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CONTINUA IN THE STONE-CECH REMAINDER OF R^2

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In this paper it is shown that $\beta R^2 - R^2$ contains 2^c nonhomeomorphic continua. This extends the result already known for dimension three and greater.

Introduction. In [5], it is shown that for $n \geqq 3$, there are 2^c nonhomeomorphic continua in $\beta R^n - R^n$. The proof involves embedding solenoids in R^3, and hence does not work for the cases $n = 1, 2$. In this paper, we prove that $\beta R^2 - R^2$ also contains 2^c nonhomeomorphic subcontinua. While this implies the result for $(n) \geqq 3$, the construction in [5] also exhibits c continua in $\beta R^3 - R^3$ with nonisomorphic first Cech cohomology groups, and 2^c compacta in $\beta R^3 - R^3$, no two of which have the same shape. Also, it seems reasonable that the continua constructed in $\beta R^3 - R^3$ may be shown to have different shapes, or even nonisomorphic first Cech cohomology groups. In the case of $\beta R^2 - R^2$, it seems unlikely that any additional shape-theoretic results can be obtained with this construction. The case $n = 1$ is yet unsolved.

Preliminaries. Let βX denote the Stone-Cech compactification of a space X. For references, see Gillman and Jerison [1], or Walker [4]. The Stone-Cech remainder of X, $\beta X - X$, will be denoted by X^*. Note that the remainder of a closed subset of R^n is contained in $\beta R^n - R^n$. Also, the image under a rotation of R^2 of a set in R^2 of the form $\{(x, y): x \geqq 0,\ \alpha \leqq y \leqq \gamma;\ \alpha,\ \gamma \in R\}$ will be called a thickened ray.

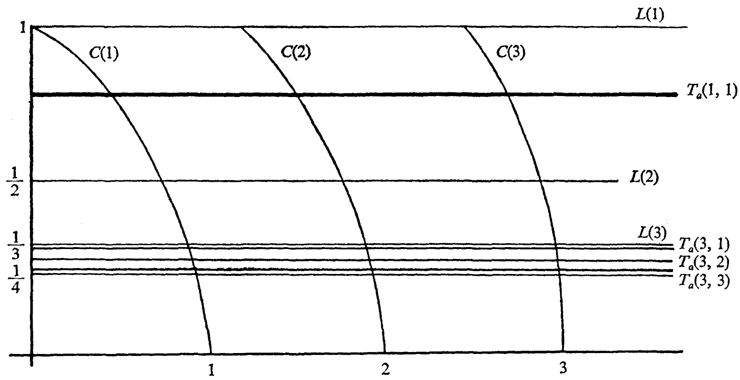
Main result.

THEOREM. *There are 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.*

Proof. For the sake of clarity, we consider first the construction of c nonhomeomorphic continua in $\beta R^2 - R^2$. We will then apply these arguments and results in the construction of 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

Consider a collection $\{P_a: a \in \mathscr{A}\}$ where each P_a is an infinite subset of positive integers; for $a \neq b$, either $P_a - P_b \neq \emptyset$ or $P_b - P_a \neq \emptyset$; and $\text{card}\,\mathscr{A} = c$. For $p \in P_a$, consider the two rays $\{(x, y): x \geqq 0,\ y = 1/p\}$ and $\{(x, y): x \geqq 0,\ y = 1/(p + 1)\}$. Between these rays, consider p disjoint thickened rays, say $T_a(p, n)$, where $n = 1, 2, \cdots, p$, and labeled so that if $n_1 < n_2$, the y-coordinate of any point in $T_a(p, n_1)$ is greater than the y-coordinate of points in $T_a(p, n_2)$.

