Haar measure dx on G and let $I_1(G) = I(G)\cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution (and that the maximal ideal space of $I_1(G)$ can be identified with F/W). We shall denote by $I^\infty(G)$ the space of C^∞-spherical functions and by $I_c^\infty(G)$ the space of compactly supported functions in $I^\infty(G)$.

For $f\in I_1(G)$ define its spherical Fourier transform, $\hat{f}$ on F by:

$$\hat{f}(\lambda) = \int_G f(x)\phi_{-\lambda}(x)dx\ , \quad \lambda\in F .$$

Then it is known that $\hat{f}$ is a W-invariant bounded function on F, holomorphic in F^0(=interior of F) and continuous on F. Also $(f*g)^\wedge = \hat{f}\cdot\hat{g}$ for $f,\ g\in I_1(G)$ where $f*g$ is the convolution of f and g and is given by

$$(f*g)(y) = \int_G f(yx^{-1})g(x)dx\ , \quad y \in G\ .$$

If $f \in I_c^\infty(G)$ then $\hat{f}$ is defined on all of $\boldsymbol{a}_c^*$ (and in fact will be an entire W-invariant function on $\boldsymbol{a}_c^*$ satisfying the Paley-Wiener growth condition—see [2]).

We shall now introduce a space of rapidly decreasing functions in $I^\infty(G)$ which we will denote by $S_1(G)$. (This is the so called L^1-Harish-Chandra-Schwartz space of spherical functions):

Let $x \in G$. Then $x = k \exp X, k \in K, X \in \boldsymbol{p}$ ($\boldsymbol{g} = \boldsymbol{k} + \boldsymbol{p}$ is the Cartan decomposition of the Lie algebra $\boldsymbol{g}$ of G). Put $\sigma(x) = ||X||$, where $||\cdot||$ is the norm induced on $\boldsymbol{p}$ by the restriction of the Killing form. For any left invariant differential operator D on G and any integer $r \geqq 0$, we define for $f \in I^\infty(G)$

$$p_{D,r}(f) = \sup_{x \in G} (1 + \sigma(X))^r |\phi_0(x)|^{-2} |Df(x)|$$

where ϕ_0 is the elementary spherical function corresponding to $\lambda = 0$. Define $S_1(G) = \{f; f \in I^\infty(G) \text{ and } p_{D,r}(f) < \infty \forall r, D\}$. $S_1(G)$ becomes a Frechet-space when equipped with topology induced by the family of semi norms $p_{D,r}$. It is known that $S_1(G) \hookrightarrow I_1(G)$ and $I_c^\infty(G) \hookrightarrow S_1(G)$ are both dense inclusions.

Now let $Z(F)$ be the space of functions f on F satisfying the following conditions: (i) f is holomorphic in F^0 and continuous on F, (ii) If $u \in P(\boldsymbol{a}_c^*)$ and $l \geqq 0$ is any integer, then $q_{u,l}(f) = \sup_{\lambda \in F^0}(1 + ||\lambda||^2)^l |(\partial(u)f)(\lambda)| < \infty$, (where $||\lambda||^2 = ||\lambda_1||^2 + ||\lambda_2||^2$, $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in a^*$ and $||\lambda_i||^2 = \langle\lambda_i, \lambda_i\rangle$). Let $\bar{Z}(F)$ denote the subspace of $Z(F)$ consisting of W-invariant functions. $Z(F), \bar{Z}(F)$ are algebras under pointwise multiplication and we topologize them by the family of semi norms q_{ul}. In this topology $Z(F), \bar{Z}(F)$ are Frechet spaces. If $a \in \bar{Z}(F)$ define the 'wave packet' ψ_a on G by:

$$\psi_a(x) = \frac{1}{|W|}\int_{a^*} a(\lambda)\phi_\lambda(x)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda\ ,$$

($|W|$ is the order of the Weyl group) .

