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ON THE CLOSED IDEALS IN $A(W)$

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This paper is about the ideal theory of the algebra of functions continuous on the closure and holomorphic in the interior of a domain on a compact Riemann surface. The description of the closed ideals in the disc algebra is shown to apply to an ideal whose hull meets the boundary of the domain in a finite union of analytic arcs. The canonical factorization into inner and outer functions in the disc is replaced by a potential theoretic decomposition theorem, thus allowing essentially the same description to be carried over. The basically local nature of the problem is used to reduce it to the previously known ideal theory of a compact bordered Riemann surface. This reduction is facilitated by a factorization theorem that is proved by potential theoretic methods.

Let W be a domain (i.e., open, connected set) on a compact Riemann surface S, let ∂W denote the boundary of W and $\bar{W} = W \cup \partial W$ its closure. Let $A(W)$ be the set of all complex valued functions that are continuous on $\bar{W}$ and holomorphic on W; $A(W)$ is a Banach algebra in the uniform norm. In the case of the unit disc, Beurling (unpublished), and Rudin [8] described the closed ideals of $A(W)$; see also [5] for an exposition of these results.

In the case of a finite Riemann surface, Voichick [10] found an analogous description. This case was also treated by Hasumi [3], and Stanton [9]. These descriptions are essentially local; thus one may ask if they extend in some form to more general domains. In this paper, we obtain corresponding results for closed ideals of a certain type. We assume that ∂W contains a subset Γ such that $W \cup \Gamma$ is a bordered Riemann surface with analytic border Γ, that W lies on one side of Γ, and that Γ has finitely many components. We describe those closed ideals in $A(W)$ whose hulls lie in $W \cup \Gamma$. We reduce our problem to the ideal theory of a finite Riemann surface by means of a factorization theorem which allows us to separate the singularities of functions in $A(W)$. The factorization theorem follows from the decomposition theorem of Parreau [6], so our methods are somewhat potential theoretic.

In § 1, we illustrate our methods in the case of an annulus in the complex plane. In § 2 we collect some facts about harmonic functions necessary for the proof of the factorization theorem in § 3 and the description of closed ideals in § 4.

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1. The annulus. In general, we reduce the ideal theory of the domain W to that of a simpler domain. To avoid obscuring the main ideas with technical points, we indicate how to reduce the ideal theory of $A(W)$, where W is an annulus in the complex plane, to that of the disc algebra.

Let $W = \{z: r < |z| < 1\}$ and let Γ be the boundary of W. Let I be a closed ideal in $A(W)$. Let $W_1 = \{z: |z| < 1\}$, $W_2 = \{z: |z| > r\} \cup \{\infty\}$, and let Γ_1, Γ_2 be their boundaries. Now $A(W_j)$, $j = 1,2$, are just copies of the disc algebra. Our method depends on two observations: (1) If f is a nonzero element in $A(W)$, then there exist an integer n and functions f_j in $A(W_j)$, $j = 1, 2$, such that $f = z^n f_1 f_2$. If ρ is previously given, with $r < \rho < 1$, then f_1, f_2 can be chosen so that f_1 does not vanish for $|z| < \rho$ and f_2 does not vanish for $|z| > \rho$. (One proves this by considering $\log|f|$.) (2) A function f in $A(W)$ that does not vanish on the hull of I is invertible modulo I; that is, there is a function g in $A(W)$ such that $gf - 1$ is in I. (One proves this by observing that the spectrum of the quotient Banach algebra $A(W)/I$ is the hull of I. Recall that the hull of I is the set $\{p \in \bar{W}: f(p) = 0 \text{ for all } f \text{ in } I\}$.)

Choose ρ, $r < \rho < 1$, so that the hull of I does not meet the circle $|z| = \rho$. Let F be a nonzero function in I and factor F as above: $F = z^n F_1 F_2$. By observation (2), $F_1 + F_2$ is invertible modulo I. Let $I_1 = \{f \in A(W): fF_2 \in I\}$ and $I_2 = \{f \in A(W): fF_1 \in I\}$. Since $F_1 + F_2$ is invertible modulo I, $I = I_1 \cap I_2$. Let $J_k = \{f \in A(W_k): f|W \in I_k\}$, $k = 1, 2$; J_k is a closed ideal in $A(W_k)$. Using the ideal theory of the disc algebra, we find closed sets $E_k \subset \Gamma_k$ of linear measure zero and inner functions Φ_k on W_k such that J_k consists of all functions of the form $f\Phi_k$ where f is in $A(W_k)$ and f vanishes on E_k. Using observation (1) we see that I_k is the smallest closed ideal in $A(W)$ which contains J_k, so I_k consists of all functions of the form $f\Phi_k$ where f is in $A(W)$ and f vanishes on E_k. Thus I consists of all functions of the form $f\Phi_1\Phi_2$ where f is in $A(W)$ and f vanishes on $E_1 \cup E_2$.

