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AN M-IDEAL CHARACTERIZATION OF G-SPACES

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It is shown that a separable real Lindenstrauss space is a G-space if and only if the intersection of any family of M-ideals is an M-ideal. This result extends work of A. Gleit and U. Uttersrud.

1. Introduction. A closed subspace of a Banach space V is said to be an *M-ideal* if its annihilator is the range of an L-projection on the dual space V^*. A real Banach space V is a (Grothendieck) *G-space* if there is a compact Hausdorff space K and a set $\{(k_\alpha, k'_\alpha, \lambda_\alpha): \alpha \in A\}$ of triples, where $k_\alpha, k'_\alpha \in K$ and λ_α real, such that V is isometric to the space of all continuous functions f on K which satisfy $f(k_\alpha) = \lambda_\alpha f(k'_\alpha)$ for all $\alpha \in A$.

Ulf Uttersrud proved in [12] that in a G-space, the intersection of any family of M-ideals is an M-ideal; and he raised the question: Does this property characterize G-spaces among those real Banach spaces V in which $\ker(p)$ is an M-ideal for each extreme point p of the unit ball in V^*? In this paper we give a partial answer by showing that this property characterizes G-spaces among separable real Lindenstrauss spaces (L_1-preduals). This generalizes Alan Gleit's result that a separable simplex space is an M-space if and only if the intersection of any family of M-ideals is an M-ideal [5, Theorem 2.3]. Our general approach will follow that of [5]; what makes this possible is a theorem of J. B. Bendnar and H. E. Lacey which describes a real Lindenstrauss space in terms of a barycentric mapping [7, §21, Theorem 8]. Part of their theorem is stated below at the end of §2. The main results in this paper are Theorems 4.1 and 5.2. The former, an existence theorem, is the analog for Lindenstrauss spaces of [5, Theorem 1.4], and the latter is the M-ideal characterization of G-spaces mentioned above.

2. Conventions. Throughout, V will denote a real Lindenstrauss space and K the closed unit ball of V^* with the weak* topology. E is the set of extreme points of K, and Z is the weak* closure of E. The homeomorphism $\sigma: Z \to Z$ is defined by $\sigma(z) = -z$. We denote by C the space $C(Z)$ of all real continuous functions on Z with the uniform norm. For $f \in C$, the functions σf and odd f are defined on Z by $\sigma f(z) = f(\sigma z)$ and odd $f = (f - \sigma f)/2$. The space $C_\sigma = C_\sigma(Z)$ consists of the odd functions in C, that is, those $f \in C$ for which $f = \text{odd} f$. We shall frequently regard V as a subspace of C_σ and write $f(p)$ in place of $p(f)$, for $p \in Z$ and $f \in V$. The term measure will denote an element of C^*, that is, a regular Borel signed measure on Z. For $\mu \in C^*$, the measures $\sigma\mu$ and odd μ are defined by $\sigma\mu(B) = \mu(\sigma B)$ and odd $\mu = (\mu - \sigma\mu)/2$. An odd measure is a measure μ for which $\mu = \text{odd}\,\mu$. The space C_σ is the range of the contractive projection P defined on C by $Pf = \text{odd}\,f$. The adjoint P^* is an isometry of C_σ^* onto the space of odd measures. Thus we may regard C_σ^* as the weak* closed subspace of C^* consisting of all odd measures. For a subset T of C^*, $\bar{T}$ denotes the weak* closure of T (relative to C). Thus if $T \subseteqq C_\sigma^*$, then $\bar{T} \subseteqq C_\sigma^*$. For a subspace X of C_σ, $X^\perp$ is the annihilator of X in C_σ^*. For $z \in Z$, δ_z denotes point mass at z, and we define $\gamma_z = \text{odd}\,\delta_z$. We shall use terminology and results from [11] concerning the Choquet ordering and maximal measures. If $z \in Z$ and μ is any maximal probability measure on K representing z, we define $\pi_z = \text{odd}\,\mu$. (This is well-defined by Lazar's theorem [7, §21, Theorem 7].) For $f \in C$, the function f_π is defined on Z by $f_\pi(z) = \int_Z f\,d\pi_z$ for each $z \in Z$. Since π_z is supported by Z, we may denote $f_\pi(z)$ by $\pi_z(f)$. It is shown in the proof of the Bednar-Lacey theorem [7, §21, Theorem 8] that for each $f \in C$, the function f_π (denoted there by f_ρ) is integrable with respect to every $\mu \in C^*$. Their theorem includes the following characterization of V, which first appeared as [4, Corollary 3.3]:

$$V = \{f \in C\colon f(z) = f_\pi(z) \quad \text{for all} \quad z \in Z\}\,.$$

3. Preliminary lemmas.

LEMMA 3.1. (1) $\|f_\pi\| \leqq \|f\|$ *for each* $f \in C$.

