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THE LEVI DECOMPOSITION OF A SPLIT (B, N)-PAIR

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Let p be a prime number. If G is a finite group with a split (B, N)-pair of characteristic p then each parabolic subgroup G_J of G can be written as a semidirect product of certain subgroups C_J and L_J. Moreover G_J is the full normalizer of C_J in G.

1. **Introduction.** Let G be a Chevalley group. The set of its parabolic subgroups $\{G_J \mid J \subseteqq R\}$ is indexed by the subsets of the set R of fundamental roots of the associated Lie algebra. Each G_J admits a decomposition of the following form:

$$G_J = C_J L_J$$

where $C_J \trianglelefteq G_J$ and $C_J \cap L_J = \{1\}$. Furthermore, G_J is the normalizer of C_J in G. This decomposition of G_J into the semidirect product of C_J and L_J is called the Levi decomposition and L_J and its conjugates in C_J are called the Levi subgroups of G_J (see [2, p. 118-119]). In this paper we show that if G is a finite group with a split (B, N)-pair of characteristic p then the parabolic subgroups of G admit a similar decomposition.

The difficulty in proving the existence of the Levi decomposition for an arbitrary finite group with a split (B, N)-pair is showing that $C_J \trianglelefteq G_J$ (Lemma 1) and that L_J is itself a group with a split (B, N)-pair (Lemma A). Curtis proves these facts in [5, Proposition 1.5(a), (d), p. 669] and concludes that G admits a Levi decomposition. However, his arguments depend on the use of the commutator relations ([5, Proposition 1.4(f), p. 669]) and the proof of these relations relies heavily on the Fong-Seitz classification ([6], [7]) of split (B, N)-pairs of rank 2. It is the advantage of this present note to prove the required facts without these commutator relations, but under the assumption

$$(*) \qquad U \cap U^{w_i} \trianglelefteq U \quad \text{for all} \quad w_i \in R\,.$$

The reader should note that (*) appears as a hypothesis in [7, Theorem D, p. 238]. Moreover, in the case when p is an odd prime (*) follows using a very strong result on 2-transitive permutation groups due to Kantor and Seitz ([8, Theorem C′, p. 131]. See [10, proof of Theorem 4.5].). That result is essential to the Fong-Seitz classification (see [6, p. 2]). By assuming (*) we too then employ the Kantor-Seitz result; however, since we do not refer to the Fong-

Seitz papers, we have achieved a substantial simplification of the existing proof.

Throughout our discussion $G = (G, B, N, R, U)$ will denote a finite group with a split (B, N)-pair of characteristic p and rank n (see [4, Definition 2.1, p. B-8]). Hence G satisfies the following conditions:

(i) G has a (B, N)-pair ([4, Definition 2.1, p. B-8]) where $H = B \cap N$ and the Weyl group $W = N/H$ is generated by the set of involutions $R = \{w_1, \cdots, w_n\}$.

(ii) $H = \bigcap_{n \in N} n^{-1}Bn$.

(iii) U is a normal p-subgroup of B; $B = U \cdot H$ is a semidirect product and H is abelian with order prime to p.

Notice that (iii) tells us that we always have a Levi decomposition in the case $J = \Phi$, $G_\Phi = B$.

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2. Preliminaries. The Weyl group of a (B, N)-pair is isomorphic to the Weyl group of a root system in Euclidean space in such a way that R corresponds to the set of fundamental reflections (see [9, p. 439]). We therefore define $\Delta = \{a_i \mid w_i \in R\}$ to be the set of fundamental roots of this root system.

