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THE EQUATIONS $\Delta u = Pu(P \geqq 0)$ ON RIEMANN SURFACES AND ISOMORPHISMS BETWEEN RELATIVE HARDY SPACES

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It has been demonstrated by M. Nakai that the Banach spaces PB (the space of bounded solutions on R of the equation $\Delta u = Pu$, $P \geq 0$) and HB (the space of bounded harmonic functions on R) are isometrically isomorphic whenever the condition

$$\int_R P(z)G(z, w_0)\,dxdy < +\infty$$

is valid for some point w_0 in R ($z = x + iy$). Here, $G(z, w)$ is the harmonic Green's function on R. In this paper we shall show, under the preceding condition that the Hardy space H^p, $1 < p \leq +\infty$, of harmonic functions on a hyperbolic Riemann surface R is isometrically isomorphic to the relative Hardy space PH^p_w of quotients of solutions of $\Delta u = Pu$ by the P-elliptic measure w of R.

1. Introduction. Throughout this paper, let R be a hyperbolic Riemann surface. We consider a density P on R, that is, a non-negative Hölder continuous function on R which depends on the local parameter $z = x + iy$ in such a way the partial differential equation

$$(1.1) \qquad \Delta u = Pu\,, \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2\,,$$

is invariantly defined on R. Let $P \not\equiv 0$ on R. A real valued function u is called a P-harmonic function (or P-solution) in an open set U of R, if u has continuous partial derivatives up to the order 2 and satisfies the equation (1.1) on U. The totality of bounded P-harmonic functions on R is denoted by PB. Then, PB is a Banach space with the uniform norm

$$(1.2) \qquad \|u\| = \sup_{z \in R} |u(z)|\,.$$

Also, HB is the Banach space of the totality of bounded harmonic functions on R with the uniform norm (1.2).

Many works ([5, 6, 11, 12 and 14] among others) were done on the comparison theorem, that is, to compare the spaces PB for different choices of P. For example, in 1960 ([11]) it was proved that, if two densities P and Q on R satisfy the condition

$$(1.3)\qquad \int_R |P(z) - Q(z)| \{G^P(z, w_0) + G^Q(z, w_1)\} dxdy < +\infty$$

for some points w_0 and w_1 in R, then Banach spaces PB and QB are isometrically isomorphic, where $G^P(z, w)$ and $G^Q(z, w)$ are Green's functions of R with pole w associated with the equations (1.1) and $\Delta u = Qu (Q \geqq 0)$ respectively. Here, in particular we consider the case $Q \equiv 0$ on R. In this case we can conclude that under the assumption

$$(1.4)\qquad \int_R P(z) G(z, w_0) dxdy < +\infty$$

for some point w_0 in R, the Banach spaces PB and HB $(=QB, Q \equiv 0)$ are isometrically isomorphic, where $G(z, w)$ is the harmonic Green's function of R with pole w in R.

Let a pair (R, P), $P \not\equiv 0$, be hyperbolic. Then there exists the positive P-solution w on R which takes the constant 1 at the ideal boundary of R, which we call the P-elliptic measure of R. The P-elliptic measure w plays a role somewhat analogous to that played by the constant 1. A w-P-harmonic function is a quotient of a P-harmonic function by the P-elliptic measure w. The relative Hardy class PH^p_w, $1 \leqq p \leqq +\infty$, of w-P-harmonic functions in R is defined by the way analogous to that of Hardy class H^p of harmonic functions on R. We are interested in the comparison problem of Banach space structures of PH^p_w and H^p. In this paper, we shall give the theorem: under the assumption (1.4) the Banach spaces PH^p_w and H^p, $1 < p \leqq +\infty$, isometrically isomorphic.

Let Δ_1 and Δ_{P1} are the sets of minimal boundary points of Martin and P-Martin compactifications, respectively. And, let χ and χ_P be the harmonic measure on Δ_1 and the P-elliptic measure on Δ_{P1}, respectively. Since L. L. Naim [9] proved that H^p and PH^p_w, $1 < p \leqq +\infty$, are isometrically isomorphic to Banach spaces $L^p(\Delta_1, \chi)$ and $L^p(\Delta_{P1}, \chi_P)$ respectively, by constructing a measurable transformation defined almost everywhere on Δ_{P1} into Δ_1 we shall investigate a relation between χ and χ_P under the assumption (1.4), and so, we can find an isomorphism from PH^p_w onto H^p, $1 < p \leqq +\infty$.

2. Preliminaries. In 1941 Martin [7] introduced a compactification in the investigation of nonnegative harmonic functions. Nakai [10] extended the Martin theory to the setting of P-harmonic functions. The results of these theories were established by Hervé [4] in the setting of Brelot's axiomatic potential theory. We shall use extensively the Martin compactification R^* and the Nakai's P-Martin compactification R^*_P of R. We denote by Δ_{P1} (resp. Δ_1) the set of minimal points of $R_P^* - R$ (resp. $R^* - R$) and by K_a^P (resp. K_b) the associated Martin kernel of $a \in \Delta_{P1}$ (resp. $b \in \Delta_1$) with pole z_0. And, let

$$K^P(z, a) = K_a^P(z) , \qquad z \in R \quad \text{and} \quad a \in \Delta_{P1} ,$$

and

$$K(z, b) = K_b(z) , \qquad z \in R \quad \text{and} \quad b \in \Delta_1 .$$

Let P be a density on R which is not constantly zero on R. Almost every theorem in this paper will be proved under Nakai's integral condition:

$$\int_R P(z)G(z, w_1)dxdy < +\infty \tag{2.1}$$

at some point w_1 in R. If this condition holds at some point w_1 of R, then it does at all points of R by Harnack's inequality.

We state the definition of P-elliptic measure from the work of H. Royden [14]. By a compact region we mean a connected open set whose closure is compact and whose boundary is composed of finite number of analytic curves. Let $\{R_n\}$ be an exhaustion of R, i.e., a sequence of compact regions such that $\bar{R}_n \subset R_{n+1}$ and $R = \bigcup_{n=1}^{\infty} R_n$. We define the function w_n to be the P-solution in R_n which is identically one on ∂R_n. For $P \not\equiv 0$ we have $0 < w_n < 1$. Since the maximum principle implies that the functions w_n form a monotone decreasing sequence of positive P-solutions, this sequence converges uniformly on each compact set in R to a nonnegative P-solution w, which is called the P-elliptic measure of R.