(2) *The map* $f \to f_\pi$ *of* C *into the bounded functions on* Z *is linear.*

(3) *If* μ *is a positive measure and* ν *is a maximal measure which dominates* μ *in the Choquet ordering, then* $\int_Z f_\pi d\mu = \int_Z (\text{odd}\,f)d\nu$ *for all* $f \in C$.

(4) *If* $f \in C_\sigma$ *and* $z \in E$, *then* $f_\pi(z) = f(z)$.

(5) $V = \{f \in C_\sigma\colon f(z) = f_\pi(z)$ *for all* $z \in Z \sim E\}$.

Proof. (1) and (2) are easily verified. In (3), the conclusion holds for f the restriction to Z of a continuous convex function on K [7, p. 217] and these functions are uniformly dense in C. Using (1) and (2) as well, one may routinely verify that the conclusion holds for every $f \in C$. To prove (4), let $z \in E$ and let μ be any maximal probability measure representing z. Then $\mu = \delta_z$ [11, p. 8]. Thus for $f \in C_\sigma$, $f_\pi(z) = \pi_z(f) = (\text{odd}\,\mu)(f) = \mu(\text{odd}\,f) = \mu(f) = f(z)$.

The statement in (5) follows from (4) and the Effros-Bednar-Lacey characterization of V quoted at the end of §2.

LEMMA 3.2. *Assume E is a Borel set. Then*

$$V^{\perp} = \left\{\mu \in C_\sigma^*: \mu(f) = \int_{Z\sim E} (f - f_\pi)d\mu \text{ for all } f \in C_\sigma\right\} .$$

Proof. The inclusion $\supseteqq$ is clear by Lemma 3.1(5). The reverse inclusion follows from Lemma 3.1(4) and the fact that the annihilator of V in C^* consists of those $\mu \in C^*$ such that $\int_Z f_\pi d\mu = 0$ for all $f \in C$. (See proof of [7, §21, Theorem 8].)

LEMMA 3.3. *Let X be a Borel subset of Z such that $\sigma X = X$, and suppose $\mu, \nu \in C_\sigma^*$ are related by $\nu(f) = \int_X f d\mu$ for all $f \in C_\sigma$. Then $\nu(B) = \mu(B \cap X)$ for every Borel subset B of Z.*

Proof. If two odd measures agree on C_σ, then they are identical. The conclusion now follows from [5, Lemma 1.1].

LEMMA 3.4. *Assume V is separable. Let $\mu, \omega \in C_\sigma^*$ be related by $\omega(f) = \int_{Z\sim E} (f - f_\pi)d\mu$ for all $f \in C_\sigma$. Then $\omega(B) = \mu(B)$ for every Borel $B \subseteqq Z \sim E$.*

Proof. Let μ_1 and μ_2 be maximal measures which dominate μ^+ and μ^-, respectively, in the Choquet ordering. Let $f \in C_\sigma$. Then by Lemma 3.1, parts (4) and (3), we have

$$\begin{aligned}\omega(f) &= \int_Z (f - f_\pi)d\mu = \int_Z f d\mu - \int_Z f d\mu_1 + \int_Z f d\mu_2 \\ &= \int_Z f d\mu - \int_Z f d(\text{odd}\,\mu_1) + \int_Z f d(\text{odd}\,\mu_2) .\end{aligned}$$

Let Borel $B \subseteqq Z \sim E$. Then $\omega(B) = \mu(B) - \text{odd}\,\mu_1(B) + \text{odd}\,\mu_2(B)$ by Lemma 3.3. But $\text{odd}\,\mu_1(B) = 0 = \text{odd}\,\mu_2(B)$ because $\sigma B \subseteqq Z \sim E$ and μ_1, μ_2 are supported by E. (K is metrizable.) Thus $\omega(B) = \mu(B)$.

4. An existence theorem. In the following theorem, $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$ denotes the space of all odd measures whose total variation on $(Z \sim E) \cup Y$ is zero, and E_A is the set of those points p in Z such that evaluation at p is an extreme point of the unit in the dual A^* of A.

THEOREM 4.1. *Let V be a separable Lindenstrauss space, and* *let* $q \in Z \sim E$. *Suppose* Y *is a closed set in* Z *satisfying*

1. $Y \cap (Z \sim E) = \{q\}$.
2. $Y \cap -Y = \emptyset$.
3. $E \sim (Y \cup -Y) \neq \emptyset$.
4. $\pi_q(Y) < 1/2$.

