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TRANSITIVE GROUPS OF ISOMETRIES ON H^n

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This paper investigates transitive groups of direct isometries, without fixed points, of hyperbolic n-space H^n. For $n = 2$ there is a natural one-to-one correspondence between the set of all such groups and the set of ideal points of H^2. For $n \geqq 3$ there is an analogous collection of groups, which are in several senses the simplest but not the only such groups.

The existence of a transitive group of transformations without fixed points can be used to define an addition of points in the transformed space. The idea of sums of points in hyperbolic spaces has been used in probabilistic applications, for example by Kifer and by Karpelevich, Tutubalin and Shur. These involve a composition of measures based on the collection (not a group) of translations of H^2 or H^3. The group structure seems necessary for certain statistical questions, such as characterizations of normal distributions, which were in part the motivation for this investigation.

1. Some properties of H^n. Theorems about hyperbolic geometry can be stated and proved in a variety of settings, and some ideas are more manageable in one setting than in another. This introductory section reviews the basic properties used in the rest of the paper, grouped according to the setting in which they seem most natural. Proofs are omitted, except for a few hints where the references may not provide the needed generality.

The earliest setting for hyperbolic geometry is the synthetic or axiomatic approach. The classical works of Lobachevsky and Bolyai are still quite readable; both are included in Bonola [1]. A good recent textbook, although limited to plane geometry, is Gans [6]. Eves [5], Chapter 7, is rather brief but does suggest some very useful relationships to topics covered elsewhere in the book. The extension to higher dimensions is actually not difficult; the very brief treatment of H^3 in many standard textbooks (for example, Kulczycki [9], pg. 110-124), together with the standard results for E^n, should be sufficient to point the way.

In this synthetic setting, two lines (or planes of any dimension) are called parallel if they are nonintersecting but asymptotic. A pencil of mutually parallel lines defines an ideal point of H^n, and we may say that the ideal point lies on each of the lines, or that each of the lines passes through or contains the ideal point. Two parallel planes have only one ideal point Ω in common, and they are said to be parallel at Ω. The term "hyperparallel" is used in this paper to refer to hyperplanes or coplanar lines which are neither intersecting nor parallel. Repeated use is made of the fact that two hyperparallels have a unique common perpendicular.

The sum of measures of the interior angles of a triangle is less than π; thus, for example, the fourth angle of the Lambert quadrilateral in Fig. 4 must be acute.

Reflection in a hyperplane is an isometry, and all isometries of H^n are expressible as the products of no more than $n+1$ reflections. The proof and the extent of freedom in choosing the reflection mirrors are exactly analogous to the euclidean case. Moreover, reflections in perpendicular hyperplanes commute. The synthetic proof of this fact is not completely trivial; a proof in the projective model is suggested below.

The orthogonal trajectories of a parallel pencil of lines are called orispheres (or oricycles if $n=2$); they are open hypersurfaces of constant curvature, and are said to pass through the ideal point of the pencil. Each orisphere is isometric to a euclidean $(n-1)$-plane; for $n=3$ this was pointed out by Wachter in a letter to Gauss in 1816.

Toward the end of the 19th century several models for hyperbolic geometry were developed. A conformal model credited to Poincaré is discussed briefly in Spivak [12], pg. 10ff, and at more length for $n=2$ in Chapters 5 and 6 of Meschkowski [10]. The points of H^n are represented by the points of $\boldsymbol{R}^n$ with $z^n > 0$. Hyperplanes are represented by the intersections of this half-space with spheres and hyperplanes perpendicular to the hyperplane $z^n = 0$. The fact that angle measures and thus perpendicularity are reproduced is very useful in some situations. For example, consider reflection across a hyperplane H whose representative E is a "vertical" hyperplane. The map of $z^n > 0$ representing this reflection must be conformal, must preserve the hyperplane $z^n = 0$ and must leave each point on E fixed, while interchanging the two half-spaces into which E separates $\boldsymbol{R}^n$. Thus the representing map is euclidean reflection across E. The product of two reflections in parallel vertical hyperplanes is a euclidean translation parallel to $z^n = 0$. Also, two hyperbolic hyperplanes perpendicular to a line represented by a vertical line are represented by concentric hemispheres, and again using only conformality we see that the product of reflections in two such hyperplanes is represented by a dilatation about the common center of the hemispheres. These representations are used in the proof of Lemma 1.

