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A THEOREM ON QUASI-PURE-PROJECTIVE TORSION FREE ABELIAN GROUPS OF FINITE RANK

C. VINSONHALER AND W. WICKLESS

It is shown that a quasi-pure-projective torsion free abelian group of finite rank is either strongly indecomposable or a direct sum of isomorphic rank one groups.

The purpose of this note is to record the author's theorem that quasi-pure-projective (qpp) torsion free abelian groups of finite rank are either strongly indecomposable or homogeneous completely decomposable.

This result has been in the folklore for several years. It is cited in [1], Theorem E (a classification theorem for finite rank qpp groups) and in [3], (a characterization of locally free quasi-pure-injective groups).

Throughout, "group" will mean torsion free abelian group of finite rank. Otherwise the terminology and notations are from [2].

DEFINITION 1. A group A is a quasi-pure-projective if for any pure subgroup K of A and homomorphism $\theta: A \to A/K$, there is an endomorphism $\bar{\theta}$ of A such that $\pi\bar{\theta} = \theta$, where $\Pi: A \to A/K$ is the natural factor map.

For the proof of our result it is convenient to consider what seems to be a generalization of the qpp property. (Actually, in [3] it is shown that the property defined below is equivalent to qpp in the finite rank case).

DEFINITION 2. A group A is almost quasi-pure-projective (aqpp) if there exists a fixed integer n such that for any pure subgroup K of A and homomorphism $\theta: A \to A/K$, there is an endomorphism $\bar{\theta}$ of A such that $\Pi\bar{\theta} = n\theta$, where $\Pi: A \to A/K$ is the natural factor map.

The integer n will be called the associated integer for the aqpp group A.

A sequence of lemmas leads to the result. The proof of the first one is routine.

LEMMA 1. *A direct summand of an* aqpp *group is* aqpp.

The proof of the next lemma is given in [3], § 5. The proof can also be obtained by slight modification of Lemma 4.1 of [1].

LEMMA 2. *Let A be a strongly indecomposable* aqpp *group and B be a proper pure subgroup of A. Then A and A/B are divisible by the same primes.*

Let $\langle g_1, g_2, \cdots, g_n \rangle_*$ denote the pure subgroup generated by the elements $g_1, g_2, \cdots, g_n$. (The group in which the pure subgroup is to be taken will be obvious in each instance where the notation is used.)

The next lemma is the key to the theorem.

LEMMA 3. *Let A and B be strongly indecomposable reduced groups such that $A \oplus B$ is* aqpp. *Then given $0 \neq x \in A$ and $0 \neq y \in B$, there is a homomorphism $f: B \to \langle x \rangle_*$ such that $f(y) \neq 0$.*

Proof. Let p be a prime such that $pB \neq B$, and write $n = p^k l$, with $(p, l) = 1$, where n is the associated integer for $A \oplus B$. Now let $\{y = y_1, y_2, \cdots, y_s\}$ be a maximal rationally independent set in B and let $B_1 = \{q_1 \mid q_1 y_1 + \cdots + q_s y_s \in B,\ q_j \in Q,\ 1 \leqq j \leqq s\}$. Since $B_1 \cong B/\langle y_2, \cdots, y_s \rangle_*$, Lemmas 1 and 2 imply that $pB_1 \neq B_1$. Let t be the largest integer such that $1/p^t \in B_1$. Form $H = \langle p^{k+2t+1}x - y_1, y_2, y_3, \cdots, y_s \rangle_*$ and consider the map $\theta: A \oplus B \to (A \oplus B)/H$ given by $\theta(a \oplus b) = \beta_1/p^{k+t+1}(y_1 + H)$ where $a \in A$ and $b = \sum_{j=1}^{s} \beta_j y_j \in B$. Note that θ is defined since $y_1 + H = p^{k+2t+1}(x + H)$. Since $A \oplus B$ is aqpp, there is a lifting $\bar{\theta}: A \oplus B \to A \oplus B$ such that $\Pi\bar{\theta} = n\theta$, where Π: $A \oplus B \to (A \oplus B)/H$ is the natural projection. Then $p^{t+1}\bar{\theta}(y_1)$ has the form $p^{t+1}\bar{\theta}(y_1) = np^{t+1}/(p^{k+t+1})y_1 + h = ly_1 + h$ where $h \in H$.

Let $\prod_A$ be projection of $A \oplus B$ onto A. If $\prod_A \bar{\theta}(y_1) = 0$, then in the above equation, $h \in B$. But then $1/p^{t+1}(ly_1 + h) \in B$, contradicting the choice of t. Thus $0 \neq \prod_A \bar{\theta}(y_1) \in \prod_A \bar{\theta}(B) \subseteqq \langle x \rangle_*$.

The final lemma is the motivation for considering aqpp groups. The proof is routine.

LEMMA 4. *If A is quasi-isomorphic to $B(A \sim B)$ and A is* aqpp, *then B is* aqpp.

Our theorem now follows easily, using Lemmas $1 - 4$.

THEOREM. *Let A be an* aqpp *group. Then A is either strongly indecomposable or is a direct sum of isomorphic groups of rank one.*

Proof. If A is not strongly indecomposable, write $A \sim A_1 \oplus \cdots \oplus A_r$, $r \geqq 2$, where each A_i is nonzero and strongly indecomposable

([2]), § 92). Given $0 \neq y \in A_1$, $0 \neq x \in A_2$, apply Lemmas 3 and 4 to obtain $f: A_1 \to \langle x \rangle_*$ with $f(y) \neq 0$ and (by symmetry) $g: A_2 \to \langle y \rangle_*$ with $g(x) \neq 0$. We conclude that type (x) = type (y) and $\langle x \rangle_* \cong \langle y \rangle_*$ ([2], p. 109 (D) and Th. 85.1).

But if $f: A_1 \to \langle x \rangle_*$ with $f(y) \neq 0$ and type $(x) \equiv$ type (y), then it immediately follows that $\langle y \rangle_*$ is a quasi-summand of A_1. To see this let $K = \operatorname{Ker} f$ and note that $\langle y \rangle_* \cap K = (0)$. Since A_1/K is isomorphic to a subgroup of $\langle x \rangle_*$ we have: type $\langle y \rangle_*$=type $(\langle y \rangle_* \oplus K)/K \leqq$ type $A_1/K \leqq$ type $\langle x \rangle_*$ = type $\langle y \rangle_*$. Thus, there exists an integer m such that $m(A_1/K) \subseteqq (\langle y \rangle_* \oplus K)/K$. This gives $A_1 \sim \langle y \rangle_* \oplus K$. Hence $A_1 = \langle y \rangle_*$. Similarly, all of the A_i must be of rank one, and having the same type, are isomorphic. Thus A is quasi-isomorphic to a direct sum of isomorphic groups of rank one. This implies that A is actually isomorphic to this sum by [2], Prop. 98.1.

REFERENCES

1. D. Arnold, B. O'Brien and J. Reid, *Quasi-pure injective and projective torsion free Abelian groups of finite rank,* London Math. Soc. Proc., **38** (1979), 532-544.

2. L. Fuchs, *Infinite Abelian Groups,* Academic Press, New York.

3. C. Vinsonhaler, *Almost quasi-pure injective Abelian groups,* Rocky Mt. J. Math., **9** (1979), 569-576.

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THE UNIVERSITY OF CONNECTICUT
STORRS, CT 06268