Then $A = \{f \in V\colon f(y) = \pi_q(f) \text{ for all } y \in Y\}$ *is a nontrivial Lindenstrauss space,* A^* *is isometric to* $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$, *and* $E_A \supseteqq E \sim (Y \cup -Y)$.

The proof will be preceded by several lemmas following, to some extent, the general pattern of Gleit's proof of [5, Theorem 1.4]. Our main objective is to show that $C_\sigma^*/A^\perp$ is isometric to $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$. In Lemmas 4.2–4.4 below, we preserve the notation and hypotheses of Theorem 4.1. In particular, the separability of V implies that π_q is supported by E.

Let $S = \{\gamma_y\colon y \in Y\}$. (Recall that $\gamma_y = \text{odd}\,\delta_y$.)

LEMMA 4.2. *Let* $\mu \in \overline{\text{co}}(S)$. *Then* μ *is an odd measure on* $Y \cup -Y$ *and* $\mu(Y) = 1/2$.

Proof. Clearly μ is odd. Let $\{\mu_n\} \subseteqq \text{co}(S)$ be a sequence which converges weak* to μ. Then $\text{supp}\,\mu_n \subseteqq Y \cup -Y$ for each n, hence $\text{supp}\,\mu \subseteqq Y \cup -Y$ [2, III, §3, Proposition 6]. By Urysohn's lemma, there are $g_1, g_2 \in C$ with $g_1 = 1$ on Y, $g_1 = 0$ on $-Y$, $g_2 = 0$ on Y, and $g_2 = -1$ on $-Y$. Let $f = \text{odd}(g_1 + g_2)$. Then $f \in C_\sigma$, $f = 1$ on Y, and $f = -1$ on $-Y$. We have $\mu_n(f) = 1$ for all n, hence $\mu(f) = 1$. Thus $1 = \int_{Y \cup -Y} f\,d\mu = \mu(Y) - \mu(-Y) = 2\mu(Y)$.

The following notation will be used in Lemmas 4.3 and 4.4 and in the proof of Theorem 4.1. We also preserve the definition of S preceding Lemma 4.2. Let $D = \{f \in C_\sigma\colon f(y) = \pi_q(f) \text{ for all } y \in Y\}$. Let $F = \left\{\mu \in C_\sigma^*\colon \text{there exists an odd measure } \lambda \text{ on } Y \cup -Y \text{ such that } \mu(f) = \int_{Y \cup -Y} f\,d\lambda - 2\pi_q(f)\lambda(Y) \text{ for each } f \in C_\sigma\right\}$. Let $\alpha = 1 - 2\pi_q(Y)$, and let $T = \{\gamma_y - \pi_q\colon y \in Y\}$.

LEMMA 4.3. (1) $F = \text{span}(\overline{\text{co}}(T))$.
(2) $D^\perp = F$.

