SINGULARITIES OF SOLUTIONS TO LINEAR SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS BY APPROXIMATION METHODS

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Let the linear second-order elliptic partial differential equation be given in the normal form

$$\Delta^2 v + a(x, y)v_x + b(x, y)v_y + c(x, y)v = 0, \qquad (x, y) \in E^2$$

with real-valued coefficients that are entire functions on $\mathscr{C}^2$ and whose coefficient $c(x, y) \leq 0$ on the disk D: $x^2 + y^2 \leq 1$. Let the initial domain of definition of the real-valued regular solution $v = v(x, y)$ be D. A local Chebyshev approximation scheme is given by which global information is determined concerning the location of the singularities of the principal branch of the analytic continuation of v. This follows from an error analysis of best approximates taken over certain families of regular solutions whose singularities are in comp (D). The Bergman and Gilbert Integral Operator Method is utilized in this function-theoretic extension of the theorems of S. N. Bernstein and E. B. Saff; these theorems classify the singularities of analytic functions of one complex-variable via the growth in the error of Chebyshev approximations taken over rational functions of type (n, ν).

1. **Introduction.** The singularities of the real-valued regular (classical) solutions of the linear second-order elliptic partial differential equation

$$(1) \qquad \mathscr{L}(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + a(x, y)\frac{\partial v}{\partial x} + b(x, y)\frac{\partial v}{\partial y} + c(x, y)v = 0$$

are considered here. The real analytic coefficients continue analytically as entire functions on $\mathscr{C}^2$ when x and y continue as independent complex variables; also, the coefficient $c(x, y) \leq 0$ on $x^2 + y^2 \leq 1$.

Properties of the singularities of solutions to linear elliptic partial differential equations stir special interest in several areas of mathematical physics [5, 7], for example, in potential scattering. Using function theoretic methods, R. P. Gilbert and D. L. Colton [8] determined necessary and sufficient conditions concerning the location of singularities of regular solutions v in terms of corresponding information for a unique associated analytic function f of one complex-variable. Our principle interest is in global information concerning the singularities of v independent of the associate. This information appears by approximation methods that determine

global relationships between the singularities on v and the growth of the errors in solutions that approximate v locally.

The basis for this analysis is the Bergman and Gilbert Integral Operator Method [1, 3, 5, 6] which extends the classical theorems of S. N. Bernstein [2, 14] and E. B. Saff [13] from analytic function theory. Those classical results analyze the polar singularities of analytic f via approximation methods in much the same way that the Hadamard and Mandelbrojt theorems [4] analyze the polar singularities of f via its Taylor's coefficients.

The Hadamard and Mandelbrojt coefficient theorems have been extended to solutions of various classes of partial differential equations [3, 5, 6] via the Integral Operator Method. The ideas of Bernstein and Saff have been applied [9-11] along with these methods to study the singularities of certain second-order elliptic equations with singular coefficients, i.e., the generalized axisymmetric and biaxisymmetric potential equations. Those results also contain calculations of the order and type of entire function potentials from the convergence rates of the errors in the best local harmonic polynomial approximates. Similar calculations are not considered here for the equation (1) with entire function coefficients due to the lack of a suitable inverse operator. Next are introduced the

2. **Preliminary results.** Following standard procedure [See 1, 5, 8], the functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ analytically continue as $a(z, z^*)$, $b(z, z^*)$ and $c(z, z^*)$ by the change to the hyper-complex coordinates $z = x + iy$, $z^* = x - iy$ for $(x, y) \in \mathscr{C}^2$; reducing eqn. (1) to a complex-valued hyperbolic equation

$$(2) \qquad L(U) = \frac{\partial^2 U}{\partial z \partial z^*} + A(z, z^*)\frac{\partial U}{\partial z^*} + B(z, z^*)\frac{\partial U}{\partial z} + C(z, z^*)U = 0 \ ,$$

$$U(z, z^*) = v[(z + z^*)/2, (z - z^*)/2i]$$
$$A(z, z^*) = [a(z, z^*) + ib(z, z^*)]/4$$
$$B(z, z^*) = [a(z, z^*) - ib(z, z^*)]/4$$
$$C(z, z^*) = c(z, z^*)/4 \ .$$