Using a similar argument and proceeding by induction on the connectivity of the domain, we can extend this result to a finitely connected plane domain. It is then possible to treat the case of a finite Riemann surface by choosing a neighborhood of its border that is conformally equivalent to a plane domain. We omit the details of this method because the results themselves are well known.

2. Potential theoretic preliminaries. In this section we study the boundary behavior of the terms in Parreau's decomposition of

$\log|f|$, where f is in $A(W)$. We first recall some elementary facts about analytic arcs and the Schwarz reflection principle. We then state Parreau's theorem and study the individual terms in the decomposition. Parreau's theorem is in [6], and a generalization appears in [4]. General facts about quasi-bounded and singular harmonic functions can be found in [1] and [2].

Henceforth we shall assume that ∂W contains a nonempty subset Γ such that $W \cup \Gamma$ is a bordered Riemann surface with analytic border. Thus Γ is a union of simple, regular analytic arcs and curves, and W lies on one side of Γ. In order to be explicit, we recall some facts about analytic arcs. A simple regular analytic arc on S is a univalent real analytic mapping α of an open interval I of real numbers into S with nowhere vanishing differential. Since α extends to be a univalent holomorphic mapping of a neighborhood of I in the complex plane, it carries complex conjugation over to an anti-conformal mapping which is defined in a neighborhood of $\alpha(I)$ and fixes each point of $\alpha(I)$. We shall often refer to $\alpha(I)$ as the curve α, and we shall call this anti-conformal mapping *reflection* in α. Assume that α is contained in ∂W and that W lies on one side of α. We shall say that an open set R on S is *symmetric* with respect to α and W if $R \sim \alpha = R_+ \cup R_-$, where $R_+ \subset W$, $R_- \cap W = \varnothing$, and R_+ is carried onto R_- by reflection in α. We shall use the notations R_+, R_- with this meaning.

Let p_0, q_0 be points of α, and let β denote the subarc joining p_0 and q_0. Now $p_0 = \alpha(a)$, $q_0 = \alpha(b)$ for some a, $b \in I$. We may assume that $a < b$ and that α inherits its orientation from W. Let $c > 0$, and let

$$V = \{z = x + iy\colon\ a < x < b,\ -c < y < c\}$$
$$V' = \{z = x + iy\colon\ a < x < b,\ -2c < y < 2c\}$$
$$V'' = \{z = x + iy\colon\ a < x < b,\ -2c < y < -c\}\ .$$

Choose c so small that α is univalent on the closure of V'. Let $R = \alpha[V]$, $R' = \alpha[V']$, $R'' = \alpha[V'']$. We may also assume that R' is symmetric with respect to β and W. Let D be the interior of $S \sim R'_-$. Notice that $R'' \cap D = \varnothing$. The domains D and R have the following properties:

(i) $W \subset D$, ∂D is a piecewise analytic Jordan curve, and $\partial W \cap \partial D = \bar{\beta}$;

(ii) R is simply connected, R is symmetric both with respect to β and W and with respect to β and D, and $R \cap \partial W = \beta = R \cap \partial D$. We shall refer to R and D as *auxiliary domains* associated with W, α, p_0, q_0.

Let u be a positive harmonic function on an open set Ω, and let

v be a nonnegative harmonic function on Ω. We say that v is *u-singular* and write $v \perp u$ if the greatest harmonic minorant of u and v is zero. We say that v is *u-quasi-bounded* and write $v \ll u$ if there is an increasing sequence $\{v_n\}$ of nonnegative harmonic functions such that $\lim v_n = v$ and $0 \leqq v_n \leqq nu$ for each n. More generally, we say that a harmonic function is *u-singular* (u-quasi-bounded) if it is a linear combination of nonnegative u-singular (u-quasi-bounded) functions. If $u = 1$ we speak of *singular* and *quasi-bounded* harmonic functions. In the case of the unit disc, a positive harmonic function is singular if and only if it is the Poisson integral of a measure singular with respect to Lebesgue measure on the unit circle, and it is quasi-bounded if and only if it is the Poisson integral of a positive Lebesgue integrable function on the unit circle.