Proof. To prove (1), we first show that $\overline{\text{co}}(T) \subseteqq F$. Let $\nu \in \overline{\text{co}}(T)$ and let $\lambda = \nu + \pi_q$. Then clearly $\lambda \in \overline{\text{co}}(S)$. Hence by Lemma 4.2, λ is an odd measure on $Y \cup -Y$ and $\lambda(Y) = 1/2$. Therefore $\nu \in F$. Thus $\text{span}(\overline{\text{co}}(T)) \subseteqq F$ because F is a linear subspace of C_σ^*. For the reverse inclusion, let $\mu \in F$ and assume $\mu \neq 0$. Then there is a nonzero odd measure λ on $Y \cup -Y$ such that $\mu(f) = \int_{Y \cup -Y} f\,d\lambda - 2\pi_q(f)\lambda(Y)$ for all $f \in C_\sigma$. Since λ is odd, we have $\sigma\lambda^+ = \lambda^-$, hence $\lambda = \mathrm{odd}(2\lambda^+)$ [4, p. 443]. Also, $||2\lambda^+|| = ||\lambda||$ because $||2\lambda^+|| = 2\lambda^+(Z) = |\lambda|(Z) + \lambda(Z) = |\lambda|(Z) = ||\lambda||$. Let λ_1 and λ_2 be defined by $\lambda_1(B) = 2\lambda^+(B \cap Y)$ and $\lambda_2(B) = 2\lambda^+(B \cap -Y)$ for all Borel $B \subseteqq Z$. Then $\lambda_1 + \lambda_2 = 2\lambda^+$, hence $\mathrm{odd}\,\lambda_1 + \mathrm{odd}\,\lambda_2 = \lambda$. Thus λ_1 and λ_2 cannot both be zero. We consider first the case where one of λ_i is zero. Suppose $\lambda_1 = 0$. Then $\lambda_2 = 2\lambda^+$ is a positive measure on $-Y$, and $||\lambda_2||/||\lambda|| = 1$. We then have $\lambda_2/||\lambda|| \in \overline{\mathrm{co}}(\{\delta_{-y}: y \in Y\})$ [11, p. 3], hence $\mathrm{odd}\,\lambda_2/||\lambda|| \in \overline{\mathrm{co}}(\{\gamma_{-y}: y \in Y\}) = \overline{\mathrm{co}}(-S) = -\overline{\mathrm{co}}(S)$. Thus $-\lambda/||\lambda|| \in \overline{\mathrm{co}}(S)$. Then $(-\lambda/||\lambda||) - \pi_q \in \overline{\mathrm{co}}(T)$, and also $2\lambda(Y) = -||\lambda||$ by Lemma 4.2. Hence $\mu = \lambda - 2\lambda(Y)\pi_q = -||\lambda||((-\lambda/||\lambda||) - \pi_q)$, and so $\mu \in \mathrm{span}(\overline{\mathrm{co}}(T))$. A similar argument will show that if $\lambda_2 = 0$, then $\lambda/||\lambda|| \in \overline{\mathrm{co}}(S)$, and $\mu = ||\lambda||((\lambda/||\lambda||) - \pi_q)$ is in $\mathrm{span}(\overline{\mathrm{co}}(T))$. We now consider the case $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Then $\lambda_1/||\lambda_1|| \in \overline{\mathrm{co}}(\{\delta_y: y \in Y\})$ and $\lambda_2/||\lambda_2|| \in \overline{\mathrm{co}}(\{\delta_{-y}: y \in Y\})$. Hence $\mathrm{odd}\,\lambda_1/||\lambda_1|| \in \overline{\mathrm{co}}(S)$ and $-\mathrm{odd}\,\lambda_2/||\lambda_2|| \in \overline{\mathrm{co}}(S)$. Then $||\lambda_1|| = 2\,\mathrm{odd}\,\lambda_1(Y)$ and $||\lambda_2|| = -2\,\mathrm{odd}\,\lambda_2(Y)$ by Lemma 4.2. Hence $||\lambda_1|| - ||\lambda_2|| = 2\lambda(Y)$. Thus $\lambda = ||\lambda_1||((\mathrm{odd}\,\lambda_1/||\lambda_1||) - \pi_q) - ||\lambda_2||((-\mathrm{odd}\,\lambda_2/||\lambda_2||) - \pi_q) + 2\lambda(Y)\pi_q$, and so $\mu = \lambda - 2\lambda(Y)\pi_q \in \mathrm{span}(\overline{\mathrm{co}}(T))$. Hence $F = \mathrm{span}(\overline{\mathrm{co}}(T))$. We now prove (2). Clearly $T \subseteqq D^\perp$, hence $F \subseteqq D^\perp$ by (1). Thus $D^\perp = \bar{F}$. To show that F is weak* closed, it suffices to show it is norm closed [3, V. 5.9]. We proceed as in Part A of the proof of [5, Theorem 1.4]. Consider a $\mu \in F$. Then there is an odd measure λ on $Y \cup -Y$ such that $\mu(f) = \int_{Y \cup -Y} f\,d\lambda - 2\pi_q(f)\lambda(Y)$ for each $f \in C_\sigma$. By Lemma 3.3, for each Borel set $B \subseteqq Z$, $\mu(B) = \lambda(B \cap (Y \cup -Y)) - 2\pi_q(B)\lambda(Y)$. In particular, $\mu(Y) = \lambda(Y)(1 - 2\pi_q(Y)) = \alpha\lambda(Y)$. Thus for each Borel set $B \subseteqq Y \cup -Y$, we have $\lambda(B) = \mu(B) + 2\pi_q(B)\mu(Y)/\alpha$. Hence μ uniquely determines λ. Further, since $|\lambda| = |\mu + (2\mu(Y)/\alpha)\pi_q|$ on $Y \cup -Y$, we get $|\lambda|(Y) \leqq |\mu|(Y) + 2|\mu|(Y)|\pi_q|(Y)/\alpha \leqq ||\mu|| + 2||\mu||\,||\pi_q||/\alpha$. Hence $||\lambda|| = |\lambda|(Y \cup -Y) = 2|\lambda|(Y) \leqq ||\mu||(2 + 4/\alpha)$. It can now be easily verified that F is norm closed. Hence $F = D^\perp$.

LEMMA 4.4. (1) $D^\perp + V^\perp = \{\mu + \nu: \mu \in D^\perp, \nu \in V^\perp$, *and* $\mu = 0$ *on* $Z \sim E\}$.