A change of dependent variables

$$V(z, z^*) = U(z, z^*) \exp \left\{ \int_0^{z^*} A(z, \zeta)d\zeta - h(z) \right\}$$

for an arbitrary entire function h gives the *Bergman canonical form* of eqn. (2) [1, 8],

$$(3) \qquad \mathscr{B}(V) = \frac{\partial^2 V}{\partial z \partial z^*} + D(z, z^*)\frac{\partial V}{\partial z^*} + F(z, z^*)V = 0 \ ,$$

$$F = A_z + AB - C , \qquad D = h' - \int_0^{z^*} A_\zeta d\zeta + B .$$

It is known [8] that regular V has a local representation $V = b_2[f]$ about the origin that is defined from a unique b_2-*associated* analytic function $f = f(z)$ by the integral operator $b_2[f]$,

$$(4) \qquad v(z, z^*) = b_2[f(\sigma)] = \int_{\mathscr{L}} E(z, z^*, t) f(\sigma) d\mu(t) ,$$

$$\sigma = z(1 - t^2)/2 , \qquad d\mu(t) = dt/(1 - t^2)^{1/2}$$

where $\mathscr{L}$ is the contour $t = e^{i\theta}$ from -1 to $+1$. The Bergman E-function follows:

$$E(z, z^*, t) = 1 + \sum_{n=1}^{\infty} t^{2n} z^n \int_0^{z^*} P^{(2n)}(z, \zeta) d\zeta ,$$

$$P^{(2)} = -2F , \qquad (2n + 1)P^{(2n+2)} = -2\Big[P_z^{(2n)} + DP^{(2n)} + F \int_0^{z^*} P^{(2n)} d\zeta \Big] ,$$

$n = 1, 2, \cdots$. The principal branch of the function element $V(z, z^*)$ continues analytically from its initial domain of definition by contour deformation to a (larger) domain of associated as described in the "Envelope Method" [see 5, 6]. Using this method, Gilbert and Colton [Theorem 1; [8]] show that the (principal branch) of $V(z, \bar{z})$ is singular at $z = \alpha$ if, and only if, the b_2-associate f is singular at $z = \alpha/2$.

The b_2-associates in this paper are approximated by real rational functions of type (n, ν),

$$r_{n,\nu}(z) = p_n(z)/q_\nu(z) , \qquad n, \nu = 0, 1, \cdots ,$$

the ratio of real-valued relatively prime polynomials of degrees n and ν. The functions $r_{n,0}(z)$ are simply the polynomials $p_n(z)$. Corresponding to these b_2-associates we find multi-valued function elements

$$\Psi_{n,\nu}(z, z^*) = b_2[p_n(\sigma)/q_\nu(\sigma)]$$

and

$$\Phi_n(z, z^*) = b_2[p_n(\sigma)]$$

$n, \nu = 0, 1, \cdots$ whose principal branches are selected to approximate $V(z, z^*)$. These approximates may be viewed as "rational" functions under the quasi-multiplication

$$\Psi_{n,\nu} = \Psi_{n,0} * \Psi_{0,\nu} \equiv b_2[r_{n,0} \cdot (1/r_{\nu,0})] .$$

The sets $\mathscr{R}_{n,\nu}$ and $\mathscr{P}_n \equiv \mathscr{R}_{n,0}$ compose respectively the sets of all

rational functions of type (n, ν) and all polynomials of degree n. The sets corresponding to the images of $\mathscr{R}_{n,\nu}$ and $\mathscr{P}_n$ under b_2 are

$$\mathscr{R}_{n,\nu} = \{\Psi_{n,\nu} \colon \Psi_{n,\nu} = b_2[r_{n,\nu}], \ r_{n,\nu} \in \mathscr{R}_{n,\nu}\}$$

and

$$\mathscr{P}_n = \{\Phi_n \colon \Phi_n \in \mathscr{R}_{n,0}\} \ ,$$

$n, \nu = 0, 1, 2, \cdots$. The essential measures become the growth of the errors in the (mini-max) best "rational" approximates defined by the Chebyshev norms

$$e_{n,\nu}(f) = \inf \{\| f - r_{n,\nu} \| \colon r_{n,\nu} \in \mathscr{R}_{n,\nu}\} \ ,$$

$$\| f - r_{n,\nu} \| = \sup \{| f(z) - r_{n,\nu}(z)| \colon z \in D\} \ ,$$

where the disk $D_\rho = \{z \in \mathscr{C} \colon |z| \leqq \rho\}$ with $D_1 \equiv D$ and

$$E_{n,\nu}(V) = \inf \{\|| V - \Psi_{n,\nu} \|| \colon \Psi_{n,\nu} \in \mathscr{R}_{n,\nu}\} \ ,$$

$$\|| V - \Psi_{n,\nu} \|| = \sup \{| V(z, z^*) - \Psi_{n,\nu}(z, z^*)| \colon (z, z^*) \in D^2\} \ ,$$