Let $L(n) = \{(x, y): x \geqq 0,\ y = 1/n\}$, and let $C(n) = \{(x, y): x^2 + y^2 = n,\ x \geqq 0,\ 0 \leqq y \leqq 1\}$. Hence we have the following situation:



The continuum X will be formed as follows. Let T_a denote the union of the Stone-Cech remainders of the thickened rays $T_a(p, n)$, L the union of the remainders of the rays $L(n)$, and C the remainder of the union of the curves $C(n)$. X will be the closure in βR^2 of the union of these sets, i.e. $X = \bar{T}_a \cup \bar{L} \cup C$. One can verify that X is a continuum $\beta R^2 - R^2$. (Note that X is not the Stone-Cech remainder of the closure in R^2 of the union of the rays and curves.) For a different subset P_b of positive integers, we define T_b analogously, and let $Y = \bar{T}_b \cup \bar{L} \cup C$. Then Y is also a continuum in $\beta R^2 - R^2$.

We will show that X and Y are not homeomorphic. Suppose h is a homeomorphism from X onto Y. We begin by showing that $h(\bar{T}_a) = \bar{T}_b$.

Suppose $x \in T_a^*(p, n) = \beta(T_a(p, n)) - T_a(p, n))$ for some $p \in P_a$, $1 \leqq n \leqq p$, so that x is not an element of $\overline{C - T_a}$. Then, since $T_a^*(p, n) \cap \bar{L} = \emptyset$, there is a neighborhood $N(x)$ of x in X such that $N(x) \subseteqq T_a^*(p, n)$. Suppose $h(x)$ is not an element of $\bar{T}_b$. Then $h(x) \in \bar{L}$ or $h(x) \in C - (\bar{L} \cup \bar{T}_b)$. But $C - (L \cup \bar{T}_b)$ is open in Y, so each point of $C - (\bar{L} \cup \bar{T}_b)$ has a neighborhood of dimension $\leqq 1$, since $\dim(C) = 1$. Since any neighborhood of x has dimension 2 (by claim 2, Theorem 6 of [5]), $h(x)$ cannot be an element of $C - (\bar{L} \cup \bar{T}_b)$. Hence, $h(x) \in \bar{L}$. Then $h(N(x))$ is a neighborhood of $h(x)$, which implies there is a point $y \in L$ such that $y \in h(N(x))$. But since $y \in L$, y has neighborhoods of dimension $\leqq 1$, while every neighborhood of $h^{-1}(y)$ has dimension 2, since $h^{-1}(y) \in N(x)$ and $N(x) \subseteqq T_a^*(p, n)$. This is a contradiction, and so $h(x) \in \bar{T}_b$.

By an argument similar to the proof of claim 3, Theorem 6 of

[5], every point of $T_a^*(p, n)$ is a limit point of such points x, so $h(T_a^*(p, n)) \subseteqq \bar{T}_b$, for every (p, n) with $p \in P_a$, $1 \leqq n \leqq p$. Therefore, $h(T_a) \subseteqq \bar{T}_b$, which implies $h(\bar{T}_a) \subseteqq \bar{T}_b$. Similarly, $h(\bar{T}_b) \subseteqq \bar{T}_a$, and so $h(\bar{T}_a) = \bar{T}_b$.

Now, h must take the isolated components of $\bar{T}_a$ to the isolated components of $\bar{T}_b$. These are precisely the sets $T_a^*(p, n)$ and $T_b^*(q, m)$, respectively. So, for every (p, n) with $p \in P_a$, $1 \leqq n \leqq p$, we have $h(T_a^*(p, n)) = T_b^*(q, m)$ for some $q \in P_b$, $1 \leqq m \leqq q$.

Since $a \neq b$, either $P_a - P_b \neq \emptyset$ or $P_b - P_a \neq \emptyset$, so without loss of generality assume $P_b - P_a \neq \emptyset$, and let $q \in P_b - P_a$. For some (p, n), $p \in P_a$, $1 \leqq n \leqq p$, $h(T_a^*(p, n)) = T_b^*(q, 1)$. We may assume $p < q$ since $p \neq q$. Then there are integers m, m' such that $1 \leqq m \leqq q$, $1 \leqq m' \leqq q$, with $h^{-1}(T_b^*(q, m)) = T_a^*(p, i)$ for some i, and $h^{-1}(T_b^*(q, m')) = T_a^*(p', n')$ for some $p' \in P_a$, $p' \neq p$, $1 \leqq n \leqq p'$, and $|m - m'| = 1$. Now, $T_b^*(q, m)$ and $T_b^*(q, m')$ separate Y into two connected components and one disconnected component (since $|m - m'| = 1$). However, $h^{-1}(T_b^*(q, m)) = T_a^*(p, i)$ and $h^{-1}(T_b^*(q, m')) = T_a^*(p', n')$ separate X into three connected components, since $p \neq p'$ This is a contradiction; hence X and Y are not homeomorphic.