Let $\nu\colon N \to W$ be the natural epimorphism. For each subset $J \subseteq R$, the parabolic subgroup $G_J = (G_J, B, N_J, J, U)$ is an unsaturated split (B, N)-pair of characteristic p and rank $|J|$ where $W_J = \langle w_i \mid w_i \in J\rangle$ and $N_J = \nu^{-1}(W_J)$ (see [1, Proposition 1, p. 28]). The group $G_J = BN_JB$ is unsaturated (see [10]) since $\bigcap_{n \in N_J} n^{-1}Bn$ may be larger than H; that is, $\bigcap_{n \in N_J} U^n > 1$. Any $w \in W$ can be written as a minimal product of the generators in R. We denote by $l(w)$ the length of such an expression. For each $J \subseteq R$, w_J will denote the unique element of maximal length in W_J. In the case $J = R$ we write w_0 for w_R. If X is any subset of G and $g \in G$, then $X^g = g^{-1}Xg$.

DEFINITIONS. Let $w \in W$. Then ${}_wU^- = U \cap U^{w_0w}$, ${}_wU^+ \cap U^w$. Write ${}_{w-1}U^-$ as U_w^- and ${}_{w-1}U^+$ as U_w^+. In the case $w = w_i$ we write U_i for ${}_{w_i}U^-$. Let $V_i = U_i^{w_i}$ and $V = U^{w_0}$. Set $G_i = \langle U_i, V_i\rangle$ and $H_i = H \cap G_i$.

Let $(w_i) \in N$ be such that $(w_i)H = w_i (w_i \in R)$. As in [3, Lemma 2.2, p. 351] we choose

(a) $$(w_i) \in G_i \quad \text{for each} \quad w_i \in R\,.$$

In which case

(b) $$G_i = U_iH_i \cup U_iH_i(w_i)U_i$$

for all $w_i \in R$ ([3, Lemma 2.7, p. 351]).

DEFINITIONS. For each $J \subseteqq R$ let $L_J = \langle H, (U_i)^w \mid w \in W_J, w_i \in J \rangle$. Set $U_J = {}_{w_J}U^-$, $B_J = HU_J$.

Notice that $L_J = \langle H, (G_i)^w \mid w \in W_J, w_i \in J \rangle$ and that

(c) $$L_J = \langle H, G_i \mid w_i \in J \rangle$$

by our choice of representatives (a).

LEMMA A. *If G is a finite group with a split (B, N)-pair then (L_J, B_J, N_J, J, U_J) is a split (B, N)-pair for any $J \subseteqq R$.*

Curtis proved Lemma A using the commutator relations ([5, Proposition 1.4(f), Proposition 1.5(d), p. 669]). The following proof was suggested to the author by J. A. Green. We first require:

LEMMA B. *Let $w \in W_J$. Then ${}_wU^- \subseteqq U_J$. In particular $U_i \subseteqq U_J$ for all $w_i \in J$.*

Proof. Write $w_J = vw$ with $l(v) + l(w) = l(w_J)$. By [10, Lemma 2.2], $U_J = {}_wU^-({}_vU^-)^w$ and the result follows.

Proof of Lemma A. We verify the (B, N)-pair axioms as given, for example in [4, p. B-8]:

(i) $L_J = \langle B_J, N_J \rangle$ and $B_J \cap N_J \leqq N_J$.

Let $w_J = w_{i_1} \cdots w_{i_q}$ be a reduced expression for w_J with all $w_{i_s} \in J$ $(1 \leqq s \leqq q)$. Then

$$U_J = (U_{i_q})(U_{i_{q-1}})^{w_{i_q}} \cdots (U_{i_1})^{w_{i_2} \cdots w_{i_q}}$$

by [4, Proposition 3.3(vi), p. B-13]. Hence $U_J \leqq L_J$ and $B_J \subseteqq L_J$ since $H \subseteqq L_J$. By (a) and (c) L_J contains each (w_i) where $w_i \in J$ so that $N_J \subseteqq L_J$. Hence $\langle B_J, N_J \rangle \subseteqq L_J$. Conversely, if $w_i \in J$ then $U_i \subseteqq U_J \subseteqq \langle B_J, N_J \rangle$ by Lemma B and $Hw \subseteqq N_J \subseteqq \langle B_J, N_J \rangle$ all $w \in W_J$. Therefore $(U_i)^w \subseteqq \langle B_J, N_J \rangle$ all $w \in W_J$ and $L_J \subseteqq \langle B_J, N_J \rangle$. We also have that $H \subseteqq B_J \cap N_J \subseteqq B \cap N = H$ and (i) is proved.