(2) $A^\perp = D^\perp + V^\perp$.

Proof. To prove (1), let $\mu \in D^\perp$. We will show $\mu = \mu_1 + \mu_2$, where $\mu_1 \in D^\perp$, $\mu_2 \in V^\perp$, and $\mu_1(B) = 0$ for every Borel $B \subseteqq Z \sim E$. By Lemma 4.3(2), we have $\mu(f) = \lambda(f) - 2\pi_q(f)\lambda(Y)$ for all $f \in C_\sigma$, where λ is an odd measure on $Y \cup -Y$. Let $\lambda_1 = \lambda - 2\lambda(\{q\})\gamma_q$ and let $\mu_1 = \lambda_1 - 2\lambda_1(Y)\pi_q$. Then λ_1 is an odd measure on $Y \cup -Y$ and $\lambda_1(\{q\}) = 0$. Thus $\mu_1 \in D^\perp$ by Lemma 4.3(2), and $\mu_1(f) = \mu(f) -$

$2\lambda(\{q\})(\gamma_q(f) - \pi_q(f))$ for all $f \in C_\sigma$. Letting $\mu_2 = 2\lambda(\{q\})(\gamma_q - \pi_q)$, we have $\mu_2 \in V^\perp$ by Lemma 3.1(5). Let Borel $B \subseteqq Z \sim E$. Then $\lambda_1(B) = 0 = \pi_q(B)$, hence $\mu_1(B) = 0$. This establishes (1). To prove (2), note that $A = D \cap V$, hence $A^\perp$ is the weak* closure of $D^\perp + V^\perp$. To show $D^\perp + V^\perp$ is weak* closed, we may use sequences (by the Krein-Smulian theorem) because C is separable and $D^\perp + V^\perp$ is convex [3, V. 7.16]. Let $\{\omega_n\} \subseteqq D^\perp + V^\perp$ converge weak* to ω. Then the sequence $\{||\omega_n||\}$ is bounded [3, V. 4.3]. Further, by (1) we have for each n, $\omega_n = \mu_n + \nu_n$, where $\mu_n \in D^\perp$, $\nu_n \in V^\perp$, and $\omega_n = \nu_n$ on $Z \sim E$. Let $f \in C_\sigma$ with $||f|| \leqq 1$. Then by Lemma 3.2, $\omega_n(f) = \mu_n(f) + \int_{Z\sim E}(f - f_\pi)d\omega_n$. Hence

$$\begin{aligned} |\mu_n(f) &= \left|\int_Z f d\omega_n - \int_{Z\sim E}(f - f_\pi)d\omega_n\right| \\ &= \left|\int_E f d\omega_n + \int_{Z\sim E} f_\pi d\omega_n\right| \\ &\leqq |\omega_n|(E) + |\omega_n|(Z \sim E) = |\omega_n|(Z) = ||\omega_n|| \,. \end{aligned}$$

Thus $||\mu_n|| \leqq ||\omega_n||$, hence the sequence $\{||\mu_n||\}$ is bounded. Then the weak* closure of $\{\mu_n\}$ is compact and metrizable, hence some subsequence of $\{\mu_n\}$ converges weak* to an element of $D^\perp$. Assume, for simplicity of notation, that $\{\mu_n\}$ cnnverges weak* to $\mu \in D^\perp$. The sequence $\{||\nu_n||\}$ is bounded because $||\nu_n|| \leqq ||\omega_n|| + ||\mu_n||$ for each n. Hence some subsequence of $\{\nu_n\}$ converges weak* to an element $\nu \in V^\perp$. Then $\omega = \mu + \nu \in D^\perp + V^\perp$.

Proof of Theorem 4.1. We first show that A^* is isometric to $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$. Since A^* is isometric to $C_\sigma^*/A^\perp$, it will suffice to construct an isometry of $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$ onto $C_\sigma^*/A^\perp = C_\sigma^*/(D^\perp + V^\perp)$ (Lemma 4.4(2)). Let $\theta\colon C_\sigma^*\{Z; (Z \sim E) \cup Y\} \to C_\sigma^*/A^\perp$ be defined by $\theta(\mu) = \mu + A^\perp$ for each $\mu \in C_\sigma^*\{Z; (Z \sim E) \cup Y\}$. Clearly θ is linear, and we claim that θ is a bijection. For, suppose $\theta(\mu) = 0$. Then