The P-elliptic measure w is either identically zero or else everywhere positive. In the second case we say that the pair (R, P) is hyperbolic provided $P \not\equiv 0$.

The P-elliptic measure w may be characterized as the largest P-solution which is bounded by 1.

For the P-elliptic measure w of R, there exists a unique measure χ_P supported by Δ_{P1} such that

$$w(z) = \int_{\Delta_{P1}} K^P(z, a)d\chi_P(a) , \qquad z \in R , \tag{2.2}$$

which is called the P-elliptic measure on the P-Martin boundary.

And, the harmonic measure is denoted by χ, that is, the measure which represents the constant function 1 and is supported by Δ_1:

$$1 = \int_{\Delta_1} K(z, b)d\chi(b) , \qquad z \in R .$$

DEFINITION 2.1. We introduce the set Δ_{P0} of point a in Δ_{P1} such that

$$\int_R P(z)G(z, w_1)K^P(z, a)dxdy < +\infty \tag{2.3}$$

for some point w_1 in R, and hence for every point in R.

DEFINITION 2.2. We introduce the set Δ'_{0P} of points b in Δ_1 such that

$$\int_R P(z)G(z, w_1)K(z, b)dxdy < +\infty$$

for some point w_1 in R.

LEMMA 2.1. *Let u be a positive P-solution on R such that*

$$\int_R P(z)G(z, w_1)u(z)dxdy < +\infty \tag{2.4}$$

for some point w_1 in R, and let μ be the canonical measure on Δ_{P1} which represents u:

$$u(z) = \int_{\Delta_{P1}} K^P(z, a)d\mu(a) , \qquad z \in R .$$

Then, $\Delta_{P1} - \Delta_{P0}$ is a measurable set of μ-measure zero.

Proof. For each positive integer n, let E_n be a set of points a in Δ_{P1} such that

$$\int_R P(z)G(z, w_1)K^P(z, a)dxdy \geqq n ,$$

where w_1 is a fixed point in R. Since E_n is measurable and, by Fubini's theorem,

$$\begin{aligned} n\mu(E_n) &\leqq \int_{\Delta_{P1}} \left\{\int_R P(z)G(z, w_1)K^P(z, a)dxdy\right\} d\mu(a) \\ &= \int_R P(z)G(x, w_1)\left\{\int_{\Delta_{P1}} K^P(z, a)d\mu(a)\right\} dxdy \\ &= \int_R P(z)G(z, w_1)u(z)dxdy , \end{aligned}$$

we have

$$\begin{aligned} \mu(\Delta_{P1} - \Delta_{P0}) &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &\leqq \mu(E_n) \\ &\leqq \frac{1}{n}\int_R P(z)G(z, w_1)u(z)dxdy \end{aligned}$$

for every positive integer n. Hence, it follows that $\mu(\Delta_{P1} - \Delta_{P0}) = 0$.

LEMMA 2.2. *Let v be a positive harmonic function on R such that*

$$\int_R P(z)G(z, w_1)v(z)dxdy < +\infty$$

for some w_1 in R and ν be the canonical measure on Δ_1 which represents v:

$$v(z) = \int_{\Delta_1} K(z, b)d\nu(b) , \qquad z \in R .$$

Then, $\Delta_1 - \Delta'_{0P}$ is a measurable set of ν-measure zero.

Proof. This can be shown by the same proof as that of Lemma 2.1.

THEOREM 2.3. *Let P be a density on R which satisfies Nakai's integral condition* (2.1). *Then, the P-elliptic measure of the set $\Delta_{P1} - \Delta_{P0}$ is zero:*

$$\chi_P(\Delta_{P1} - \Delta_{P0}) = 0 .$$

Proof. Since $w < 1$ on R, from (2.1) it follows that

$$\int_R P(z)G(z, w_1)w(z)dxdy \leqq \int_R P(z)G(z, w_1)dxdy < +\infty .$$

Therefore, by the fact that w is represented as the integral (2.2) by the measure χ_P, Lemma 2.1 gives this theorem.

Lemma 2.2 gives the following:

THEOREM 2.4. *Under the same assumption as that in Theorem 2.3, the harmonic measure of the set $\Delta_1 - \Delta'_{0P}$ is zero.*

Proof. The constant function 1 and the harmonic measure χ play the roles of v and ν in Lemma 2.2.

3. Relations between minimal P-solutions and minimal harmonic functions. To give an isomorphism between PB and QB, Nakai [11] has defined the transform $T^{PQ}f$ for a function f as follows:

$$T^{PQ}f(z) = f(z) + \frac{1}{2\pi}\int_R (P(w) - Q(w))G^Q(w, z)f(w)dudv ,$$

where $w = u + iv$. And, Lahtinen [5], Nakai, and Satō [15] showed some properties of the transformation T^{PQ}. In this paper we consider only the case in which Q is identically zero on R. Some usefull properties of T^{P0} will be shown in this section.

DEFINITION 3.1. Let P be a density on R and f be a continuous function on R for which

$$\int_R P(w)G(w, z_0)|f(w)|dudv < +\infty , \qquad w = u + iv , \tag{3.1}$$

is true at some point z_0 in R (then it holds at all points z in R). Then, the linear transformation $T^{P0}f$ of f is well defined by

$$T^{P0}f(z) = f(z) + \frac{1}{2\pi}\int_R P(w)G(w, z)f(w)dudv . \tag{3.2}$$

By changing the role of P and 0 we define also the transformation T^{0P}. For a continuous function g on R such that

$$\int_R P(w)G^P(w, z_0)|g(w)|dudv < +\infty \tag{3.3}$$

for some point z_0 in R, $T^{0P}g$ is defined by

$$T^{0P}g(z) = g(z) - \frac{1}{2\pi}\int_R P(w)G^P(w, z)g(w)dudv . \tag{3.4}$$

To derive properties of T^{P0} we consider an auxiliary sequence of transformations T_n^{P0}, $n = 1, 2, \cdots$, of a real valued continuous function f defined on the closure $\bar{R}_n$ of R_n as follows:

$$T_n^{P0}f(z) = f(z) + \frac{1}{2\pi}\int_{R_n} P(w)G(R_n, w, z)f(w)dudv ,$$

where $G(R_n, w, z)$ is the harmonic Green function on R_n. It is evident that, if f is a P-solution on R_n, then $T_n^{P0}f$ is a continuous function on $\bar{R}_n$ which is harmonic on R_n and satisfies

$$T_n^{P0}f|\partial R_n = f|\partial R_n$$

(see, for example, Nakai [11] or Lahtinen [5]).