$n, \nu = 0, 1, \cdots, D^2 = D \times D$ and the errors in the best "polynomial" appropriates

$$e_n(f) = e_{n,0}(f) \ , \qquad E_n(V) = E_{n,0}(V) \ , \qquad n = 0, 1, \cdots \ .$$

This brings us to the main objective.

3. The singularities of $V(z, \bar{z})$: $\mathscr{B}(V) = 0$. The study of the singularities of V and U reveals equivalence because $V(z, z^*)$ is singular at $(z_0, z_0^*) \in \mathscr{C}^2$ if, and only if, $U(z, z^*)$ is singular at (z_0, z_0^*). Furthermore, $z^* = \bar{z}$ if, and only if, $(x, y) \in E^2$ so the singularities of v may be studied by noting those of $V(z, \bar{z})$.

The first objective recognizes those entire function elements V whose analytic continuations from their initial domains of definition have no singularities located at finite distances from the origin. This is accomplished via a function-theoretic extension of the Bernstein theorem so that we naturally select a polydisk as the initial domain of definition in the following theorem.

THEOREM 1. *Let $V(z, z^*)$ be a regular solution of $\mathscr{B}(V) = 0$ on the polydisk D^2. Then the function element $V(z, \bar{z})$ has an analytic continuation as an entire function solution if, and only if,*

$$(5) \qquad\qquad \lim_{n \to \infty} [E_n(V)]^{1/n} = 0 \ .$$

Proof. For $V(z, z^*)$ a regular solution of $\mathscr{B}(V) = 0$ on the

polydisk D^2, we need to establish the necessity of condition (5) by assuming the function element $V(z, \bar{z})$ has analytic continuation as an entire function. By [Theorem 1 [8]], the same holds true of the b_2-associate $f = f(z)$ so that the equation

$$V(z, \bar{z}) = b_2[f(\sigma)]$$

is global as is the equation

$$V(z, \bar{z}) - \Phi_n(z, \bar{z}) = b_2[f(\sigma) - p_n(\sigma)] , \qquad n = 0, 1, \cdots$$

for each $p_n \in \mathscr{P}_n$ and $\Phi_n = b_2[\mathscr{P}_n]$. Because the functions $V - \Phi_n$ are regular on D^2, the contour $\mathscr{L}$ is homologous to $\mathscr{L}_0 = \{t = e^{i\theta}: \theta$ decreases from π to $0\}$ and $|\sigma| \leq 1$ if $(z, t) \in D \times \mathscr{L}_0$. We find the estimates

$$|V(z, z^*) - \Phi_n(z, z^*)| \leq \int_{\mathscr{L}_0} |E(z, z^*, t)| |f(\sigma) - p_n(\sigma)| d|\mu|(t)$$

$$\leq c(E) \|f - p_n\| , \qquad n = 0, 1, 2, \cdots ,$$

$$c(E) = \sup \left\{ \int_{\mathscr{L}_0} |E(z, z^*, t)| d|\mu|(t): |z|, |z^*| \leq 1 \right\}$$

on D^2. The constant $c(E)$ is finite knowing that $E(z, z^*, t)$ is continuous on $D^2 \times \mathscr{L}_0$, a consequence of the entire function coefficients in eqn. (5). The appraisals

$$(6) \qquad \qquad \||V - \Phi_n\|| \leq c(E) \|f - p_n\|$$

and

$$E_n(V) \leq c(E) e_n(f) , \qquad n = 0, 1, 2, \cdots ,$$

follow.