We now fix a positive harmonic function u in Ω. If v is a nonnegative harmonic function in Ω, then there exist uniquely determined nonnegative harmonic functions q and s on Ω such that

$$v = q + s ,$$

where q is u-quasi-bounded and s is u-singular. In case $u = 1$, we shall call s the *singular part* of v. This decomposition is due to Parreau [6]; for a proof in this generality, see [2, § 2].

The following lemmas state some properties of harmonic functions in a form convenient for later use.

LEMMA 1. *Let p_0, q_0 be points of a subarc α of Γ, and let R, D be corresponding auxiliary domains. Let U be a quasi-bounded harmonic function defined in R_+. Assume that the nontangential boundary values of U vanish almost everywhere (with respect to linear measure) on $R \cap \alpha$. Then U extends to be a harmonic function in R.*

Proof. It suffices to prove that $\lim_{q \to p} U(q) = 0$ for each p in $R \cap \alpha$, for then we can apply the Schwarz reflection principle. For this purpose we may assume that R_+ is the unit disc and that $R \cap \alpha$ is an arc on the unit circle. Since U is quasi-bounded it is the Poisson integral of its nontangential boundary values. Since these boundary values vanish almost everywhere on the arc $R \cap \alpha$, we can extend U continuously to this arc by setting it equal to zero there [11, vol. I, p. 97].

LEMMA 2. *With the notation of the previous lemma, let $\{U_k\}$ be a sequence of nonnegative bounded harmonic functions each of which vanishes almost everywhere on $R \cap \alpha$. Assume that $U = \Sigma U_k$ converges in R_+. Then U extends to be a harmonic function in R.*

Proof. Again we may assume that R_+ is the unit disc. Each U_k is then the Poisson integral of a function f_k on the unit circle, and f_k vanishes almost everywhere on $R \cap \alpha$. Now U is the Poisson integral of $f = \Sigma f_k$, and f vanishes almost everywhere on $R \cap \alpha$. The lemma follows.

LEMMA 3. *With the notation of Lemma* 1, *let s be a singular harmonic function in W. Assume that s is quasi-bounded on R_+. Then s extends to be a harmonic function in $W \cup R$.*

Proof. We may assume that s is nonnegative. It is sufficient to show that the nontangential limits of s vanish almost everywhere on $R \cap \alpha$. Otherwise there would be a compact subset E of $R \cap \alpha$ and a number $c > 0$ such that E has positive linear measure and $\lim_{q \to a} s(q) \geqq c$ for all $a \in E$. (Here we mean nontangential limit.) Let χ be the characteristic function of E. Let u be the solution of the Dirichlet problem in W obtained by Perron's method from the boundary data $c\chi$. Then u is a positive bounded harmonic function in W and $u \leqq s$. This contradiction shows that s vanishes on $R \cap \alpha$ and so it extends to $W \cup R$.

LEMMA 4. *With the notation of Lemma* 1, *let s' be a nonnegative singular harmonic function on D. Let $s' = u + s$ where u is quasi-bounded on W and s is singular on W. Then*

$$\lim_{p \to q} u(p) = 0 \tag{1}$$

for all $q \in \beta$.

Proof. We may assume that s' is nonnegative. By the technique used in proving Lemma 3, we see that the nontangential boundary values of s', and therefore of u, vanish a.e. on β. The assertion now follows from Lemma 1.

Parreau [6] has proved that

$$\log |f| = Q - s - b\,, \qquad f \in A(W) \tag{2}$$

where Q is a quasi-bounded harmonic function on W, s is a nonnegative singular harmonic function on W, and b is a potential on W. The functions Q, s, b are uniquely determined by f. When we want to indicate their dependence on f, we write Q_f, s_f, b_f. We call (2) the *canonical decomposition* of $\log |f|$ (or of f). Now b is the potential of a measure $\mu = \mu_f$ on W. It follows from (2) that μ is a discrete measure and that $\mu(p)$ is the order of the zero of f at p for every p in W. Hence

$$b(p) = \Sigma\mu(q)G_W(p, q)\ , \tag{3}$$

where G_W is the Green's function of W. In the case of the unit disc, the canonical decomposition is equivalent to the factorization of f into its outer, singular, and Blaschke factors; indeed Q, s, b are just the logarithms of the moduli of these factors.

We need some facts about potentials of discrete measures, i.e., series of the form (3).