The following lemma is a special case of Lahtinen's lemma in [5] in which P is acceptable in the sense of his definition.

LEMMA 3.1 (*Lahtinen*). *Let f be a P-solution on R and $\{f_n\}$ a* *sequence of P-solutions each defined on* $\bar{R}_n$ *such that* $\lim_{n\to+\infty} f_n = f$. *If there exists a function u continuous on R such that* $|f_n| \leqq u$ *for each positive integer n and u fullfils the inequality obtained by replacing f by u in* (3.1) *at some point in R, then* $T^{P0}f$ *is well defined and has the following properties*: (1) $\lim_{n\to+\infty} T_n^{P0} f_n = T^{P0} f$, (2) $T^{P0}f$ *is harmonic on R.*

By changing the roles of P and 0, the transformation T_n^{0P} is defined and we can state the following:

LEMMA 3.1′. *Let g be a harmonic function on R and* $\{g_n\}$ *be a sequence of harmonic functions each defined on* R_n *such that* $\lim_{n\to+\infty} g_n = g$. *If there exists a function v continuous on R such that* $|g_n| \leqq v$ *for each positive integer n and v fullfils the inequality* (3.3) *at some point in R, then* $T^{0P}g$ *is well defined and has the following properties*: (1) $\lim_{n\to+\infty} T_n^{0P} g_n = T^{0P}g$, (2) $T^{0P}g$ *is P-harmonic on R.*

LEMMA 3.2. *Let P be a density on R. If P satisfies Nakai's condition*

$$\int_R P(z)G(z, w_0)dxdy < +\infty \tag{3.5}$$

for some point w_0 *in R, then we have*

$$\begin{aligned} G(z, w) &= G^P(z, w) + \frac{1}{2\pi}\int_R P(\zeta)G(\zeta, z)G^P(\zeta, w)d\xi d\eta \\ &= G^P(w, z) + \frac{1}{2\pi}\int_R P(\zeta)G(\zeta, w)G^P(\zeta, z)d\xi d\eta , \end{aligned} \tag{3.6}$$

for each point (z, w) *in* $R \times R$ *with* $w \neq z$, *where* $\zeta = \xi + i\eta$.

Proof. Green's formula implies that, for (z, w) in $R_n \times R_n$ with $z \neq w$,

$$\begin{aligned} G(R_n, z, w) &= G^P(R_n, z, w) \\ &\quad + \frac{1}{2\pi}\int_{R_n} P(\zeta)G(R_n, \zeta, w)G^P(R_n, \zeta, z)d\xi d\eta , \end{aligned} \tag{3.7}$$

where $G^P(R_n, z, w)$ is Green's function of R_n related to the differential equation (1.1) and $G(R_n, z)$ is the harmonic Green function of R_n.

In order to apply Lebesgue's dominated covergence theorem, let

$$F(z, w, \zeta) = P(\zeta)G(\zeta, w)G^P(\zeta, z) .$$

And, let U and V be small discs with centers z and w respectively such that $U \cap V = \emptyset$. Then, by the minimum principle, Nakai's condition (3.5) gives that

$$\int_V F(z, w, \zeta) d\xi d\eta \leqq \sup_{\zeta \in \partial U} G^P(\zeta, z) \times \int_R P(\zeta) G(\zeta, w) d\xi d\eta < +\infty$$

and, by $G^P(\zeta, z) < G(\zeta, z)$ (which follows from the definition of the Green function),

$$\begin{aligned}\int_{R-V} F(z, w, \zeta) d\zeta d\eta &\leqq \sup_{\zeta \in \partial V} G(\zeta, w) \times \int_R P(\zeta) G^P(\zeta, z) d\xi d\eta \\ &< \sup_{\zeta \in \partial V} G(\zeta, w) \times \int_R P(\zeta) G(\zeta, z) d\xi d\eta < +\infty ,\end{aligned}$$

from which it follows that

$$\int_R F(z, w, \zeta) d\xi d\eta = \int_V F(z, w, \zeta) d\xi d\eta + \int_{R-V} F(z, w, \zeta) d\xi d\eta < +\infty . \tag{3.8}$$

Therefore, since

$$P(\zeta) G(R_n, \zeta, w) G^P(R_n, \zeta, z) \leqq F(z, w, \zeta)$$

for each positive integer n and

$$\lim_{n \to +\infty} P(\zeta) G(R_n, \zeta, w) G^P(R_n, \zeta, z) = P(\zeta) G(\zeta, w) G^P(\zeta, z) ,$$

Lebesgue's dominated convergence theorem shows (3.6) as n tends to $+\infty$ in (3.7).

LEMMA 3.3. *Let f be a continuous function on R such that*

$$\int_R P(z) G(z, w_1) |f(z)| dxdy < +\infty \tag{3.9}$$

for some point w_1 in R. Then, it holds that

$$\int_R P(z) G^P(z, w_0) |T^{P0} f(z)| dxdy < +\infty \tag{3.10}$$

for all points w_0 in R.

Proof. By the definition of $T^{P0}f$, we have

$$(3.11)\qquad \begin{aligned} &\int_R P(z)G^P(z, w_0)\,|\,T^{P0}f(z)\,|\,dxdy \\ &\leqq \int_R P(z)G^P(z, w_0)\,|\,f(z)\,|\,dxdy \\ &+ \int_R P(z)G^P(z, w_0)\left\{\frac{1}{2\pi}\int_R P(w)G(w, z)\,|\,f(w)\,|\,dudv\right\}dxdy\ . \end{aligned}$$

Here, the first term of the right side of this inequality is finite by the inequality $G^P(z, w_0) < G(z, w_0)$ on R.