We now estimate the norm $e_n(f)$. Let p_n^* be the mini-max polynomial for $e_n(f)$ and $\Phi_n^* = b_2[p_n^*]$. The entire function $f - p_n^*$ expands on $[-1, +1]$ in a series of Chebyshev polynomials

$$(7) \qquad t_n(x) = \frac{1}{2} \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (2z)^{n-2k} , \qquad n = 0, 1, 2, \cdots$$

that is analytically continued to the ellipse $\mathscr{E}_\rho = \{z \in \mathscr{C}: |z - 1| + |z + 1| < 2\rho\}$, $\rho > 4$ as

$$f(z) - p_n^*(z) = 2 \sum_{k=n+1}^{\infty} \alpha_k t_k(z) , \qquad \alpha_k = \alpha_k(f) .$$

The bounds $|\alpha_k| \leq \text{Sup} \{|f(z)|: z \in \mathscr{E}_\rho\}\rho^{-k}$, $k = 0, 1, 2, \cdots$ establish as in [12], leading to the estimate

$$e_n(f) = \| f - p_n^* \| \leqq \kappa(\rho)(2/\rho)^n , \qquad n = 0, 1, \cdots ,$$

similar to that in [9]. Then eqn. (6) gives

$$E_n(V) \leqq \| | V - \Phi_n^* | \| \leqq c(E) \| f - p_n^* \| \leqq c(E)\kappa(\rho)(2/\rho)^n , \qquad n = 0, 1, \cdots .$$

Thus $\lim_{n \to \infty} [E_n(V)]^{1/n} \leqq 2/\rho$ for all $\rho > 4$ establishing eqn. (5) as $\rho \to \infty$.

For the sufficiency, let $V(z, z^*)$ be regular on D^2 and assume that the Bernstein limit eqn. (5) is satisfied. The function V satisfies the Goursat data [1, 8],

$$V(z, 0) = g(z) = \int_{\mathscr{L}_0} f(\sigma)d\mu(t) , \qquad V(0, z^*) = g(0) , \qquad z \in D .$$

Moreover, the analytic g is singular at $z = 2\alpha$ if, and only if, f is singular at $z = \alpha$ (see [8]). Because of the result just referenced, we then sufficiently establish that $V(z, \bar{z})$ is an entire function by establishing the same for $g(z)$. To that end we observe the identities

$$g(z) - p_n(z) = V(z, 0) - \Phi_n(z, 0) , \qquad z \in D$$

$p_n \in \mathscr{P}_n$, $n = 0, 1, 2, \cdots$ and the inequalities

$$(8) \qquad \| g - p_n \| \leqq \| V - \Phi_n \| \leqq \| | V - \Phi_n | \|$$

from which the bounds

$$\mu(g - p_n) \leqq \| | V - \Phi_n | \| ,$$

$$\mu(h) = \sup \{|h(x)| \colon -1 \leqq x \leqq +1\}$$

$n = 0, 1, \cdots$ follow. The classical Bernstein theorem proves that if $\varepsilon_n(g) = \inf \{\mu(g - p_n) \colon p_n \in \mathscr{P}_n\}$ and $[\varepsilon_n(g)]^{1/n} \to 0$ as $n \to \infty$, $g = g(z)$ is entire. This conclusion follows becauseof eqn. (8) we have an estimate

$$[\varepsilon_n(g)]^{1/n} \leqq [E_n(V)]^{1/n} , \qquad n = 1, 2, \cdots$$

whose larger member satisfies the Bernstein limit.

Prior to considering the second problem where at least one of the singularities of V is located at a finite distance from the origin, we observe that when $n > \nu$,

$$\Psi_{n, \nu}(z, \bar{z}) = \Phi_{n(\nu)}(z, \bar{z}) + \Psi_{\omega, \nu}(z, \bar{z}) ,$$

and the principal branch $\Psi_{n, \nu}(z, \bar{z}) \rightrightarrows \Phi_{n(\nu)}(z, \bar{z})$ (uniformly) on compacta of $\mathscr{C}$ as the poles of $\Psi_{n, \nu}$ recede to infinity. In this case the "rational" approximates interpret as reducing to "polynomial" approximates. We further note that the initial domain on which the principal branch of the "meromorphic" function element V is regular,

in analogy with the corresponding results in function theory, is the polydisk D^2. The principal branch of V analytically continues via contour deformation. Moreover, this function element is said to have a *pole-like singularity* of order ν at $z = \alpha$ if the b_2-associate f has a pole of *order* ν counted with multiplicity at $z = \alpha/2$; the element also has a pole-like singularity of *infinite-order* at $z = \alpha$ if f has an essential singularity at $z = \alpha/2$. We now consider the general results.