$$\mu \in (D^\perp + V^\perp) \cap C_*^\sigma\{Z; (Z \sim E) \cup Y\} \,.$$

By Lemmas 4.4(1) and 3.2, there is $\mu_1 \in D^\perp$ such that $\mu(f) = \mu_1(f) + \int_{Z\sim E}(f - f_\pi)d\mu$ for all $f \in C_\sigma$. But $|\mu|(Z \sim E) = 0$, hence $\mu = \mu_1$. Then by Lemma 4.3(2), $\mu = \lambda - 2\lambda(Y)\pi_q$, where λ is an odd measure on $Y \cup -Y$. Then $\mu(Y) = \alpha\lambda(Y)$. Hence $\lambda(Y) = 0$ since $\mu(Y) = 0$. Therefore $\mu = \lambda$. But $|\mu|(Y \cup -Y) = 0$, hence $\lambda = 0$, and so $\mu = 0$. Hence θ is one-to-one. To show that θ is onto, let $\nu \in C_\sigma^*$ and let $\nu_\pi \in C_\sigma^*$ be defined by $\nu_\pi(f) = \int_{Z\sim E} f_\pi d\nu$ for all $f \in C_\sigma$ Define a measure

λ_ν on $Y \cup -Y$ by

$$\lambda_\nu(\{q\}) = \lambda_\nu(\{-q\}) = 0 , \quad \text{and} \quad \lambda_\nu(B) = \nu_\pi(B) + \nu(B) + 2\pi_q(B)\lambda_\nu(Y)$$

for each Borel subset B of $(Y \cup -Y) \sim \{q, -q\}$, where $\lambda_\nu(Y) = \lambda_\nu(Y \sim \{q\})$ is found by consistency. Let $\omega_\nu \in C_\sigma^*$ be defined by

$$\omega_\nu(f) = \int_{Z\sim E}(f - f_\pi)d\nu + \int_{Y \cup -Y} f d\lambda_\nu - 2\pi_q(f)\lambda_\nu(Y)$$

for each $f \in C_\sigma$. Then $\omega_\nu \in V^\perp + D^\perp$ by Lemmas 3.1(5) and 4.3. Thus $\nu - \omega_\nu \in \nu + A^\perp$. To show $\nu - \omega_\nu \in C_\sigma^*\{Z; (Z \sim E) \cup Y\}$, let B be any Borel subset of $(Z \sim E) \cup Y$ and let $B_1 = B \cap (Y \sim \{q\})$. Then $\omega_\nu(B) = \omega_\nu(B \cap (Z \sim E)) + \omega_\nu(B_1) = \nu(B \cap (Z \sim E)) + \omega_\nu(B_1)$ by Lemmas 3.3 and 3.4. And $\omega_\nu(B_1) = -\nu_\pi(B_1) + \lambda_\nu(B_1) - 2\pi_q(B_1)\lambda_\nu(Y) = \nu(B_1)$ by Lemma 3.3 and the definition of λ_ν. Hence $\omega_\nu(B) = \nu(B \cap (Z \sim E)) + \nu(B_1) = \nu(B)$. Thus $\nu - \omega_\nu \in C_\sigma^*\{Z; (Z \sim E) \cup Y\}$. Since $\theta(\nu - \omega_\nu) = \nu + A^\perp$, we have that θ is onto.

To prove that θ is an isometry, let $\mu \in C_\sigma^*\{Z; (Z \sim E) \cup Y\}$. Then $\|\theta(\mu)\| = \|\mu + A^\perp\| = \inf\{\|\nu\|: \nu \in \mu + A^\perp\}$. Thus $\|\theta(\mu)\| \leqq \|\mu\|$. For the reverse inequality, we show $\|\nu\| \geqq \|\mu\|$ for all $\nu \in \mu + A^\perp$. Let $\nu \in \mu + A^\perp$. Then $\theta(\nu - \omega_\nu) = \nu + A^\perp = \mu + A^\perp$, hence $\nu - \omega_\nu = \mu$. Let $X = Y \cup -Y$. Then $\|\mu\| = \|\nu - \omega_\nu\| = |\nu - \omega_\nu|(E \sim X) \leqq |\nu|(E \sim X) + |\omega_\nu|(E \sim X)$. If B is any Borel subset of $E \sim X$, then by Lemma 3.3, $\omega_\nu(B) = \nu(B \cap (Z \sim E)) - \nu_\pi(B) + \lambda_\nu(B \cap X) - 2\pi_q(B)\lambda_\nu(Y) = -\nu_\pi(B) - 2\pi_q(B)\lambda_\nu(Y)$. Hence $\omega_\nu = -\nu_\pi - 2\lambda_\nu(Y)\pi_q$ on $E \sim X$, consequently, $|\omega_\nu| = |\nu_\pi + 2\lambda_\nu(Y)\pi_q|$ on $E \sim X$. From the definition of λ_ν we have

$$\lambda_\nu(Y) = \lambda_\nu(Y \sim \{q\}) = (\nu_\pi(Y \sim \{q\}) + \nu(Y \sim \{q\}))/\alpha .$$