To apply Fubini's theorem to the second term we define a function $F(z, w)$ by

$$F(z, w) = \frac{1}{2\pi}P(z)P(w)G^P(z, w_0)G(w, z)\,|\,f(w)\,|\ .$$

Since, by Lemma 3.2,

$$(3.12)\qquad \begin{aligned} &\int_R \left\{\int_R F(z, w)dxdy\right\}dudv \\ &= \int_R \left\{\frac{1}{2\pi}\int_R P(z)G^P(z, w_0)G(z, w)dxdy\right\}P(w)\,|\,f(w)\,|\,dudv \\ &= \int_R P(w)\left\{G(w, w_0) - G^P(w, w_0)\right\}\,|\,f(w)\,|\,dudv \\ &< \int_R P(w)G(w, w_0)\,|\,f(w)\,|\,dudv \\ &< +\infty\ . \end{aligned}$$

Fubini's theorem shows that the second term of the right side of (3.11) is equal to (3.12). Hence, we established the lemma.

LEMMA 3.4. *Let f be a positive P-harmonic function on R which satisfies the same condition as that in Lemma 3.3. Then we have*

$$T^{0P}(T^{P0}f) = f \quad on \quad R\ .$$

Proof. Lemma 3.3 shows the inequality (3.10), and so, Lemma 3.1 and 3.1′ imply that $T^{0P}(T^{P0}f)$ is well defined and is P-harmonic, since

$$T^{P0}f = \lim_{n\to+\infty} T^{P0}_n f$$

and

$$T^{P0}_n f < T^{P0}f$$

for every n. Furthermore,

$$T^{0P}(T^{P0}f) = \lim_{n\to+\infty} T_n^{0P}(T_n^{P0}f) \tag{3.13}$$

on R.

The definitions of T_n^{P0} and T_n^{0P} give that

$$T_n^{0P}(T_n^{P0}f)\,|\,\partial R_n = T_n^{P0}f\,|\,\partial R_n = f\,|\,\partial R_n\,.$$

Since $T_n^{0P}(T_n^{P0}f)$ is P-harmonic on R_n, the maximum principle implies that

$$T_n^{0P}(T_n^{P0}f) = f \quad \text{on} \quad R_n\,,$$

which completes the proof by (3.13).

LEMMA 3.5. *Let g be a continuous function on R such that*

$$\int_R P(z)G(z, w_1)\,|\,g(z)\,|\,dxdy < +\infty \tag{3.14}$$

for some point w_1 in R. Then, it follows that

$$\int_R P(z)G(z, w_0)\,|\,T^{0P}g(z)\,|\,dxdy < +\infty\,,$$

where w_0 is any point in R.

Proof. This can be provided in the same way as that of Lemma 3.3.

DEFINITION 3.2. We define the space P_0 (resp. H'_P) consisting of positive P-solutions f (resp. positive harmonic functions g) on R with the property (3.9) (resp. (3.14)), and define the space H_P consisting of positive harmonic functions g on R such that

$$\int_R P(z)G^P(z, w_1)g(z)dxdy < +\infty$$

for some point w_1 in R.

LEMMA 3.6. *Let g be a harmonic function in H_P such that $T^{0P}g$ belongs to the space P_0. Then, it follows that*

$$T^{P0}(T^{0P}g) = g \quad \textit{on} \quad R\,.$$

Proof. Since the function $T^{0P}g$ satisfies the condition (3.1) in Definition 3.1, $T^{P0}(T^{0P}g)$ is well defined, and Lemmas 3.1, 3.1′ show the equality in this lemma by the same way as that in the proof of Lemma 3.4.

LEMMA 3.7. $H'_P \subset T^{P0}(P_0) \subset H_P$.

Proof. Lemmas 3.3, 3.5 and 3.6 show this lemma.

THEOREM 3.8. *T^{P0} is a one-to-one transformation from P_0 onto $T^{P0}(P_0)$, and T^{0P} coinsides with its inverse transformation.*

Proof. Lemma 3.4 shows this theorem.

LEMMA 3.9. *Let g and g_1 be harmonic functions on R. If $g \leqq g_1$ on R and g, g_1 belong to the space H_P. Then, it follows that*

$$T^{0P}g \leqq T^{0P}g_1 \quad on \quad R .$$

Proof. Since

$$T_n^{0P}g \,|\, \partial R_n = g \,|\, \partial R_n \leqq g_1 \,|\, \partial R_n = T_n^{0P}g_1 \,|\, \partial R_n ,$$

the maximum principle for P-solutions shows that

$$T_n^{0P}g \leqq T_n^{0P}g_1 \quad \text{on} \quad R_n$$

for each n. Thus, from Lemma 3.1′ it follows that

$$T^{0P}g \leqq T^{0P}g_1 \quad \text{on} \quad R .$$

THEOREM 3.10. *If a minimal P-solution K_a^P belongs to the space P_0 (i.e., $a \in \Delta_{P0}$), then $T^{P0}K_a^P$ is a minimal harmonic function on R, that is, there exists a unique point b in Δ_1 such that*

$$T^{P0}K_a^P = T^{P0}K_a^P(z_0)K_b \quad on \quad R .$$

Proof. Let g be a positive harmonic function on R such that

$$0 < g \leqq T^{P0}K_a^P \quad \text{on} \quad R . \tag{3.15}$$

By Lemmas 3.4 and 3.9 we have

$$0 < T^{0P}g \leqq T^{0P}(T^{P0}K_a^P) = K_a^P \quad \text{on} \quad R ,$$

and so, $T^{0P}g = \alpha K_a^P$ on R, where α is a positive constant. Then, since (3.15) implies $g \in H_P$ by Lemma 3.3, from Lemma 3.6 it follows that

$$g = T^{P0}(T^{0P}g) = \alpha T^{P0}K_a^P ,$$

which shows that $T^{P0}K_a^P$ is a minimal harmonic function on R.

LEMMA 3.11. *Let f and f_1 be P-harmonic functions on R. If $f \leqq f_1$ on R and f, f_1 belong to the space P_0, then it follows that*

$$T^{P0}f \leqq T^{P0}f_1 \quad on \ R .$$

Proof. By Lemma 3.1 this can be proved similarly as Lemma 3.8.