THEOREM 2. *Let $V(z, z^*)$ be a regular solution of $\mathscr{B}(V) = 0$ on the polydisk D^2 and let the principal branch of $V(z, \bar{z})$ analytically continue as a solution with atmost ν pole-like singularities in the disk D^2. Then there exists a "rational" approximating sequence $\{\Psi_{n,\nu}\}_{n=0}^{\infty}$ for which*

$$\limsup_{n\to\infty} |||V - \Psi_{n,\nu}|||^{1/n} \leq 1/\rho \qquad (<1) .$$

Proof. Let $V(z, z^*) = b_2[f(\sigma)]$ on D^2 and $\Psi_{n,\nu} = b_2[r_{n,\nu}]$, $n = 0, 1, 2, \cdots$ where the $\Psi_{n,\nu}(z, z^*)$ are regular on D^2. Then

$$|V(z, z^*) - \Psi_{n,\nu}(z, z^*)| \leq c(E)\|f - r_{n,\nu}\|$$

and

$$(9) \qquad |||V - \Psi_{n,\nu}||| \leq c(E)\|f - r_{n,\nu}\| , \qquad n = 0, 1, \cdots$$

so that

$$\limsup_{n\to\infty} |||V - \Psi_{n,\nu}|||^{1/n} \leq \limsup_{n\to\infty} \|f - r_{n,\nu}\|^{1/n} , \qquad \nu = 0, 1, \cdots .$$

To establish the reverse estimate, the Goursat data is utilized to give

$$(10) \qquad \|f - r_{n,\nu}\| \leq |||V - \Psi_{n,\nu}||| , \qquad n = 0, 1, \cdots$$

and

$$\limsup_{n\to\infty} \|f - r_{n,\nu}\|^{1/n} \leq \limsup_{n\to\infty} |||V - \Psi_{n,\nu}|||^{1/n} ,$$

so that

$$(11) \qquad \limsup_{n\to\infty} |||V - \Psi_{n,\nu}|||^{1/n} = \limsup_{n\to\infty} \|f - r_{n,\nu}\|^{1/n}$$

(ν-fixed). By the Gilbert-Colton result, the b_2-associate of $V(z, \bar{z})$, f regular on D, has ν-singularities in D_ρ. Applying a result of J. L. Walsh [15, 16] demonstrates the existence of a sequence $\{r_{n,\nu}\}_{n=0}^{\infty}$ for which

(12) $$\limsup_{n\to\infty} \| f - r_{n,\nu} \|^{1/n} \leqq 1/\rho \qquad (<1) .$$

Combining eqn. (11) and eqn. (12) with the image of this sequence under b_2 completes the proof.

We next secure the converse of this result in

THEOREM 3. *Let $V(z, z^*)$ be a regular solution of $\mathscr{B}(V) = 0$ on the polydisk D^2. If $\{\Psi_{n,\nu}\}_{n=0}^{\infty}$ is a sequence (ν-fixed) of regular approximates $\Psi_{n,\nu}(z, z^*)$ on D^2 for which*

$$\limsup_{n\to\infty} \|| V - \Psi_{n,\nu} \||^{1/n} \leqq 1/\rho \quad (<1) ,$$

then the principal branch of $V(z, \bar z)$ may be analytically continued to $D_{2\rho}$ with atmost ν pole-like singularities.

Proof. According to reasoning in the previous theorem which leads to eqn. (10), the following appraisal is valid

$$\limsup_{n\to\infty} \| f - r_{n,\nu} \|^{1/n} \leqq 1/\rho \quad (<1)$$

for $r_{n,\nu}(z) = \Psi_{n,\nu}(z, 0)$ on $z \in D$. By Saff's converse [13] to Walsh's result, the associate f analytically continues to D_ρ with atmost ν-poles. Applying, once again, the Gilbert-Colton result shows that the principal branch of $V(z, \bar z)$ has analytic continuation to $D_{2\rho}$ with atmost ν pole-like singularities.

Arriving at the main result, we characterize the singularities, located at a given distance from the origin, of the analytic continuations of V from its initial domain of definition. This is an extension of Saff's mini-max characterization of the analytic continuation properties of analytic functions.

