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A CHARACTERIZATION OF THE LOCAL RADON-NIKODYM PROPERTY BY TENSOR PRODUCTS

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In this paper, results are presented that characterize the collection of all vector valued measures expressible as an indefinite Bochner integral. More precisely, if X is a Banach space, an X-valued vector measure, τ, defined on a measurable space (S, Ω) is expressible as a Bochner integral if and only if τ belongs to $ca(S, \Omega)\hat{\otimes}_\pi X$, where $\hat{\otimes}_\pi$ denotes the strong (or projective) tensor product of two Banach spaces. Other related results are given.

Introduction. Throughout this paper, (S, Ω) will denote a measurable space and X a Banach space. By $ca(\Omega)$ $[cafv(\Omega; X)]$ we mean the Banach space of all real valued (resp., X-valued) countably additive set functions with finite variation, equipped with the total variation norm $|\cdot|$. Generally, we use the basic notions and notation in Dunford and Schwartz [2].

A vector valued measure $\tau \in cafv(\Omega: X)$ is said to have the Radon-Nikodym property if whenever $\lambda \in ca(\Omega)$ is a positive measure such that $\tau \ll \lambda$ (that is, $|\tau(E)| \to 0$ whenever $\lambda(E) \to 0$), then there exists a Bochner integrable function $f: S \to X$, (see pages 144-154 in [2]) such that

$$\tau(E) = \int_E f d\lambda \text{ for all } E \in \Omega .$$

In this case, f is called the Radon-Nikodym derivative of τ with respect to λ. The space of Bochner integrable functions from S into X with respect to a scalar measure λ is denoted $B(S, \Omega, \lambda; X)$; the space of all X-valued measures on Ω that have the RN (Radon-Nikodym) property is denoted $RNca(\Omega; X)$, and forms a closed linear subspace of $cafv(\Omega; X)$.

The Main Results. The RN property of a measure is important in classifying certain tensor products of spaces of measures. In preparation for this, we establish an important lemma.

LEMMA 1. *Suppose $\tau \in cafv(\Omega; X)$ such that $\tau \ll \lambda$ and $\lambda \ll \nu$, for some two positive measures λ and ν on Ω. If τ has a Radon-Nikodym derivative with respect to ν, then it has a derivative with respect to λ.*

Proof. By the Lebesgue Decomposition theorem, there exists positive measures μ and σ such that $\nu = \mu + \sigma$ and $\mu \ll \lambda$ and $\sigma \perp \lambda$. Since $\sigma \perp \lambda$, there exists a set $E_0 \in \Omega$ with $\sigma(E_0) = 0$ and $\lambda(S - E_0) = 0$. From $\mu \ll \lambda$, there exists an $h \in L_1^+(S, \Omega, \lambda)$ such that

$$\mu(E) = \int_E h d\lambda \text{ for all } E \in \Omega .$$

Let f denote the derivative of τ with respect to ν, then for $E \in \Omega$.

$$\tau(E) = \int_E f d\nu = \int_E f d\mu + \int_E f d\sigma = \int_E f h d\lambda + \int_E f d\sigma .$$

It is easily seen that $\int_E f d\sigma = \int_{EE_0} f d\sigma + \int_{E-E_0} f d\sigma = 0$ for all $E \in \Omega$. Thus, $\tau(E) = \int_E f h d\lambda$ and, therefore, fh is the Radon-Nikodym derivative of τ with respect to λ.

THEOREM 2. *Let $\{\tau_k\} \subseteq cafv(\Omega; X)$ be a sequence of vector measures such that $\sum_{k=1}^{\infty} |\tau_k|(S) < +\infty$. If τ_k has the RN property for each k, then so does $\tau = \sum_{k=1}^{\infty} \tau_k$.*

Proof. Suppose $\lambda \in ca(\Omega)$ is a positive measure such that $\tau \ll \lambda$. Note that $\sum |\tau_k|(E)$ converges absolutely for each $E \in \Omega$, consequently, $\sum |\tau_k|$ defines a σ-additive measure on Ω such that $\tau_n \ll \sum |\tau_k|$ for each n.

Define $\nu = \lambda + \sum |\tau_k|$. Then ν is a positive measure on Ω such that $\lambda \ll \nu$; consequently, $\tau \ll \lambda \ll \nu$. It suffices, in view of Lemma 1 to show that τ has a derivative with respect to ν.

Indeed, for each n, $\tau_n \ll \nu$ and τ_n has the RN property implies there exists a function $f_n \in B(S, \Omega, \nu; X)$ such that $\tau_n(E) = \int_E f_n d\nu$. It is easily seen that $\sum_{n=1}^{\infty} f_n$ converges in $B(S, \Omega, \nu; X)$. Therefore, if we define $f = \sum f_n$ it is seen that

$$\tau(E) = \sum \tau_n(E) = \sum \int_E f_n d\nu = \int_E \sum f_n d\nu = \int_E f d\nu .$$

Thus, f is the derivative of τ with respect to ν.