THEOREM 3.12. *If a minimal harmonic function K_b belongs to $T^{P0}(P_0)$, then $T^{0P}K_b$ is a minimal P-harmonic function on R and is contained in the space P_0.*

Proof. This can be proved similarly as Theorem 3.10 by Lemma 3.11.

Theorems 3.10 and 3.12 can be paraphrased by saying that the transformation $T^{P0}: P_0 \to H_P$ gives a one-to-one mapping from the set of minimal P-harmonic functions in P_0 onto the set of minimal harmonic functions in $T^{P0}(P_0)$.

The following theorem says that the P-elliptic measure w of R is transformed into the constant function 1 on R by T^{P0}.

THEOREM 3.13. *If the pair (R, P) is hyperbolic, then $T^{P0}w = 1$ on R.*

Proof. By $w = \lim_{n\to+\infty} w_n$, Lemma 3.1 implies that

$$T^{P0}w = \lim_{n\to+\infty} T_n^{P0}w_n = 1 \quad \text{on} \quad R .$$

4. Relation between the P-elliptic and harmonic measures.

DEFINITION 4.1. Let Δ_{0P} be a set consisting of points b in Δ_1 such that the minimal harmonic function K_b belongs to the set $T^{P0}(P_0)$:

$$\Delta_{0P} = \{b \in \Delta_1 \colon K_b \in T^{P0}(P_0)\} .$$

In the following, it will be shown that Δ_{0P} is measurable. We shall use the same notations χ_P and χ for the restrictions of the P-elliptic and harmonic measures to the measurable sets Δ_{P0} and Δ_{0P}

respectively, and consider two measure spaces (Δ_{P0}, χ_P) and (Δ_{0P}, χ).

The purpose of this section is to show that there exists a measur ability preserving transformation t from (Δ_{P0}, χ_P) onto (Δ_{0P}, χ) such that $\chi_P \circ t^{-1}$ is absolutely continuous with respect to χ.

Theorems 3.10 and 3.12 give the following definitions.

DEFINITION 3.2. We define a transformation

$$t_{P0}\colon \Delta_{P0} \longrightarrow \Delta_{0P}$$

by assigning to a in Δ_{P0} a point $b = t_{P0}(a)$ in Δ_{0P} such that $T^{P0}K_a^P(z_0)$ $K_b = T^{P0}K_a^P$ on R.

DEFINITION 3.3. We define a transformation

$$t_{0P}\colon \Delta_{0P} \longrightarrow \Delta_{P0}$$

by assigning, to b in Δ_{0P}, a point $a = t_{0P}(b)$ in Δ_{P0} such that $T^{0P}K_b(z_0)K_a^P = T^{0P}K_b$ on R.

It is clear, by Theorems 3.8, 3.10 and 3.12, that t_{0P} is the inverse of t_{P0}: $t_{0P} = t_{P0}^{-1}$.

LEMMA 4.1. *Under Nakai's condition*:

$$\int_R P(z)G(z, w_1)dxdy < +\infty \tag{4.1}$$

for some point w_1 in R (then it holds at all points in R), the function $T^{P0}K_a^P(w_0)$ of a in Δ_{P0} is lower semi-continuous on Δ_{P0}, where w_0 is any fixed point in R.

Proof. Let $D_r(w_0)$ be the disc centered at w_0 and having radius r. By Harnak's inequality there is a positive constant α such that, for all z in $D_r(w_0)$ and for all points a in Δ_{P0},

$$\alpha^{-1}K^P(w_0, a) \leqq K^P(z, a) \leqq \alpha K^P(w_0, a) .$$

Thus, for each point a in Δ_{P0}

$$\begin{aligned} &\int_{D_r(w_0)} P(z)G(z, w_0)K^P(z, a)dxdy \\ &\qquad \leqq \alpha K^P(w_0, a) \times \int_{D_r(w_0)} P(z)G(z, w_0)dxdy . \end{aligned} \tag{4.2}$$

Since (4.1) implies that

$$\lim_{r\to 0}\int_{D_r(w_0)} P(z)G(z, w_0)dxdy = 0 ,$$

for any $\varepsilon > 0$ we can find a positive number $\delta = \delta(\varepsilon)$ such that

$$\int_{D_\delta(w_0)} P(z)G(z, w_0)dxdy < \varepsilon . \tag{4.3}$$

The function $K^P(z, a)$ is finitely continuous on $(\bar{R}_n - D_\delta(w_0)) \times \Delta_{P0}$, so that for any $\varepsilon > 0$ there exists a neighborhood $U(a)$ of a such that

$$|K^P(z, a') - K^P(z, a)| < \varepsilon$$

for $a' \in U(a) \cap \Delta_{P0}$ and $z \in \bar{R}_n - D_\delta(w_0)$. Therefore, from (4.2) and (4.3) it follows that

$$\begin{aligned}
&\left|\int_{R_n} P(z)G(z, w_0)K^P(z, a')dxdy - \int_{R_n} P(z)G(z, w_0)K^P(z, a)dxdy\right| \\
&\quad \leqq \int_{R_n - D_\delta(w_0)} P(z)G(z, w_0)|K^P(z, a') - K^P(z, a)|dxdy \\
&\qquad + \int_{D_\delta(w_0)} P(z)G(z, w_0)K^P(z, a')dxdy \\
&\qquad + \int_{D_\delta(w_0)} P(z)G(z, w_0)K^P(z, a)dxdy \\
&\quad \leqq \varepsilon \times \int_{R_n - D_\delta(w_0)} P(z)G(z, w_0)dxdy \\
&\qquad + \alpha(K^P(w_0, a') + K^P(w_0, a)) \times \int_{D_\delta(w_0)} P(z)G(z, w_0)dxdy \\
&\quad \leqq \varepsilon \times \int_R P(z)G(z, w_0)dxdy + \varepsilon \times \alpha(K^P(w_0, a') + K^P(w_0, a)) .
\end{aligned}$$

This inequality shows the continuity of the function on Δ_{P0}:

$$\int_{R_n} P(z)G(z, w_0)K^P(z, a)dxdy ,$$

by which the relation

$$\lim_{n\to+\infty}\int_{R_n} P(z)G(z, w_0)K^P(z, a)dxdy = \int_R P(z)G(z, w_0)K^P(z, a)dxdy$$

implies that $T^{P0}K_a^P(w_0)$ is lower semi-continuous on Δ_{P0}.