Thus

$$\begin{aligned} |\omega_\nu|(E \sim X) \leqq |\nu_\pi|(E \sim X) &+ 2|\nu_\pi(Y \sim \{q\}) \\ &+ \nu(Y \sim \{q\})||\pi_q|(E \sim X)/\alpha . \end{aligned}$$

And $|\pi_q|(E \sim X) \leqq \alpha$ because

$$\begin{aligned} |\pi_q|(E \sim X) + 2\pi_q(Y) &\leqq |\pi_q|(E \sim X) + 2|\pi_q|(Y) \\ &= |\pi_q|(E \sim X) + |\pi_q|(X) = |\pi_q|(E) = \|\pi_q\| \leqq 1 . \end{aligned}$$

Hence

$$\begin{aligned} |\omega_\nu|(E \sim X) &\leqq |\nu_\pi|(E \sim X) + 2|\nu_\pi|(Y \sim \{q\}) + 2|\nu|(Y \sim \{q\}) \\ &= |\nu_\pi|(E \sim X) + |\nu_\pi|(X \sim \{q, -q\}) + |\nu|(X \sim \{q, -q\}) \\ &= |\nu_\pi|(E) + |\nu|(X \sim \{q, -q\}) \leqq \|\nu_\pi\| + |\nu|(X \sim \{q, -q\}) . \end{aligned}$$

Also $||\nu_\pi|| \leqq |\nu|(Z \sim E)$ because for $||f|| \leqq 1$, $|\nu_\pi(f)| = \left|\int_{Z\sim E} f_\pi d\nu\right| \leqq |\nu|(Z \sim E)$. Therefore $|\omega_\nu|(E \sim X) \leqq |\nu|(Z \sim E) + |\nu|(X \sim \{q, -q\})$, and so $||\mu|| \leqq |\nu|(E \sim X) + |\nu|(Z \sim E) + |\nu|(X \sim \{q, -q\}) = |\nu|(Z) = ||\nu||$, thus completing the proof that θ is an isometry.

Continuing to denote $Y \cup -Y$ by X, let $C^*\{Z; (Z \sim E) \cup X\}$ be the space of all measures whose total variation on $(Z \sim E) \cup X$ is zero. Gleit has shown that this is an L-space [6, Proposition 1.1]. The space $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$ is the range of the contractive projection P on $C^*\{Z; (Z \sim E) \cup X\}$ defined by $P(\mu) = \text{odd}\,\mu$. Thus $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$ is isometric to an L_1-space [7, §17], hence A is a Lindenstrauss space.

For each $p \in Z \sim \{0\}$, the evaluation functional (measure) γ_p is an extreme point of the unit ball in C_σ^* [7, §10, Lemma 3]. Let $p \in E \sim X$. Then $\gamma_p \in C_\sigma^*\{Z; (Z \sim E) \cup Y\}$, hence γ_p is an extreme point of the unit ball in $C_\sigma^*\{Z; (Z \sim E) \cup Y\}$. Further, γ_p is mapped onto $\gamma_p | A = p | A$ by the composition of isometries

$$\theta\colon C_\sigma^*\{Z; (Z \sim E) \cup Y\} \longrightarrow C_\sigma^*/A^\perp \quad \text{and} \quad C_\sigma^*/A^\perp \longrightarrow A^* .$$

Thus $p \in E_A$. Hence $E_A \supseteqq E \sim X$.

Finally, A is nontrivial because $E \sim X \neq \varnothing$; and this concludes the proof of Theorem 4.1.

5. The characterization. In this section, ω_z denotes the measure defined by $\omega_z = 2\pi_z^+$ for each $z \in Z$. Properties of ω_z were studied and used effectively in [4]. In the proof of Lemma 5.1 below, I_β^0 and I^0 denote the annihilators of I_β and I, respectively, in V^*. This lemma is the analog for Lindenstrauss spaces of [5, Lemma 2.2].