($c(\lambda)$ is the well known c-function of Harish-Chandra and one knows that $c(\lambda)^{-1}c(-\lambda)^{-1}$ is a continuous function on $\boldsymbol{a}^*$ of at most polynomial growth. Further if $d\mu$ is the measure on $\boldsymbol{a}^*$ defined by $d\mu = |W|^{-1}c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$, then one knows that the map $f \to \hat{f}$ is an isometry of $I(G) \cap L^2(G)$ onto $L^2(a^*, d\mu)^W$ where the superscript W indicates Weyl-group invariants in $L^2(a^*, d\mu)$). We are now finally in a position to state the theorem of Trombi-Varadarajan [7]:

THEOREM 2.1. (i) *If* $f \in S_1(G)$, *then* $\hat{f} \in \bar{Z}(F)$.

(ii) *If* $a \in \bar{Z}(F)$ *then the integral defining the wave packet* ψ_a *converges absolutely and in fact* $\psi_a \in S_1(G)$ *and* $\hat{\psi}_a = a$.

(iii) *The map* $f \to \hat{f}$ *is a topological linear isomorphism of* $S_1(G)$ *onto* $\bar{Z}(F)$.

Before closing this section we introduce some more function spaces and state a proposition due to Gangolli-Warner [4]. (As the authors point out in [4] this proposition can be obtained by a careful examination of the proof of Theorem 2.1 of Trombi-Varadarajan.)

Let m, l be nonnegative integers and let us put $\bar{Z}_{m,l}(F)$ for the space of functions f on F such that (i) f is holomorphic in F^0, continuous on F, and invariant under the action of W (ii) If $u \in P(\mathfrak{a}_c^*)$ and degree $u \leqq m$, then

$$q_{u,l}(f) = \sup_{\lambda \in F^0} (1 + \|\lambda\|^2)^l |(\partial(u)f)(\lambda)| < \infty .$$

Put $\bar{Z}_m(F) = \bigcap_{l \geqq 0} \bar{Z}_{m,l}(F)$. Then tne following proposition is contained in Proposition 3.3 and Corollary 3.4 of Gangolli-Warner [4].

PROPOSITION 2.2. *Let* G *be a noncompact connected semi-simple Lie group with finite center. Then* $\exists$ *an integer* m_G *(depending only on the group* G*) such that if* $a \in \bar{Z}_{m_G}(F)$, *then:*

(i) *The integral defining the wave packet* ψ_a *converges absolutely.*

(ii) $\psi_a \in I_1(G)$.

3. An analogue of the Wiener-Tauberian theorem. Before we state and prove the main theorem we will first prove a couple of preliminary lemmas which will be used in the proof of the main theorem. The first lemma is a very mild strengthening of Proposition 2.2 and the second lemma is a slight generalization of Lemma 5.2 for the case of $G = \mathrm{SL}(2, \boldsymbol{R})$ in [1].

LEMMA 3.1. *There exists an integer* m_G *(depending only on the group* G*) such that if* $a \in \bar{Z}_{m_G}(F)$ *then all the following conditions are satisfied*

(i) *The integral defining* ψ_a *(the wave packet) converges absolutely.*

(ii) $\psi_a \in I_1(G)$.

(iii) $\hat{\psi}_a = a$.

Proof. From Proposition 2.2 it follows that we can find an integer m_G such that if $a \in \bar{Z}_{m_G}(F)$ then (i) and (ii) are satisfied. We will show that (iii) is also satisfied. Observe first that if $a \in \bar{Z}_{m_G}(F)$, then since $(\forall l)$ it decays faster than $1/(1+\|\lambda\|^2)^l$ on $\boldsymbol{a}^*$ and since $c(\lambda)^{-1}c(-\lambda)^{-1}$ has at most polynomial growth, a is integrable with respect to the measure $c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$ on $\boldsymbol{a}^*$. To prove that $\hat{\psi}_a = a$, we first show that

$$(*)\qquad \forall b \in \bar{Z}(F),\ \int_{\boldsymbol{a}^*} a(\lambda)b(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda = \int_{\boldsymbol{a}^*} \hat{\psi}_a(\lambda)b(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda\ .$$