LEMMA 4.2. *The function $T^{0P}K_b(w_0)$ of b in Δ_{0P} is upper semi-continuous on Δ_{0P}, where w_0 is a fixed point in R.*

Proof. Applying the inequality

$$\int_R P(z)G^P(z, w)dxdy \leqq 2\pi$$

for all w in R, which is stated in Myberg [8], we can prove this lemma in the same way as that of the proof of Lemma 4.1. This Myberg's inequality plays the role of Nakai's condition (4.1) in Lemma 4.1.

Let Δ_P and Δ be the P-Martin and Martin ideal boundaries of R, respectively. We identify these ideal boundaries Δ_P and Δ with subsets of the product space of the real lines. Let $\{w_i\}$ be a countable dense set of points in R. To a point a in Δ_P (resp. b in Δ) we assign a point $m_P(a)$ (resp. $m_0(b)$) of the product space $\prod_{i=1}^{\infty} I_i$ (I_i is the real line for all i) whose ith coordinate is $K^P(w_i, a)$ (resp. $K(w_i, b)$) for each i. Then, the mappings

$$m_P: \Delta_P \longrightarrow \prod_{i=1}^{\infty} I_i$$

and

$$m_0: \Delta \longrightarrow \prod_{i=1}^{\infty} I_i$$

are continuous and one-to-one, and also their inverse mappings

$$m_P^{-1}: m_0(\Delta_P) \longrightarrow \Delta_P$$

and

$$m_0^{-1}: m_0(\Delta) \longrightarrow \Delta$$

are continuous. Therefore, the mappings

$$m_P: \Delta_P \longrightarrow m_P(\Delta_P)$$

and

$$m_0: \Delta \longrightarrow m_0(\Delta)$$

are homeomorphisms.

For a point $m_P(a)$ in $m_P(\Delta_{P0})$ we assign a point in $m_0(\Delta_{0P})$ whose ith coordinate is $K(w_i, t_{P0}(a))$ for each i; this mapping will be denoted by

$$s_{P0}: m_P(\Delta_{P0}) \longrightarrow m_0(\Delta_{0P}) .$$

And, the mapping

$$s_{0P}: m_0(\Delta_{0P}) \longrightarrow m_P(\Delta_{P0})$$

is defined by the same way as that of s_{P0}, that is, for a point $m_0(b)$ in $m_0(\Delta_{0P})$ we assign a point in $m_P(\Delta_{P0})$ whose ith coordinate is $K^P(w_i, t_{0P}(b))$ for each i. It is evident that s_{0P} is the inverse mapping of s_{P0}.

In the following we shall always assume Nakai's condition (4.1).

THEOREM 4.4. *The mapping*

$$t_{P0}: \Delta_{P0} \longrightarrow \Delta_{0P}$$

is measurability preserving.

Proof. Since m_0^{-1} is continuous on $m_0(\Delta)$ and, by Lemma 4.1, the ith coordinate of the point $s_{P0} \circ m_P(a)$, $a \in \Delta_{P0}$:

$$K(w_i, t_{P0}(a)) = T^{P0}K_a^P(w_i) \times \{T^{P0}K_a^P(z_0)\}^{-1}$$

is a measurable function on Δ_{P0} for each i, that is, $s_{P0} \circ m_P$ is measurable on Δ_{P0}, the relation

$$m_0^{-1} \circ s_{P0} \circ m_P = t_{P0} \quad \text{on} \quad \Delta_{P0}$$

shows that t_{P0} is measurable on Δ_{P0}.

Similarly, from

$$t_{P0}^{-1} = t_{0P} = m_P^{-1} \circ s_{0P} \circ m_0 \quad \text{on} \quad \Delta_{0P}$$

and Lemma 4.2, it follows that t_{P0}^{-1} is measurable on Δ_{0P}. Then the transformation $t_{P0}: \Delta_{P0} \rightarrow \Delta_{0P}$ is measurability preserving.

LEMMA 4.4. *Δ_{0P} is measurable, so that (Δ_{0P}, χ) is a measure space, where χ also denotes the restriction to Δ_{0P} of the harmonic measure on Δ_1.*

Proof. Since Δ_{P0} is measurable, this follows from the preceding lemma and the fact that $\Delta_{0P} = t_{0P}^{-1}(\Delta_{P0})$.

THEOREM 4.5. *The set $\Delta_1 - \Delta_{0P}$ is of harmonic measure zero:*

$$\chi(\Delta_1 - \Delta_{0P}) = 0 .$$

Proof. Since Δ'_{0P} consists of points b in Δ_1 such that the minimal harmonic function K_b belongs to the set H'_P (where Δ'_{0P} and H'_P are defined in §§2 and 3, respectively), Lemma 3.7 shows that

$$\Delta_1 - \Delta_{0P} \subset \Delta_1 - \Delta'_{0P} ,$$

which gives $\chi(\Delta_1 - \Delta_{0P}) = 0$ by Theorem 2.4.

LEMMA 4.6. *Let u be a P-harmonic function in P_0, and let μ be the canonical measure representing u:*

$$u(z) = \int_{\Delta_{P1}} K^P(z, a) d\mu(a) .$$

Then,

$$T^{P0}u(z) = \int_{\Delta_{P0}} T^{P0} K^P_a(z) d\mu(a) , \qquad z \in R .$$

Proof. For a point z in R, let F_z be a function defined by

$$F_z(w, a) = P(w) G(w, z) K^P(w, a)$$

for (w, a) in $R \times \Delta_{P0}$. Since Lemma 2.1 shows that $\Delta_1 - \Delta_{P0}$ has μ-measure zero, it follows that

$$\int_R \left\{\int_{\Delta_{P0}} F_z(w, a) d\mu(a)\right\} dudv = \int_R P(w) G(w, z) u(w) dudv < +\infty .$$