LEMMA 5. *Let ν be a discrete measure on W and let*

$$U(p) = \Sigma\nu(q)G_W(p, q)$$

be its potential. If U is finite at one point of $W \sim \operatorname{supp}\nu$, then U is harmonic in $W \sim \operatorname{supp}\nu$. Let α be an arc of $\Gamma \sim \operatorname{supp}\nu$. Then we can extend U harmonically across α by setting $U = 0$ on α.

Proof. The first assertion follows from Harnack's theorem. For the second, we observe that each term of the series vanishes on α and is bounded near any compact subset of α. The conclusion then follows from Lemma 2.

3. Factorization theorem.

THEOREM 1. *Let W be a domain on a compact Riemann surface S. Assume that ∂W contains a simple, free, one-sided, regular analytic arc α. Let p_0, q_0 be points of α, let β be the subarc of α joining them, and let D and R be corresponding auxiliary domains. Let $f \in A(W)$, and assume that $f(p_0) \neq 0$ and $f(q_0) \neq 0$. Then there exist functions g and h such that*

(i) *g is a bounded holomorphic function in D, g extends continuouly to $\partial D \sim \{p_0, q_0\}$, and g does not vanish on $\bar{D} \sim \bar{W}$,*

(ii) *h is a bounded holomorphic function in $W \cup R$, h extends continuously to $\partial(W \cup R) \sim \{p_0, q_0\}$, and h does not vanish on R,*

(iii) *$f = gh$ on W.*

Proof. Let $\log|f| = Q - s - b$ be the canonical decomposition of $\log|f|$. We shall show that there exist harmonic functions Q', s', and a potential b' on D such that $Q' - Q$, $s' - s$, $b' - b$ have harmonic extensions to R.

Let $E = \{p \in \beta\colon f(p) = 0\}$; E is a totally disconnected compact subset of β with linear measure zero. Let Q' be a real-valued function on ∂D which is continuous on $\partial D \sim E$, equal to Q on β, and is smooth on $\partial D \sim \beta$. (Here "smooth" means that if a neighborhood of ∂D in $D \cup \partial D$ is mapped conformally onto an annulus with ∂D going to a circle, then Q' is smooth in terms of the parameter inherited from that circle.) Denote the harmonic extension of Q' to D by Q' also. Then Q' has local harmonic conjugates which extend continuouly to $\partial D \sim E$ except perhaps at p_0 and q_0. Moreover Q' is quasi-bounded on D and $Q' - Q$ vanishes on $\beta \sim E$. By Lemma 1, $Q' - Q$ extends to be harmonic on $W \cup R$.

Let χ be the characteristic function of R_+, and let $\nu = \chi\mu$. Let b' be the potential of ν in D, i.e.,

$$b'(p) = \Sigma \nu(q) G_D(p, q) ,$$

where G_D is the Green's function of D. It follows from Lemma 5 that b' is harmonic on $D \sim \operatorname{supp} \nu$, and it follows from Lemma 2 that $b' - b$ extends to be harmonic on R.

Now the singular function s vanishes on $\beta \sim E$, so we can extend s to be harmonic in $(W \cup R) \sim E$ by refection in β. Since R_+ is simply connected, there is a holomorphic function k on R_+ such that $s = \operatorname{Re}(k)$ there. Now β is an analytic arc, and k is purely imaginary on $\beta \sim E$. Thus we may extend k by reflection to be holomorphic in $R \sim E$; then $s = \operatorname{Re}(k)$ in $R \sim E$. Therefore $*ds$ has no periods in $R \sim E$. In particular, for any cycle γ in $R \sim E$, $\int_\gamma *ds = 0$. By [7, Theorem 3], there is a harmonic function U in $S \sim E$ such that $s - U$ extends to be harmonic in R and so in $W \cup R$. Now U is bounded below on $\partial D \sim \beta$, $s - U$ is bounded on β, and $s \geqq 0$. Hence U is bounded below on β as well. By the maximum principle U is bounded below on D. By adding a constant to U, we may assume that $U \geqq 0$ on D.

On D we have

$$U = u + s'$$

where u is quasi-bounded on D and s' is a nonnegative singular harmonic function on D. On R_+,

$$s' - s = U - s - u ,$$

and so $s' - s$ is quasi-bounded. Thus $s' - s$ extends to be harmonic in $W \cup R$, by Lemma 3.