LEMMA 5.1. *Let V be a Lindenstrauss space. Let $q \in Z \sim E$, $q \neq 0$. Suppose there exist $p \in \operatorname{supp} \omega_q \cap E$ and a net $\{q_\beta\} \subseteqq E \sim \{p, -p\}$ which converges weak* to q. Suppose, further, there is an element $f \in V$ such that $p(f) \neq 0$ and $q_\beta(f) = 0$ for all β. Then there exists a family of M-ideals I_β such that $\cap I_\beta$ is not an M-ideal. If the net is a sequence, then the family of M-ideals is countable.*

Proof. Let $I_\beta = \{g \in V\colon q_\beta(g) = 0\}$. Then $I_\beta^0 = \text{span}(q_\beta)$, hence I_β is an M-ideal [9, Theorem 5.8]. Let $I = \cap I_\beta$, and suppose I were an M-ideal. Then I^0 would be a weak* closed L-summand in V^* containing q. Let L_q be the intersection of all weak* closed L-summands containing q, and let $H_q = L_q \cap K$. Then L_q is a weak* closed L-summand [1, Proposition 1.13] and H_q is the smallest weak* closed biface containing q [1, pp. 168, 169]. We have $\operatorname{supp} \omega_q \subseteqq H_q$ [4, Lemma 5.6], hence $p \in L_q$. Then, since $L_q \subseteqq I^0$ and $f \in I$, it follows that $p(f) = 0$. But this contradicts the hypothesis $p(f) \neq 0$, so we conclude that I is not an M-ideal.

THEOREM 5.2. *Let V be a Lindenstrauss space, and consider the statements:*

(1) *V is a G-space.*

(2) *The intersection of any family of M-ideals is an M-ideal.*

(3) *The intersection of any countable family of M-ideals is an M-ideal.*

One has that $(1) \Rightarrow (2) \Rightarrow (3)$. *If V is separable, then* $(3) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ was proved by Uttersrud [12, Theorem 10].

$(2) \Rightarrow (3)$ is obvious.

Not $(1) \Rightarrow$ not (3) (V separable). Suppose V is not a G-space. Then there exists $q \in Z \sim [0, 1]E$ [4, Theorem 6.3]. Since $q \neq 0$, $\omega_q/\|\omega_q\|$ is a maximal probability measure [4, p. 444], hence is supported by E. Thus $\operatorname{supp} \omega_q \cap E \neq \emptyset$. Then since $q \notin [0, 1]E$ and $\omega_q(f) = f(q)$ for all $f \in V$, there must be two linearly independent points, say p_1 and p_2, in $\operatorname{supp} \omega_q \cap E$. Since $q \in Z \sim E$, there is a sequence $\{q_n\} \subseteqq E \sim \{\pm p_1, \pm p_2\}$ which converges weak* to q. Let $Y = [q_n: n = 1, 2, \cdots\} \cup \{q\}$. We may assume $Y \cap -Y = \emptyset$. We also have $\pi_q(Y) < 1/2$. To see this, let μ be any maximal probability measure representing q. Then $\omega_q \leqq \mu$ [4, p. 443], hence $\operatorname{supp} \omega_q \subseteqq \operatorname{supp} \mu$. Since Y is closed and $p_1, p_2 \notin Y$, it follows that $\mu(Y) < 1$. Then, since $\pi_q = \operatorname{odd} \mu$, we have $\pi_q(Y) < 1/2$. Let $A = \{f \in V: f(y) = \pi_q(f)$ for all $y \in Y\}$. Then by Theorem 4.1, A is a nontrivial subspace of V and $p_1, p_2 \in E_A$. We note that $p_1|A \neq \pm p_2|A$ because $\gamma_{p_1} \neq \pm\gamma_{p_2}$. (See the end of the proof of Theorem 4.1.) Hence there are $f_1, f_2 \in A$ with $f_1(p_1) = 1 = f_2(p_2)$ and $f_1(p_2) = 0 = f_2(p_1)$. We consider the two cases $f_2(q) = 0$, $f_2(q) \neq 0$. Suppose $f_2(q) = 0$. Let $f = f_2$. Then $f \in V$ because $f \in A$, and $f(q_n) = 0$ for all n because $f(q) = 0$ and $f \in A$. Taking $p_2 = p$ in Lemma 5.1, we see that Lemma 5.1 implies that (3) is not true. Now suppose $f_2(q) \neq 0$. Let $f = f_1 - (f_1(q)/f_2(q))f_2$. Then $f \in V$ and $f(q) = 0$. Hence $f(q_n) = 0$ for all n because $f \in A$. Also, $f(p_1) = f_1(p_1) \neq 0$. Thus with $p = p_1$, Lemma 5.1 implies that (3) is not true.

REMARK. In [10, p. 78], there is an example of a Lindenstrauss space which is not a G-space and which illustrates well the above proof.

COROLLARY 5.3 [5, *Theorem* 2.3]. *A separable simplex space is an M-space if and only if the intersection of any family of M-ideals is an M-ideals.*

Proof. This follows from Theorem 5.2 above and the diagram of classes of Lindenstrauss spaces in [8, p. 181].

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