Using the conformal model, Spivak ([12], pg. 20) sketches a proof that orispheres are isometric to euclidean $(n-1)$-planes.

The setting for many of the arguments and all of the figures of this paper is the projective model, in which lines are represented by open chords of a sphere $\mathscr{S}$ in E^n. Parallel lines are represented by chords with a common endpoint, which represents the ideal point common to the lines. Hyperplanes are represented by the intersections of the interior of $\mathscr{S}$ with euclidean hyperplanes, and isometries of H^n by projective mappings of E^n which leave $\mathscr{S}$ and its interior invariant. Details of the model are developed quite fully for $n = 2$ in Buchmann [2].

The most significant feature of the projective model which is not immediately apparent is the relationship between hyperbolic perpendiculars and reciprocation. To any secant hyperplane E of a sphere $\mathscr{S}$ in E^n is associated its reciprocal or pole E' with respect to $\mathscr{S}$; this is the common point (ordinary or ideal) of all hyperplanes tangent to $\mathscr{S}$ at points of $\mathscr{S} \cap E$. In the projective model, the euclidean lines representing hyperbolic perpendiculars to the hyperplane H represented by E are concurrent (or parallel) in point E'. In particular, if E passes through the center of $\mathscr{S}$ then E' is ideal, and euclidean perpendiculars to E represent hyperbolic perpendiculars to H. A euclidean k-plane E has a reciprocal E' which is a certain $(n - k - 1)$-plane; and if E intersects $\mathscr{S}$ then the euclidean $(n - k)$-planes through E' and points of E inside $\mathscr{S}$ represent hyperbolic $(n - k)$-planes perpendicular to the hyperbolic k-plane represented by E.

Let H be a hyperbolic hyperplane represented by E. Reflection across H is represented by the projective map of E^n which maps each point P to its harmonic conjugate with respect to the segment between E and E' along the line PE'. If E passes through the center of $\mathscr{S}$, this projective map is euclidean reflection across E. Suppose H_1 and H_2 are perpendicular hyperbolic hyperplanes. We may choose a projective model so that the representing euclidean hyperplanes E_1 and E_2 both pass through the center of $\mathscr{S}$. Since H_1 and H_2 are perpendicular, so are E_1 and E_2 (this holds only at the center of $\mathscr{S}$; elsewhere the model is not conformal). Because reflections in perpendicular euclidean hyperplanes commute, so do reflections in perpendicular hyperbolic hyperplanes.

The hyperbolic hyperplanes in a pencil Π, mutually parallel at Ω, are represented by euclidean hyperplanes having a common $(n - 2)$-plane T tangent to $\mathscr{S}$. (Thus note that, if $n > 2$, not all hyperplanes through Ω are parallel.) The poles of all euclidean hyperplanes through T lie on a line t tangent to $\mathscr{S}$ and perpendicular to T at the point of tangency. Thus any secant k-plane through t represents a hyperbolic k-plane which is perpendicular to each hyperbolic hyperplane in the pencil Π. Also, if H is any hyperbolic hyperplane not containing Ω, then the pole of its representing euclidean hyperplane is a point not on the hyperplane spanned by t and T, and thus there is one and only one hyperplane in Π which is perpendicular to H.

This projective model, with points on and outside $\mathscr{S}$ interpreted as ideal and "ultra-ideal" points of H^n, is clearly a realization of the projective space of lines through the origin in $\boldsymbol{R}^{n+1}$. Specifically, each ordinary point of H^n corresponds to a line λz $(\lambda \in \boldsymbol{R})$ for some $z \in \boldsymbol{R}^{n+1}$ such that $\Phi(z, z) < 0$, where $\Phi(z, w) = -z^0w^0 + z^1w^1 + \cdots + z^nw^n$. Linear transformations of $\boldsymbol{R}^{n+1}$ which preserve Φ induce isometries of H^n; the transformation T and its composition with the mapping $z \to -z$ induce the same isometry on H^n. This interpretation allows one to use information about groups of linear transformations to study groups of isometries of H^n; this is the approach taken in Chen and Greenberg [4].