Let a homology basis for D be given; it is also a homology basis for $D \cup R$. There is a function v harmonic on the closure of $D \cup R$ whose conjugate differential $*dv$ has prescribed periods. Hence we can choose v so that the only periods of $*d(Q' - s' - b' + v)$ occur at the singularities of b' and are integral multiples of 2π. Hence there is a holomorphic function g on D such that

$$\log |g| = Q' - s' - b' + v .$$

Now g extends continuously to $\partial D \sim \{p_0, q_0\}$. Note that the only zeros of g are in the closure of R_+, and that in R_+, f and g have the same zeros with the same multiplicities. Thus the function $h = f/g$ is holomorphic in W and does not vanish on R_+. Since

$$\log |h| = Q - Q' + s' - s + b' - b - v ,$$

$\log |h|$ extends to be harmonic in R, and so h extends to be holomorphic in $W \cup R$. Now h is continuous on the closure of R_+, except perhaps at p_0 and q_0. The extension of h is obtained essentially by reflection, so h is continuous on the closure of $W \cup R$, except perhaps at p_0 and q_0.

REMARK. Let k be holomorphic in a neighborhood of $\bar{D} \cup \bar{R}$ and vanish only at p_0 and q_0. Then

$$k^2 f = (kg)(kh)$$

where kg is in $A(D)$, kh is in $A(W \cup R)$ and kh does not vanish on R. Since $\log |k|$ is quasi-bounded on $D \cup R$, comparing canonical decompositions yields $s_f = s_g + s_h$ and $b_f = b_g + b_h$.

REMARK. An obvious extension of Theorem 1 also holds if α is assumed to be a finite union of simple analytic arcs.

4. Ideal theory of $A(W)$. Let I be a closed ideal in $A(W)$. We associate three data with I: the set $E_I = \{p \in \bar{W}: f(p) = 0$ for all f in $I\}$, called the *hull* of I; the greatest harmonic minorant S_I of $\{s_f\colon f \in I\}$; and the discrete measure $\mu_I = \inf\{\mu_f\colon f \in I\}$.

THEOREM 2. *Let I be a closed ideal in $A(W)$. Assume that ∂W contains a finite union Γ of analytic arcs such that $W \cup \Gamma$ is a bordered Riemann surface and $E_I \cap \partial W = E_I \cap \Gamma$. Then*

$$I = \{f \in A(W)\colon f|E_I = 0, s_f \geqq S_I, \mu_f \geqq \mu_I\} .$$

REMARKS. In the case of the unit disc, this description reduces to the familiar one in terms of inner functions. One sees this by considering the analogy between the outer, singular, and Blaschke factors and the terms in Parreau's decomposition. This description remains valid for finite Riemann surfaces [3, 8, 9].

Proof. It is sufficient to consider the case in which Γ consists of exactly one arc. Let $K = \{f \in A(W)\colon f | E_I = 0,\ s_f \geqq S_I,\ \mu_f \geqq \mu_I\}$. Then $I \subset K$. To show that $K \subset I$, choose p_0, q_0 in Γ so that $E_I \cap \Gamma$ lies in the subarc β of Γ with endpoints p_0, q_0. Construct auxiliary domains D, R corresponding to p_0, q_0, and Γ. Let $J=\{\Phi \in A(D): \Phi| W \in I\}$. Now J is a closed ideal in $A(D)$, and D is a finite Riemann surface. Hence J is determined by its data: its hull E_J, the singular harmonic function S'_J and the discrete measure μ_J. (In this proof we shall be dealing with relationships between harmonic functions on W and functions on D. For clarity we shall use a prime to indicate that a harmonic function is defined on D.) We shall show that $E_J = E_I$, $\mu_J = \mu_I$, and that on W, $S'_J = S_I + u$ where u is quasi-bounded and vanishes on Γ. We also note that since D is bounded by a piecewise analytic arc, and since E_J avoids the corners of this arc, the function S'_J can be extended continuously to $D \cup (\partial D \sim E_J)$ by setting it equal to zero on $\partial D \sim E_J$.