The integral on the left hand side exists since both a, b decay faster than $1/(1+\|\lambda\|^2)^l$ on $\boldsymbol{a}^*$ and $c(\lambda)^{-1}c(-\lambda)^{-1}$ has at most polynomial growth. The integral on the right hand side exists because $\hat{\psi}_a$ is a bounded function (being the spherical Fourier transform of an integrable function) and b is a rapidly decreasing function. The proof of (*) is a straightforward application of Fubini's theorem keeping in mind the following facts (i) Since $b \in \bar{Z}(F)$, $\psi_b \in S_1(G)$ and is hence integrable and further $\hat{\psi}_b = b$ (ii) ψ_a is an integrable function on G and $a(\lambda)$ is integrable with respect to $c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$. Since (*) is true $\forall b \in \bar{Z}(F)$ and since $\bar{Z}(F)$ contains 'enough' functions it follows easily that

$$a(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1} = \hat{\psi}_a(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1} \quad \text{a.e. on } \boldsymbol{a}^*$$

with respect to Lebesgue measure. But the zeros of $c(\lambda)^{-1}c(-\lambda)^{-1}$ must have zero Lebesgue measure in $\boldsymbol{a}^*$ and hence it follows that $a = \hat{\psi}_a$.

LEMMA 3.2. *Let k be a fixed nonnegative integer and let $\phi(z) = e^{\langle z,z\rangle^k}$, $z \in F$. Define X by $X = \{h;\ h \in \bar{Z}(F)$ and $h\phi \in \bar{Z}(F)\}$. Then X is a dense linear subspace of $\bar{Z}(F)$.*

Proof. Let $\psi_n(z) = e^{-\langle z,z\rangle^{k+1/n}}$. Then since $\langle\cdot,\cdot\rangle$ is W-invariant, ψ_n, ϕ are W-invariant. It is easy to see that ψ_n, $\phi\psi_n \in \bar{Z}(F)$. (To see this observe that $F = \boldsymbol{a}^* + iC_\rho$. Clearly ψ_n, $\phi\psi_n$ are rapidly decreasing on $\boldsymbol{a}^*$, but if $z \in F$ the 'imaginary' part of z varies only over a compact set.) Hence if $f \in \bar{Z}(F)$, $f\phi\psi_n \in \bar{Z}(F)$. Now it is easy to see that as $n \to \infty$, $f\psi_n \to f$ in the topology of $\bar{Z}(F)$. But since $f\psi_n\phi \in \bar{Z}(F)$, $f\psi_n \in X$ and the lemma is proved.

We are now in a position to state and prove our main theorem.

THEOREM 3.3. *Let $f \in I_1(G)$ and suppose*

(i) $\hat{f}$ is nowhere vanishing on F.

(ii) $\exists$ a positive integer k such that for every $u \in P(\boldsymbol{a}_c^*)$ with degree $u \leq m_G$ (where m_G is as in Lemma 3.1) we have

$$\sup_{z \in F^0} |\partial(u)[(\hat{f}(z))^{-1}e^{-\langle z,z\rangle^k}]| < \infty .$$

Then the ideal generated by f is dense in $I_1(G)$.

Proof. (Note: condition (ii) says that '$\hat{f}$ does not vanish too fast at ∞'.) Let X be as in Lemma 3.2. Let $Y = \{\psi_a; a \in X\}$. Since by Lemma 3.2 X is dense in $\bar{Z}(F)$, by Theorem 2.1, Y is dense in $S_1(G)$. Hence since $S_1(G) \hookrightarrow I_1(G)$ is a dense inclusion, Y is a dense subspace of $I_1(G)$. We will show that every $h \in Y$ can be written as $h = f * g$, with $g \in I_1(G)$ and this will prove the theorem. Now if $h \in Y$, $\hat{h} \in X$ and $\hat{h} = \hat{f} \cdot \hat{f}^{-1}\hat{h}$.

(Note that since $\hat{f}$ does not vanish on F, $\hat{f}^{-1}$ is well defined on F.)