Then Fubini's theorem gives that

$$\begin{aligned} T^{P0}u(z) &= \int_{\Delta_{P0}} K^P(z, a) d\mu(a) + \int_R \left\{\int_{\Delta_{P0}} \frac{1}{2\pi} F_z(w, a) d\mu(a)\right\} dudv \\ &= \int_{\Delta_{P0}} \left\{K^P(z, a) + \frac{1}{2\pi} \int_R P(w) G(w, z) K^P(w, a) dudv\right\} d\mu(a) \\ &= \int_{\Delta_{P0}} T^{P0} K^P_a(z) d\mu(a) , \qquad z \in R . \end{aligned}$$

By the uniqueness in the Martin integral representation, we obtain the following usefull theorem:

THEOREM 4.7. *Let u be a P-harmonic function in P_0, and let v denote the harmonic function $T^{P0}u$. If the measures which represent u and v are denoted by μ_u and μ_v, respectively:*

$$u(z) = \int_{\Delta_{P1}} K^P(z, a) d\mu_u(a) ,$$

$$v(z) = \int_{\Delta_1} K(z, b) d\mu_v(b) ,$$

then the measure assigned to the restricted measure $\mu_u | \Delta_{P0}$ of μ_u by the measurability preserving transformation t_{P0} is absolutely con-

tinuous with respect to μ_v; $d\mu = T^{P0}K^P_{t^{-1}_{P0}(f)}(z_0)d\mu \circ t^{-1}_{P0}(b)$.

Proof. Since $t_{P0}\colon \Delta_{P0} \to \Delta_{0P}$ is measurable, from Lemma 4.6 and the definition of t_{P0}, it follows that

$$v(z) = T^{P0}u(z) = \int_{\Delta_{P0}} T^{P0}K^P_a(z)d\mu_u(a)$$
$$= \int_{\Delta_{P0}} K(z, t_{P0}(a))T^{P0}K^P_a(z_0)d\mu_u(a) = \int_{\Delta_{0P}} K(z, b)T^{P0}K^P_{t^{-1}_{P0}(b)}(z_0)d\mu_u \circ t^{-1}_{P0}(b) .$$

From the uniqueness of the canonical measure which represents v, this theorem follows.

Theorems 4.7 and 4.3 are reduced to the following theorem:

THEOREM 4.8. *Let* μ_u *and* μ_v *be measures defined in Theorem 4.7. Then,* t_{P0} *is a measurablity preserving transformation from the measure space* (Δ_{P0}, μ_u) *onto the measure space* (Δ_{0P}, μ_v) *such that* $d\mu_v = T^{P0}K^P_{t^{-1}_{P0}(b)}(z_0)d\mu_u \circ t^{-1}_{P0}(b)$.

COROLLARY 4.9. *Let* (R, P) *be a hyperbolic pair.* t_{P0} *is a measurability preserving transformation from the measure space* (Δ_{P0}, χ_P) *onto the measure space* (Δ_{0P}, χ) *such that* $d\chi = T^{P0}K^P_{t^{-1}_{P0}(b)}(z_0)d\chi_P \circ t^{-1}_{P0}(b)$.

Proof. By Theorem 3.13, Theorem 4.8 shows this corollary.

5. Comparisons between relative Hardy spaces. In [13] Parreau gave a characterization for harmonic functions in Hardy space on a Riemann surface, using the Martin boundary ([7]) and related kernel; in [9] L. L. Naim proved the similar results for the axiomatic functions of Brelot, using essentially Gowrisankaran's results ([3]) on axiomatic Martin boundary and fine limits. Since typical examples of Brelot's axiomatic setting are given by harmonic functions and by solutions of the differential equation $\Delta u = Pu (P \geqq 0)$ on an open Riemann surface R, any result established in [9] for Brelot's axiomatic setting holds for each of these two special cases without further verification. Restating definitions and theorems in [9] in the case of harmonic functions and P-solutions, we recall the definitions of Hardy spaces, the relative Hardy spaces and some theorems for functions in these spaces.

For an exhaustion $\{R_n\}$ of R and a fixed point z_0 in R, we denote by μ^P_{n,z_0} and μ_{n,z_0} the P-elliptic measure and harmonic measure on ∂R_n relative to z_0 and R_n, respectively. Clearly,

$$\int_{\partial R_n} d\mu^P_{n,z_0} \leqq 1 \quad \text{and} \quad \int_{\partial R_n} d\mu_{n,z_0} = 1$$

for all positive integers n.

DEFINITION 5.1. A hamonic function g on R belongs to the Hardy space H^p, $1 \leqq p \leqq +\infty$, if and only if the L^p-norms with respect to the harmonic measures μ_{n,z_0}, of the restrictions of g to the boundaries ∂R_n, are uniformly bounded in n. In other words, g belongs to H^p if and only if there exists a constant M, independent of n, such that $\|g_{p,n}\| \leqq M$ for all n, where

$$\|g\|_{p,n} = \left\{\int_{\partial R_n} |g|^p d\mu_{n,z_0}\right\}^{1/p}, \qquad 1 \leqq p < +\infty ,$$

and

$$\|g\|_{\infty,n} = \sup_{\partial R_n} |g| .$$

We proceed to define the relative Hardy spaces for the equation $\Delta u = Pu$ and harmonic functions. For a fixed positive P-harmonic function u on R we define the relative u-P-elliptic measure with respect to $z_0 \in R_n$ and R_n by

$$\mu^{P,u}_{n,z_0} = \frac{u}{u(z_0)} \times \mu^P_{n\ z_0} ,$$

and for a fixed positive harmonic function v on R we define the relative v-harmonic measure with respect to $z_0 \in R_n$ and R_n by

$$\mu^v_{n,z_0} = \frac{v}{v(z_0)} \times \mu_{n,z_0} .$$

For the positive P-harmonic function u u-P-harmonic functions are quotients of P-harmonic functions on R by u, and for the positive harmonic function v v-harmonic functions are quotients of harmonic functions on R by v.