(ii) The finite group $W_J \cong N_J/(N_J \cap B_J) = N_J/H$ is generated by the set J of involutions.

(iii) For all $w_i \in J$ and $w \in W_J$ there holds $w_i B_J w \subseteqq B_J w B_J \cup B_J w_i w B_J$. To prove (iii) we need only show that

(1) $$(w_i)u(w) \in B_J w B_J \cup B_J w_i w B_J$$

for any $u \in U_J$. By [4, Proposition 3.3(iii), p. B-13] we may write $u = u_1 u_2$ with $u_1 \in U_w^-$, $u_2 \in U_w^+$. By Lemma B, $u_1 \in U_J$. So $u_2 \in U_J$ and $u_2 \in U_w^+ \cap U_J = U \cap U^{w^{-1}} \cap U \cap U^{w_0 w_J}$ and $(w)^{-1}u_2(w) \in U^w \cap U \cap U^{w_0 w_J w} \subseteqq {}_{w_J w}U^- \subseteqq U_J$ by Lemma B. Therefore

$$(w_i)u(w) = (w_i)u_1(w)(w)^{-1}u_2(w) \in (w_i)u_1(w)B_J .$$

It is therefore sufficient to prove (1) for any $u(=u_1) \in U_w^+$. We examine the following two cases:

Case I. $l(w_iw) = l(w) + 1$. By [4, Proposition 3.3(i), p. B-13] and Lemma B

$$(U_w^-)^{w_i} \subseteqq ({}_{w_i}U^-)\cdot(U_w^-)^{w_i} = U_{w_iw}^- \subseteqq U_J .$$

If $u \in U_w^-$ then

$$(w_i)u(w)B_J = (w_i)u(w_i)^{-1}(w_i)(w)B_J \subseteqq U_Jw_iwB_J .$$

Hence if

$$(2) \qquad l(w_iw) = l(w) + 1 \quad \text{then} \quad (w_i)u(w) \in B_Jw_iwB_J$$

for any $u \in U_J$.

Case II. $l(w_iw) = l(w) - 1$. Writing $v = w_iw$ we then have $w = w_iv$ with $l(v) + 1 = l(w_iv)$ and as above

$$U_w^- = U_{w_iv}^- = {}_{w_i}U^-(U_v^-)^{w_i} = (U_v^-)^{w_i}U_i .$$

Therefore $(U_w^-)^{w_i} = (U_v^-)(U_i)^{w_i} \subseteqq U_J(U_i)^{w_i}$ by Lemma B. If $u \in U_w^-$ we have $(w_i)u(w) = (w_i)u(w_i)^{-1}(w_i)(w) \in B_JgvB_J$ for some $g \in G_i$. From (b) *either* g lies in $U_iH_i \subseteqq B_J$ in which case $(w_i)u(w_i) \in B_Jw_iwB_J$ *or* g lies in $U_iH_i(w_i)H_i \subseteqq B_Jw_iB_J$ in which case $(w_i)u(w) \in B_Jw_iB_JvB_J \subseteqq B_Jw_ivB_J = B_JwB_J$ using (2) (with v replacing w). Thus (1) holds in Case II.

(iv) For all $w_i \in J$, $w_iB_Jw_i \neq B_J$. Now $U_i \subseteqq B_J$ so that $w_iB_Jw_i \supseteqq (U_i)^{w_i}$. If (iv) were false then $w_iB_Jw_i = B_J \supseteqq (U_i)^{w_i}$ so that $(U_i)^{w_i} \subseteqq U$ contrary to [4, Proposition 3.3(v), p. B-13].