We now present the main result of this paper which constitutes a generalization of a theorem of Gil de Lamadrid (Theorem 4.2 [3]). In his paper, he identifies $C^*(H) \hat{\otimes}_\pi X$ as the class of all regular X-valued Radon measures of bounded variation which can be represented as an absolutely convergent series of "step measures." In his paper, H is a compact Hausdorff space, and, of course $C^*(H)$ is the space of all regular Radon measures on H.

THEOREM 3. *Let (S, Ω) be a measurable space and X a Banach space, then $ca(\Omega)\hat{\otimes}_\pi X = RNca(\Omega; X)$ isometrically.*

Indication of Proof. In [5], we show that $ca(\Omega)\hat{\otimes}_\pi X$ can be isometrically embedded in $cafv(\Omega; X)$ by the canonical isomorphism

$$\sum_{i=1}^{k} \mu_i \otimes x_i \longrightarrow \sum_{i=1}^{k} x_i\mu_i(\cdot) .$$

To prove $ca(\Omega)\hat{\otimes}_\pi X = RNca(\Omega; X)$, let $\tau \in RNca(\Omega; X)$. Put $\lambda = |\tau|$, then $\tau \ll \lambda$. Since τ has the RN property, there exists a function $f \in B(S, \Omega, \lambda; X)$ such that $\tau(E) = \int_E f d\lambda$ for all $E \in \Omega$.

Because f is Bochner integrable, f can be written in the form $f = \sum_{n=1}^{\infty} x_n \xi_{E_n} \lambda$ — a.e, where $x_n \in X$, $E_n \in \Omega$, and $\sum_{n=1}^{\infty} |x_n| \cdot \lambda(E_n) < +\infty$ (see Brooks [1]). Here ξ_E is the characteristic function of the set E.

Define $\tau_n \colon \Omega \to X$ for each positive integer n by $\tau_n(E) = x_n \cdot \lambda(EE_n)$. τ_n is easily seen to have the RN property and $\tau_n \in ca(\Omega)\hat{\otimes}_\pi X$. Furthermore,

$$\sum_{k=1}^{\infty} |\tau_k|(S) = \sum_{k=1}^{\infty} |x_k| \lambda(E_k) < +\infty . \tag{1}$$

Thus, we have

$$\tau(E) = \int_E f d\lambda = \int_E \sum x_k \xi_{E_k} d\lambda = \sum x_k \lambda(EE_k) ,$$

or,

$$\tau(E) = \sum_{k=1}^{\infty} \tau_k(E) \text{ for each } E \in \Omega . \tag{2}$$

As remarked above $\tau_k \in ca(\Omega)\hat{\otimes}_\pi X$, hence $\sum_{k=1}^{n} \tau_k \in ca(\Omega)\hat{\otimes}_\pi X$ also. Note that (1) implies that the sequence $\{\sum_{k=1}^{n} \tau_k\}$ is Cauchy in $ca(\Omega)\hat{\otimes}_\pi X$, because the variation norm is the same as the π-norm. But by (2), $\sum_{k=1}^{\infty} \tau_k$ converges setwise to τ, therefore in variation (π-norm). Thus $\tau \in ca(\Omega)\hat{\otimes}_\pi X$.

Conversely, if $\tau \in ca(\Omega)\hat{\otimes}_\pi X$, by the general theory of projective tensor products (see Trèves [6]), there exists $x_n \in X$ and $\lambda_n \in ca(\Omega)$ such that $\sum_{k=1}^{\infty} |x_k| |\lambda_k|(S) < +\infty$ and $\tau(E) = \sum_{k=1}^{\infty} x_k \lambda_k(E)$ for all $E \in \Omega$. Write $\tau_k = x_k \lambda_k$, then clearly τ_k has the RN property, $\tau = \sum \tau_k$ and $\sum |\tau_k| < +\infty$. By Theorem 2, τ has the RN property.

COROLLARY 1. *A measure $\tau \in cafv(\Omega; X)$ has the RN property if and only if τ is expressible as the indefinite Bochner integral with respect to some positive measure.*

Recall that a Banach space X has the Radon-Nikodym property if it's true that any X-valued vector measure of finite variation can be expressed as an indefinite Bochner integral.

COROLLARY 2. *A Banach space X has the Radon-Nikodym property if and only if $ca(S, \Omega) \hat{\otimes}_\pi X = cafv(S, \Omega; X)$ for every measurable space (S, Ω).*

REMARKS. In particular, if X is reflexive or a separable dual space, then $ca(\Omega) \hat{\otimes}_\pi X = cafv(S, \Omega; X)$ for every measurable space (S, Ω). It has been shown that $ca(S, \Omega) \hat{\otimes}_\pi X$ is the Banach space, with total variation norm, of all X-valued measures on Ω with the RN property; for sake of completeness, it has been shown in [4] and [5], that $ca(\Omega) \hat{\otimes}_\varepsilon X$, where $\hat{\otimes}_\varepsilon$ is the weak (or inductive) tensor product, is the Banach space of all X-valued vector measures with relatively norm compact range, equipped with the semi-variation norm. In conclusion, the following question is posed: can the criterion of Corollary 2 be used to give an "external" proof of the fact that reflexive Banach spaces and separable dual spaces have the Radon-Nikodym property?

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