So far, we have constructed c continua in $\beta R^2 - R^2$ no two of which are homeomorphic. We will now modify the construction to obtain 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

Let $S \subseteqq \mathscr{A}$ such that card $S = c$. There is a one-to-one correspondence between elements of S and real numbers r such that $0 \leqq r < 2\pi$. So, each $a \in S$ corresponds to a unique $r_a \in [0, 2\pi)$. Let $h_{r_a}: R^2 \to R^2$ be a rotation of R^2 by r_a radians. For each element, a, of S we will construct a continuum in the manner of the first section, except along the ray $h_{r_a}(\{(x, y): x \geqq 0,\ y = 0\})$. We will then take the union of these along with the Stone-Cech remainder of the set $\bigcup_{n \geqq 1}\{(x, y): x^2 + y^2 = n\}$. More precisely, let $R_a(p, n) = h_{r_a}(T_a(p, n))$, $p \in P_a$, $1 \leqq n \leqq p$, and $Q_a(n) = h_{r_a}(L(n))$. Then, let R_S denote the union of the Stone-Cech remainders of the thickened rays $R_a(p, n)$, where $a \in S$, $p \in P_a$, $1 \leqq n \leqq p$; Q the union of the remainders of the rays $Q_a(n)$; and K the remainder of the union of the circles $\{(x, y): x^2 + y^2 = n\}$, $n \geqq 1$. Let X be the closure in βR^2 of the union of the sets, i.e., $X = \bar{R}_S \cup \bar{Q} \cup K$. One can verify that X is a continuum. For another subset T of $\mathscr{A}$ such that $T \neq S$ and card $T = c$, we define R_T analogously, and let $Y = \bar{R}_T \cup \bar{Q} \cup K$. Then Y is also a continuum in $\beta R^2 - R^2$.

We will show that X and Y are not homeomorphic. Suppose h is a homeomorphism from X onto Y, and consider $\bar{R}_S \cup \bar{Q}$. Fix $a \in S$, and let N_1, N_2 be neighborhoods of the ray $h_{r_a}(\{(x, y): x \geqq 0,\ y = 0\})$ of radius 2,3 respectively. Let $f: R^2 \to [0, 1]$ be a continuous function such that $f(N_1) = 0$ and $f(R^2 - N_2) = 1$. Then f has a continuous extension, βf, to all of βR^2. For $p \in P_a$, $1 \leqq n \leqq p$ and $m \geqq 1$, since $R_a(p, n)$ and $Q_a(m)$ are contained in N_1, $\beta f(R_a^*(p, n))$ and $\beta f(Q_a^*(n))$ are both 0. On the other hand, if $a \neq a' \in S$, $q \in P_{a'}$, $1 \leqq n' \leqq q$, and $m' \geqq 1$, then outside of some compact set (that depends on a') $R_{a'}(q, n')$ and $Q_{a'}(m')$ are subset of N_2. Therefore, $\beta f(R_{a'}^*(q, n'))$ and $\beta f(Q_{a'}^*(m'))$ are both 1. This implies that the closure of the union of all sets of the form $R_a^*(p, n)(p \in P_a$, $1 \leqq n \leqq p)$ and $Q_a^*(m)$ $(m \geqq 1)$ is isolated in $\bar{R}_S \cup \bar{Q}$. Hence, an argument identical to the one in the preceding section shows that $h(\bar{R}_S) = \bar{R}_T$.

Now, h must take the isolated components of $\bar{R}_S$ to the isolated components of $\bar{R}_T$. These are precisely the sets $R_a^*(p, n)$, $a \in S$, and $R_b^*(q, m)$, $b \in T$, respectively. So for every $a \in S$ and (p, n) with $p \in P_a$, $1 \leqq n \leqq p$, we have $h(R_a^*(p, n)) = R_b^*(q, m)$, for some $b \in T$, $q \in P_b$, $1 \leqq m \leqq q$.