The (B, N)-pair is saturated since

$$(U_J)^{w_J} \cap U_J = U \cap U^{w_0w_J} \cap U^{w_J} \cap U^{w_0} \subseteqq U \cap V = \{1\} .$$

3. *Proof of the Theorem.* We now state our theorem and prove it by a succession of lemmas. Assume (*) holds:

THEOREM (*Levi Decomposition*). *Let $G = (G, B, N, R, U)$ be a finite group with a split (B, N)-pair of characteristic p. For each subset $J \subseteqq R$, there exist subgroups C_J and L_J such that*

(a) *$G_J = C_JL_J$ where $C_J \trianglelefteq G_J$ and $C_J \cap L_J = \{1\}$.*

(b) *The normalizer in G of C_J is G_J.*

Fix $J \subseteqq R$. Let $C_J = \bigcap_{n \in N_J} U^n$. Then $C_J = U \cap U^{w_J}$ and $C_J^w = C_J$ for all $w \in W_J$ by [10, Lemma 2.1 and the subsequent remark]. It can easily be shown that C_J is generated by certain root subgroups of G as in [5, p. 669] or [2, p. 119]. In the special case $J = R$ we know that $C = C_R = U \cap U^{w_0} = \{1\}$ since G is saturated.

LEMMA 1. $C_J \trianglelefteq G_J$.

Proof. The result follows by [10, Lemmas 4.2 and 4.3].

LEMMA 2. *Let* $L_J = \langle H, G_i \mid w_i \in J \rangle$. *Then* $G_J = \langle C_J, L_J \rangle$.

Proof. Notice that $\langle C_J, L_J \rangle = \langle C_J, B_J, N_J \rangle$ by (i) in the proof of Lemma A so that $\langle C_J, L_J \rangle = \langle C_J, U_J, N_J \rangle = \langle B, N_J \rangle$ by [4, Proposition 3.3(ii), p. B-13] and the result follows.

Since $C_J \trianglelefteq G_J$,

LEMMA 3. $G_J = C_J L_J$.

LEMMA 4. $U \cap L_J \subseteqq U_J$.

Proof. By Lemma A, we have a Bruhat Decomposition

$$L_J = \bigcup_{w \in W_J} B_J w B_J \,.$$

If $w \neq 1$, $w \in W_J$ then $B_J w B_J$ does not intersect B since $B_J w B_J \subseteqq BwB$ and $BwB \cap B$ is empty. Hence $B \cap L_J \subseteqq B_J$ and Lemma 4 follows.

LEMMA 5. $C_J \cap L_J = \{1\}$.

Proof. $C_J \cap L_J = C_J \cap U \cap L_J \subseteqq C_J \cap U_J = \{1\}$ by [4, Proposition 3.3(iii), p. B-13].

The proof of the following lemma is based on [2, p. 120].

LEMMA 6. *The normalizer*, $N_G(C_J)$, *of* C_J *in* G *is* G_J.

Proof. We know that $G_J \subseteqq N_G(C_J)$ so that $N_G(C_J) = G_K$ with $J \subseteqq K$. If $J \subset K$ take $w_i \in K$, $w_i \notin J$. Then by [4, Proposition 3.3(v), p. B-13], $U_i \subseteqq {}_wU^+$ for any $w \in W_J$ since $w(a_i) > 0$ all $w \in W_j$. But $C_J = \bigcap_{w \in W_J} {}_wU^+$ so that $U_i \subseteqq C_J$. Since $w_i \in G_K$, $U_i^{w_i} \subseteqq C_J^{w_i} = C_J$. On the other hand $U_i^{w_i w_0} \subseteqq U$ by [4, Proposition 3.3(v), p. B-13] since $w_i(a_i) = -a_i$. Therefore, $U_i^{w_i} \subseteqq C_J \cap V \subseteqq C \cap V = \{1\}$ and $U_i = \{1\}$, contrary to the (B, N)-pair axioms since for all $w_i \in R$, $w_i U w_i \neq U$ and $U = U_{iw_i} U^+$ (see [4, Proposition 3.3(iii), p. B-13]).

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