DEFINITIONS 5.2. A u-P-harmonic function f' belongs to the relative Hardy class PH^p_u, $1 \leqq p \leqq +\infty$, if and only if the L^p-norms with respect to the relative u-P-elliptic measure $\mu^{P,u}_{n,z_0}$, of the restrictions of f' to the boundaries ∂R_n are uniformly bounded in n. In other words, f' belongs to PH^p_u if and only if there exists a constant M, independent of n, such that $\|f'\|^P_{p,n} \leqq M$ for positive integers, where

$$\|f'\|^P_{p,n} = \left\{\int_{\partial R_n} |f'|^p d\mu^{P,u}_{n,z_0}\right\}^{1/P}, \qquad 1 \leqq p < +\infty ,$$

$$\|f'\|^P_{\infty,n} = \sup_{\partial R_n} |f'| .$$

DEFINITION 5.3. A v-harmonic function g' belongs to the relative Hardy space H_v^p, $1 \leqq p \leqq +\infty$, if and only if the L^p-norms with respect to the relative v-harmonic measure $\mu_{n\ z_0}^v$, of the restrictions of g' to the boundaries ∂R_n are uniformly bounded in n. In other words, g' belongs to H_v^p if and only if there exists a constant M, independent of n, such that $\|g'\|_{p,n} \leqq M$ for all n, where

$$\|g'\|_{p,n} = \left\{\int_{\partial R_n} |g'|^p d\mu_{n,z_0}^v\right\}^{1/p}, \qquad 1 \leqq p < +\infty ,$$

$$\|g'\|_{\infty,n} = \sup_{\partial R_n} |g'| .$$

Naim gave the extended characterization of functions in Hardy spaces, showing the role of uniform integrability. We shall recall her theorems and restate them in our case. In the following, fine filters defined by the P-Martin compactification and minimal P-harmonic functions K_a^P, $a \in \varDelta_{P1}$, is called P-fine filters.

THEOREM 5.1. *Let u be a fixed positive P-harmonic function on R. A u-P-harmonic function f' belongs to the space PH_u^p, $1 < p \leqq +\infty$, if and only if f' is the solution of a Dirichlet problem relative to u with the P-minimal boundary $\varDelta_{P1}$, the P-fine filters in R and boundary function $\bar{f}'$ in $L^p(\varDelta_{P1}, \mu_u)$, where μ_u represents u in the integral representation*

$$u(z) = \int_{\varDelta_{P1}} K^P(z, a) d\mu_u(a) .$$

And, the correspondence $f' \to \bar{f}'$ is an isometric isomorphism of the Banach space PH_u^p onto $L^p(\varDelta_{P1}, \mu_u)$.

THEOREM 5.2. *A harmonic function g belongs to the space H^p, $1 < p \leqq +\infty$, if and only if g is the solution of a Dirichlet problem with the minimal boundary $\varDelta_1$, the fine filters in R and boundary function $\bar{g}$ in $L^p(\varDelta_1, \chi)$. And, the correspondence $g \to \bar{g}$ is an isometric isomorphism of the Banach space H^p onto $L^p(\varDelta_1, \chi)$.*

THEOREM 5.3. *Let v be a fixed positive harmonic function on R. A v-harmonic function g' belongs to the space H_v^p, $1 < p \leqq +\infty$, if and only if g' is the solution for a Dirichlet problem relative to v with the minimal boundary $\varDelta_1$, the fine filters in R and boundary function $\bar{g}'$ in $L^p(\varDelta_1, \mu_v)$, where μ_v represents v in the integral representation*

$$v(z) = \int_{\varDelta} K(z, b) d\mu_v(b) .$$

And, H_v^p is a Banach space isometrically isomorphic to $L^p(\Delta_1, \mu_v)$.

THEOREM 5.4. *Let u and v be functions on R satisfying the same conditions as those in Theorem* 4.7. *If Nakai's condition*:

$$\int_R P(z)G(z, w_0)dxdy < +\infty \tag{5.1}$$

for some point w_0 in R is satisfied, then the Banach space PH_u^p, $1 < p \leqq +\infty$, is isometrically isomorphic to the Banach space H_v^p.

Proof. For a function f' in PH_u^p, $1 < p \leqq +\infty$, there exists a boundary function $\bar{f}'$ in $L^p(\Delta_{P1}, \mu_u)$, with which the relative Dirichlet problem gives f'. The function $\{T^{0P}K_b(z_0)\}^{1/P}\bar{f}' \circ t_{P0}^{-1}$ is defined on Δ_{0P} and satisfies, by Lemma 2.1 and Theorem 4.8, that

$$\int_{\Delta_{0P}} T^{0P}K_b(z_0)\,|\,\bar{f}' \circ t_{P0}^{-1}|^p d\mu_v = \int_{\Delta_{P0}} |\,\bar{f}'\,|^p d\mu_u = \int_{\Delta_{P1}} |\,\bar{f}'\,|^p d\mu_u\ ,$$

$$\operatorname*{ess.\,sup}_{\Delta_{0P}} |\,\bar{f}' \circ t_{P0}^{-1}| = \operatorname*{ess.\,sup}_{\Delta_{P1}} |\,\bar{f}'\,|\ ,$$

where the essential supremums are taken with respect to μ_u and μ_v respectively. This shows that $\bar{f}' \circ t_{P0}^{-1}$ belongs to $L^p(\Delta_1, \mu_v)$, since $L^p(\Delta_1, \mu_v) = L^p(\Delta_{0P}, \mu_v)$ by Lemma 2.2.

To a u-P-harmonic function f' in PH_u^p we assign the solution for the Dirichlet problem relative to v with the boundary function $\{T^{0P}K_b(z_0)\}^{1/P}\bar{f}' \circ t_{P0}^{-1}$. Then, this solution is a function in the space H_a^p by Theorem 5.3. Denoting this function by $\bar{T}_{P0}(f')$, we define a linear transformation

$$\bar{T}_{P0}\colon PH_u^p \longrightarrow H_v^p\ .$$

The fact that $\bar{T}_{P0}$ is an isometric isomorphism from PH_u^p onto H_v^p is easily verified by theorems prepared in §4.

THEOREM 5.5. *Let (R, P) be a hyperbolic pair. If Nakai's condition* (5.1) *is satisfied, then the Banach space PH_w^p, $1 < p \leqq +\infty$, is isometrically isomorphic to the Banach space H^p, where w is the P-elliptic measure.*

Proof. By Corollary 4.9 we can prove this theorem by the same way as that in the proof of Theorem 5.4.

Since $PH_w^\infty = PB$ and $H^\infty = HB$, it is clear that this theorem contains Nakai's result ([11]): under the condition (5.1) PB and HB are isometrically isomorphic.

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