Finally, note that H^n is a rank one symmetric space. Theorems proved in the more general situation may yield useful results for hyperbolic geometry; one such example is noted in § 5, where a theorem of Chen [3] implies the existence of an ideal point fixed under each element of a certain group of isometries of H^n. Homogeneous symmetric spaces of negative sectional curvature are probably the most likely setting for generalization of the results of this paper. I wish to thank the referee for suggesting several such generalizations, and for bringing to my attention the papers by Chen and Greenberg.

2. Simple isometries of H^n. Any direct isometry of the hyperbolic plane can be expressed as the product of two reflections in lines, and is classified as a rotation, an asymptotic rotation, or a translation, when the lines are respectively intersecting, parallel, or hyperparallel. Since proper rotations have fixed ordinary points, we will be concerned primarily with maps of the other two types.

If $n \geqq 3$ there are direct isometries of H^n which cannot be expressed as the product of two reflections; for example, "screw displacements" in H^3 (as in E^3) generally require four reflections. Clearly the collection of all isometries expressible as the product of two reflections does not form a group if $n \geqq 3$. Several reasons for restricting attention to such isometries are discussed in § 5. The main theorem (§ 4) characterizes the transitive groups of isometries of H^n, without fixed points, all of whose elements can be expressed as products of two reflections in hyperplanes. We use the same descriptive terminology as in H^2.

A translation of H^n has a distinguished axis, the common perpendicular line of the two reflection mirrors. Points on the axis are displaced a certain distance along the axis, while any other point undergoes a greater displacement, increasing with its distance from the axis. When we speak of a translation of length t along a line

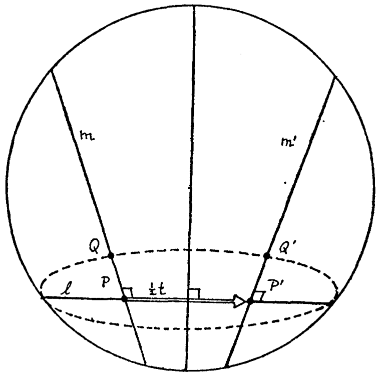

FIGURE 1

l, we mean that l is the axis of the translation and that the points of l are displaced a distance t. The translation can be expressed as the product of reflections in a pair of hyperplanes perpendicular to l, where one may be chosen arbitrarily and the other is a distance $t/2$ from it.

An asymptotic rotation about an ideal point Ω is the product of reflections in two hyperplanes which are parallel at Ω. Two such hyperplanes determine a pencil of hyperplanes, mutually parallel at Ω; the pencil may be said to be determined by a pair of hyperplanes, by one hyperplane and an ideal point on it, or by a particular asymptotic rotation. A pencil of asymptotic rotations means the collection (in fact a group) of all asymptotic rotations expressible as products of two reflections in hyperplanes belonging to a given parallel pencil Π; the same letter Π will be used to stand for either pencil. Any asymptotic rotation can be expressed as the product of reflections in two hyperplanes of the appropriate pencil, with one of them chosen arbitrarily.

Each orisphere through an ideal point Ω is displaced along itself

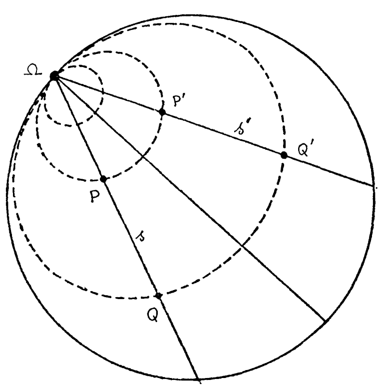

FIGURE 2

by an asymptotic rotation about Ω. This can be shown without much difficulty by synthetic means. Alternatively, consider a conformal half-space model of H^n in which Ω lies on all the "vertical" lines, that is, those perpendicular to the boundary hyperplane $z^n = 0$. Orispheres through Ω are represented by "horizontal" hyperplanes $z^n = c$. We saw in § 1 that an asymptotic rotation about Ω is represented by a euclidean translation parallel to $z^n = 0$, which displaces each horizontal hyperplane along itself. Note also that the product of two asymptotic rotations about Ω, even if they belong to different pencils, is another asymptotic rotation about Ω.