Either $S - T \neq \emptyset$ or $T - S \neq \emptyset$, so without loss of generality assume $T - S \neq \emptyset$, and let $b_0 \in T - S$. Let $q \in P_{b_0}$ and consider $R_{b_0}^*(q, 1)$. For some $a_0 \in S$, $p \in P_{a_0}$, and $1 \leqq n \leqq p$, $h(R_{a_0}^*(p, n)) = R_{b_0}^*(q, 1)$. Since $a_0 \neq b_0$, by an argument similar to the one used to show the continua in the first section were not homeomorphic, not every component of the form $R_{b_0}^*(q', m)$ can have as its inverse image under h a component of the form $R_{a_0}^*(p', n')$. Hence, there is an element a_1 of S, $p' \in P_{a_1}$, and $1 \leqq n' \leqq p'$, such that $a_1 \neq a_0$ and $h(R_{a_1}^*(p', n')) = R_{b_0}^*(q', m)$ for some $q' \in P_{b_0}$, $1 \leqq m \leqq q'$.

Now, $R_{a_0}^*(p, n)$ and $R_{a_1}^*(p', n')$ separate X into two connected components, each of which contains an infinite number of isolated components of $\bar{R}_S$. However, $h(R_{a_0}^*(p, n)) = R_{b_0}^*(q, 1)$ and $h(R_{a_1}^*(p', n')) = R_{b_0}^*(q', m)$ separate Y into either one connected and one disconnected component (in case $q = q'$, $m = 2$), or into two connected components where one contains an infinite number of isolated components of $\bar{R}_T$ and the other contains only a finite number of isolated components of $\bar{R}_T$.

Since h is an onto homeomorphism that takes the isolated components of $\bar{R}_S$ to the isolated components of $\bar{R}_T$, this is a contradiction. Hence, X and Y are not homeomorphic.

Since $\mathscr{A}$ contains 2^c subsets of cardinality c, there are 2^c choices for X, no two of which are homeomorphic. Hence, since there are at most 2^c continua in $\beta R^2 - R^2$, there are exactly 2^c nonhomeomorphic continua in $\beta R^2 - R^2$.

COROLLARY. *Let X and Y be as in the proof of the above theorem. Then there does not exist a continuous map $f: X \to Y$ that is a shape equivalence. In particular, X and Y are not* *homotopic.*

Proof. In [2], J. Keesling proved the following: Suppose Z is real compact and K is a continuum contained in $\beta Z - Z$. Then if $h(K) = L$ is any continuous map which is a shape equivalence, h is a homeomorphism. Hence, since X and Y are not homeomorphic, there does not exist such an f.

REMARK. In the first part of the proof of the theorem, it would have been simpler to let A be the union of the regular and thickened rays, along with the curves $C(n)$ and the positive x-axis, and let $X = \beta A - A \subseteq \beta R^2 - R^2$. However, in this case, any neighborhood of a point p in the remainder of the x-axis in X has dimension 2, yet is not in $\bar{T}_a$. The fact that any neighborhood of p has dimension 2 follows from the fact that if $\{B_k\}_{k=1}^{\infty}$ is a decreasing sequence of closed, n-dimensional sets in R^m, then for any point x in $B = \bigcap_{k\geqq 1} B_k^*$, any neighborhood of x in B has dimension n. To see that p is not in $\bar{T}_a$, let $h: R^2 \to [0, 1]$ where $h(\{(x, y): x \geqq 2,\ 0 \leqq y \leqq 1/(x^2)\}) = 1$, and $h(\{(x, y): x \geqq 2,\ y \geqq 1/x\}) = 0$. Then $h(T_a) = 0$ implies $\beta h(\bar{T}_a) = 0$, but $\beta h(p) = 1$. Thus, if we had used the above definition for X instead of the one given in the proof of the theorem, we would not have been able to show that the sets $T_a^*(p, n)$ were sent to the sets $T_b^*(q, m)$ under the homeomorphism.

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