To establish these relations between the data determined by I and J, let f be a nonzero element in I. Then there exist points p_1, q_1 on Γ such that p_0, p_1, q_1, q_0 occur in that order along Γ, $f(p_1) \neq 0$, $f(q_1) \neq 0$, and $E_I \cap \Gamma$ lies in the subarc of Γ with endpoints p_1, q_1. Construct auxiliary domains D_1, R_1 corresponding to p_1, q_1, and Γ. We may assume that $D \subset D_1$. Let k be holomorphic in a neighborhood of $\overline{D_1 \cup R_1}$ and vanish only at p_1, q_1. There exist $g \in A(D_1)$ and h in $A(W \cup R_1)$ such that $k^2 f = gh$, g does not vanish on $\bar{D}_1 \sim \bar{W}$, and h does not vanish on E_I. (See the remarks following Theorem 1 and note that only finitely many points of E_I lie outside R_1.) Since $k^2 f \in I$ and h is invertible modulo I it follows that $g \in I$. Therefore $g \in J$. If $p \notin \bar{W} \sim E_I$, we may choose f so that $f(p) \neq 0$. Thus $g(p) \neq 0$ so $p \in E_J$. Since g does not vanish on $\bar{D}_1 \sim W$, $E_J = E_I$. For p in $W \cap E_I$, $h(p) \neq 0$, so f and g have zeros of the same order at p. Hence $u_J = u_I$. Now on W, $S'_J = \sigma_J + u_J$, where σ_J is a singular harmonic function on W and u_J is a quasi-bounded harmonic function. We shall show that $\sigma_J = S_I$. Since $s_f = s_g + s_h$, $s_f \geqq s_g$. It follows from the uniqueness of the canonical decomposition that on W $s'_g = s_g + u_g$, where u_g is a nonnegative quasi-bounded harmonic function on W. Since $s'_g \geqq S'_J = \sigma_J + u_J$, $s_g \geqq \sigma_J$. Thus $s_f \geqq \sigma_J$, and so $S_I \geqq \sigma_J$. On the other hand, let $\mathscr{F}$ denote the set of all nonnegative superharmonic functions u' on D such that $u' \geqq S_I$ on W. Then $\mathscr{F}$ is a Perron family, hence $\tau' = \inf \mathscr{F}$ is a nonnegative harmonic function in D and $\tau' \geqq S_I$ on W. Let f be a nonzero element of J. Then $f \in I$ so $s'_f \geqq s_f \geqq S_I$ on W. Then $s'_f \in \mathscr{F}$, and so $s'_f \geqq \tau'$. Hence $S'_J \geqq \tau'$. Since $S'_J = \sigma_J + u_J \geqq \sigma' \geqq S_I$. Thus $S_I = \sigma_J$.

Let f be a nonzero element of K. We may assume that p_0, q_0 are chosen so that $f(p_0) \neq 0$ and $f(q_0) \neq 0$. Let k be a function holomorphic in a neighborhood of $\overline{D \cup R}$ that vanishes at p_0, q_0 and only there. By the remarks following Theorem 1, there exist functions $g \in A(D)$ and $h \in A(W \cup \mathrm{R})$ such that $k^2 f = gh$ and h does not vanish on E_I. (Note that only finitely many points of E_I lie outside R.) We shall prove below that $g \in J$. Then $g \in I$, so $k^2 f \in I$. Thus f is in I because k is invertible modulo I. Therefore $I = K$.

To prove that $g \in J$, we must show that g vanishes on E_J, that $\mu_g \geqq \mu_J$ and that $s'_g \geqq S'_J$. The first two of these conditions hold because h and k do not vanish on E_I. By the remarks following Theorem 1, $s_f = s_g + s_h$.

Now S_I is bounded near each point of $\partial W \sim E_I$. Since h does not vanish on E_I, s_h is bounded near each point of E_I. Thus s_h and S_I are mutually singular. Since $s_f - S_I \geqq 0$, we can decompose it with respect to S_I, obtaining

$$s_f - S_I = \mu + \phi$$

where $\mu \perp S_I$ and $\phi \ll S_I$. We can also decompose s_g with respect to S_I, obtaining

$$s_g = \nu + \psi$$

where $\nu \perp S_I$ and $\psi \ll S_I$. Equating the two decompositions of s_f with respect to S_I, we have

$$\psi = \phi + S_I \ .$$

Thus $s_g - S_I = \nu + \phi \geqq 0$. We now claim that $s'_g \geqq S'_J$. On $\partial D \sim E_J$, S'_J has boundary values zero, so

$$\liminf_{p \to q} (s'_g(p) - S'_J(p)) \geqq 0$$

for $q \in \partial D \sim E_I$. Now $s_g - S_I$ is the singular part of $s'_g - S'_J$ on W. It follows from Lemma 4 that $(s'_g - S'_J) - (s_g - S_I)$ has vanishing boundary values on β. Thus the inequality above holds on β also, and by the maximum principle $s'_g \geqq S'_J$.

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