3. The groups G_Ω. Let Ω be any ideal point of H^n. Denote by G_Ω the collection consisting of the identity, all translations along lines through Ω, and all asymptotic rotations about Ω.

LEMMA 1. *G_Ω is a transitive group of direct isometries of H^n, without fixed points.*

Proof. No element of G_Ω, except for the identity, has an ordinary fixed point.

For $n = 2$, G_Ω is the set of all direct isometries of H^2 under which Ω is fixed, and thus is clearly a group. For $n \geqq 3$ we use a conformal half-space model, as in the last paragraph of the preceding section. Lines through Ω are represented by vertical lines, and translations along them by dilatations about points in the hyperplane $z^n = 0$. The collection of all such dilatations together with translations parallel to $z^n = 0$ (which represent asymptotic rotations about Ω) forms a subgroup of the group of all conformal maps of the half-space $z^n > 0$; thus G_Ω is a group (in fact, a Lie subgroup of the group of isometries of H^n).

To see that G_Ω is transitive on H^n, note that if the line PQ passes

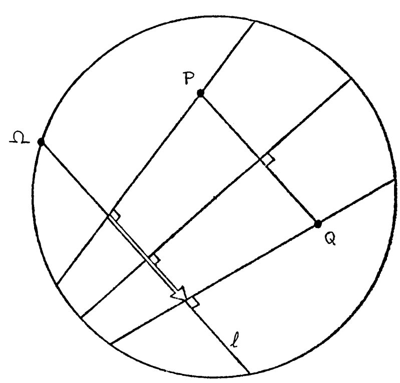

FIGURE 3

through Ω, then a certain translation along PQ maps P to Q, and if P and Q lie on the same orisphere through Ω (i.e., if the perpendicular bisector of segment PQ passes through Ω) then a certain asymptotic rotation about Ω maps P to Q. If neither of these occurs, certainly the composition of a translation and rotation will map P to Q. This is in fact a translation, which we can construct explicitly. Let l be the unique line through Ω and perpendicular to the perpendicular bisector of segment PQ (see Fig. 3). Now an appropriate translation along l maps P to Q.

4. The main theorem.

THEOREM. *A transitive group on H^n, without fixed points, consisting entirely of isometries which can be expressed as the product of two reflections in hyperplanes, must be the group G_Ω for some ideal point Ω of H^n.*

The theorem follows from three lemmas.

LEMMA 2. *There is no transitive group of translations on H^n.*

Therefore a transitive group without fixed points must contain some nontrivial asymptotic rotations.

LEMMA 3. *In a group as described in the theorem, the subgroup of rotations about any ideal point Ω contains either the identity alone, or all rotations about Ω.*

LEMMA 4. *A group which contains all rotations about an ideal point Ω, and either a translation along a line not through Ω or a rotation about some other ideal point, must contain proper rotations.*

Thus a group as described in the theorem can contain only translations along lines through an ideal point Ω and asymptotic rotations about Ω; that is, it must be a subgroup of G_Ω. But since G_Ω has no fixed points, no proper subgroup could be transitive. The theorem follows.

Proof of Lemma 2. For $n = 2$, Lemma 2 is a consequence of a stronger theorem of J. Nielsen [11], that a group of translations of H^2 is either one-dimensional (all translations having the same axis) or discrete. Note also the generalizations discussed in Chen and Greenberg [4]. The following argument is based on Nielsen's proof, with some modifications necessary for $n \geqq 3$.

Let G be a collection of translations of H^n, including the identity and inverses of all its elements, and suppose that G is transitive on H^n. We will construct a proper rotation which is the product of translations in G.

Given any $\varepsilon > 0$, there must be a translation $\tau_1 \in G$ of length $< \varepsilon$; let its axis be l. Let P be any point at a positive distance $< \varepsilon/2$ from l, and let $P' = \sigma_l P$, the reflection of P across l. There is some translation $\tau_2 \in G$ mapping P to P'. Its axis m is perpendicular to the hyperplane L of points equidistant from P and P' (note that $l \subset L$), and the length of τ_2 is less than ε.