THEOREM 4. *Let $V(z, z^*)$ be regular on the polydisk D^2 and for each ν let the minimum error in the approximation of V over sets $\{\Psi_{n,\nu}\}_{n,\nu=0}^{\infty}$ of approximates $\Psi_{n,\nu}(z, z^*)$ be given by*

$$\limsup [E_{n,\nu}(V)]^{1/n} = (2\rho_\nu)^{-1} , \qquad \nu = 0, 1, \cdots .$$

Then the sequence $\{\rho_\nu\}_{\nu=0}^{\infty}$ is nondecreasing and

(i) *D_{ρ_ν} is the maximal disk with the property that the principal branch of $V(z, \bar z)$ has analytic continuation D_{ρ_ν} with atmost ν pole-like singularities.*

(ii) *if $\rho_\nu > \rho_{\nu-1}$, the principal branch of $V(z, \bar z)$ has analytic continuation to D_{ρ_ν} with precisely ν pole-like singularities.*

(iii) *$\rho = \lim_{\nu\to\infty} \rho_\nu$ is the radius of the maximal disk D_ρ to which the principal branch $V(z, \bar z)$ has analytic continuation.*

(iv) *the analytic continuation of the principal branch of $V(z, \bar z)$*

has infinitely many pole-like singularities in the disk D_ρ if, and only if, $\rho_\nu > \rho$, $\nu = 0, 1, 2, \cdots$.

Proof. Let $V(z, z^*)$ be regular on the polydisk D^2 and for each ν let $\{\Psi_{n,\nu}\}_{n=0}^\infty$ be a sequence of regular solutions on D^2. It may be established as in eqns. (10) and (11) that

$$\| f - r_{n,\nu} \| \leqq \|\| V - \Psi_{n,\nu} \|\| \leqq c(E) \| f - r_{n,\nu} \|, \qquad n, \nu = 0, 1, \cdots$$

from which we see that

$$[e_{n,\nu}(f)]^{1/n} \leqq [E_{n,\nu}(V)]^{1/n} \leqq [c(E)]^{1/n}[e_{n,\nu}(f)]^{1/n}$$

$n \geqq 1$, $\nu \geqq 0$ from which it follows that

$$\limsup_{n\to\infty} [e_{n,\nu}(f)]^{1/n} = \limsup_{n\to\infty} [E_{n,\nu}(V)]^{1/n}$$

and

$$\limsup_{n\to\infty} [e_{n,\nu}(f)]^{1/n} = (2\rho_\nu)^{-1}, \qquad \nu = 0, 1, \cdots.$$

Then by the Saff Theorem [13], $\{\rho_\nu\}_{\nu=0}^\infty$ is nondecreasing and

(i-ii) $D_{2\rho_\nu}$ is the largest disk for which f has atmost (precisely if $\rho_\nu > \rho_{\nu-1}$) ν-poles. Apply Gilbert-Colton [Theorem 1, [8]] to obtain the corresponding information for $V(z, \bar{z})$.

(iii) $D_{2\rho}$, $\rho = \lim_{\nu\to\infty}\rho_\nu$ is the largest disk in which f is meromorphic. Thus again by [8] $V(z, \bar{z})$ has analytic continuation to D_ρ and no farther.

(iv) the b_2-associate f has infinitely many poles in D_ρ if, and only if, $\rho_\nu > \rho$, $\nu = 0, 1, \cdots$. Reason as above to complete the analysis.

4. Remarks and generalizations. It is an easy matter to interpret the preceeding results when the analytic continuations of the coefficients of eqn. (1) have singularities in the domain of association of V that are not located at infinity. In the event that the domain of association contains singularities of either the coefficients, or their analytic continuations, or if the singularities of the coefficients are located in the initial domain of definition, the preceeding results may apply to give information concerning the singularities of v. Such an example is given by the equation with singular coefficient

$$\mathscr{L}_\nu(w) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + (2\nu/y)\frac{\partial w}{\partial y} + a(x)\frac{\partial w}{\partial x} + c(x)w = 0$$

$\nu > 0$ that extends the generalized axisymmetric potential equation ($a \equiv c \equiv 0$) and in part eqn. (1). The coefficients a and c are entire functions on $\mathscr{C}$. In [5] an extension of the Vekua Method is given which presents $w = w(x, y)$ as the integral transform of $v = v(x, y)$,

a regular solution of

$$\mathscr{L}_0(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + a(x)\frac{\partial v}{\partial x} + c(x)v = 0$$

with Goursat condition: $w(x, 0) = v(x, 0)$ for all $\nu > 0$ and all real x in the common simply-connected domain of regularity. Using this information, extensions of §3 are readily suggested. The literature [1, 3, 5-7] in the references also gives methods of reformulation of the operators on conformally equivalent domains.

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