Now we claim $\hat{f}^{-1}\hat{h}$ is in $\bar{Z}_{m_G}(F)$. This follows from the definition of X and condition (ii) of Theorem 3.3 (since $\hat{f}(z)^{-1}h(z) = \hat{f}(z)^{-1} e^{-\langle z,z\rangle^k}e^{\langle z,z\rangle^k}\hat{h}(z)$). Hence by Lemma 3.1 $\psi_{\hat{f}^{-1}\hat{h}} \in I_1(G)$ and $\hat{\psi}_{\hat{f}^{-1}\hat{h}} = \hat{f}^{-1}\hat{h}$.

Claim: $h = f * \psi_{\hat{f}^{-1}\hat{h}}$. This is because

$$(f * \psi_{\hat{f}^{-1}\hat{h}})\hat{} = \hat{f}\hat{\psi}_{\hat{f}^{-1}\hat{h}} = \hat{f}\hat{f}^{-1}\hat{h} = \hat{h} .$$

Hence (by the semi simplicity of $I_1(G)$) $f * \psi_{\hat{f}^{-1}\hat{h}} = h$. Thus we have shown that every function h in a dense subspace Y of $I_1(G)$ can be writted as $h = f * g$ and this concludes the proof of our theorem.

(Note: For $G = \mathrm{SL}(2, \boldsymbol{R})$ or more generally for G a real rank one group $m_G = 2$ (see [1], [6]).)

4. The case of L^p for $1 \leqq p \leqq 2$. For $\varepsilon \geqq 0$, let $F^\varepsilon = a^* + i\varepsilon C_\rho$. Then one can introduce the spaces $Z(F^\varepsilon)$, $\bar{Z}(F^\varepsilon)$ just as in § 2. Let $I_p(G) = I(G) \cap L^p(G)$. Then one can define the so called L^p-Harish Chandra-Schwartz subspace of K-biinvariant functions i.e., $S_p(G) \subseteqq I_p(G)$ (see [7] for details). Actually the theorem of Trombi-Varadarajan is more general than stated in § 2. In fact they show that under the map $f \to \hat{f}$ the spaces $S_p(G)$ and $\bar{Z}(F^\varepsilon)$ where $\varepsilon = 2/p - 1$ are topologically isomorphic ($p \leqq 2$). Also one knows that if $p \geqq 1$ then $S_1(G) \hookrightarrow S_p(G)$ is a dense inclusion. Using this one can modify the arguments in the last section to obtain the following theorem.

THEOREM 4.1. *Let* $1 \leqq p < 2$ *and* $f \in I_p(G) \cap I_1(G)$, *such that*:

(i) $\hat{f}$ *is nowhere vanishing on* F.

(ii) $\exists$ *a positive integer* k *such that for every* $u \in P(\boldsymbol{a}_c^*)$ *with degree* $u \leqq m_G$ (m_G *as in Lemma* 3.1), *we have*

$$\sup_{z \in F^0} |\partial(u)[(\hat{f}(z))^{-1}e^{-\langle z,z\rangle^k}]| < \infty .$$

Then the set of functions of the form $g*f$, $g \in I_c^\infty(G)$, *is dense in* $I_p(G)$.

Finally we observe that the Plancharel theorem for $I_2(G)$ (i.e., the spherical Fourier transform is an isometric isomorphism of $I_2(G)$ onto $L^2(\mathfrak{a}^*, \mu)^W$, where the superscript indicates Weyl-group invariance and μ is the measure on $\mathfrak{a}^*$ defined by $d\mu = |W|^{-1}c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$) gives us the following fact: Let $f \in I_2(G)$ such that $\hat{f}$ is nonvanishing on $\mathfrak{a}^*$ except possibly on a set of μ-measure zero. Then the set of functions of the form $g*f$, $g \in I_c^\infty(G)$ is dense in $I_2(G)$. (The proof of this fact is exactly as in the case of abelian groups).

ACKNOWLEDGMENT. The author would like to express his thanks to Professor V.S. Varadarajan for his interest in this work. Some suggestions of Professor M. S. Raghunathan were very helpful for some of the proofs. The author would also like to thank Dr. M.K.V. Murthy, Dr. S. C. Bagchi, Mr. S. Kumaresan and Mr. Adi Murthy for many stimulating conversations.

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Received January 9, 1979.

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