Let M be the hyperplane through m and perpendicular to l, and choose hyperplane Q also perpendicular to l so that $\tau_1 = \sigma_Q \sigma_M$. Similarly set $\tau_2 = \sigma_S \sigma_L$. Using the fact that reflections in perpendicular hyperplanes commute, we have

$$\begin{aligned}\tau_1\tau_2\tau_1^{-1}\tau_2^{-1} &= \sigma_Q\sigma_M\sigma_S\sigma_L\sigma_M\sigma_Q\sigma_L\sigma_S = \sigma_Q\sigma_S\sigma_M\sigma_M\sigma_L\sigma_L\sigma_Q\sigma_S \\ &= (\sigma_Q\sigma_S)^2 \ .\end{aligned}$$

We claim that, if ε is sufficiently small, hyperplanes Q and S must meet, but not perpendicularly; thus $(\sigma_Q\sigma_S)^2$ is a proper rotation. This is obvious for $n = 2$ (see Fig. 4). For $n \geqq 3$, consider Fig. 4 to represent the 2-plane of l, P and P'; and suppose that s, m', q represent the intersections of hyperplanes S, M, Q with this 2-plane. Since M and Q are perpendicular to l, so are m' and q. M and S are perpendicular hyperplanes, and M is perpendicular to the 2-plane of Fig. 4, so s and m' must be perpendicular. The distance between m' and q is equal to the distance between M and Q, both being measured along l; this distance is less than $\varepsilon/2$. The distance between s and l is also less than $\varepsilon/2$, because s must pass through P'. (Note that the distance between S and L may be much smaller.) Thus for ε sufficiently small, s and q must intersect, and therefore S and Q intersect.

The hyperplane Q is perpendicular to the 2-plane of lines s and

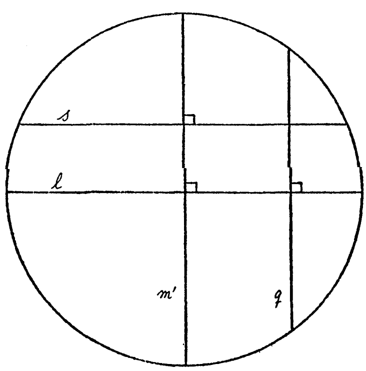

FIGURE 4

q, so if S were perpendicular to Q its intersection s with the plane of Fig. 4 would necessarily be perpendicular to Q and thus also to q. But s and q cannot be perpendicular; there are no rectangles in the hyperbolic plane. Thus Q and S are intersecting but nonperpendicular planes. This completes the proof of Lemma 2.

Proof of Lemma 3. Let G be a group as described in the theorem. Suppose that Ω is an ideal point about which there is some nontrivial rotation $\sigma_R\sigma_L$ in G. First we will show that every rotation in the pencil Π determined by R and L must be in G. Let $\mathscr{P}$ be a 2-plane through Ω, perpendicular to R and L, and let $r = \Omega\Omega'' = \mathscr{P} \cap R$ and $l = \Omega\Omega' = \mathscr{P} \cap L$. (If $n = 2$, then $r = R$, etc.) Let Σ be any oricycle in $\mathscr{P}$ through Ω; let P be any point of Σ inside the triply asymptotic triangle $\Omega\Omega'\Omega''$; and let $P' = \sigma_l P$. An element of G, mapping P to P', must be of the form $\sigma_L\sigma_M$, where M is some hyperplane through P and either parallel or hyperparallel to L. Then $m = M \cap \mathscr{P}$ is a line through P, parallel or hyperparallel to l. If $m \neq P\Omega$, m would meet r in an ordinary point, whence M would meet R in an ordinary $(n-2)$-plane, and thus $(\sigma_R\sigma_L)(\sigma_L\sigma_M) = \sigma_R\sigma_M$, a proper rotation in G. Thus we must have $m = P\Omega$, so M is parallel to L at Ω, that is, $M \in \Pi$. This means that every "sufficiently small" rotation in the pencil Π belongs to G; thus every rotation in Π belongs to G.

For $n > 2$ there are asymptotic rotations about Ω which do not belong to Π, and we must show that these also belong to G. The fact that rotations about Ω correspond to (euclidean) translations on each orisphere through Ω suggests decomposing an arbitrary rotation about Ω into a rotation in Π and another, "perpendicular" rotation, as follows.

Let Q, Q' be arbitrary points on the intersection of some hyper-

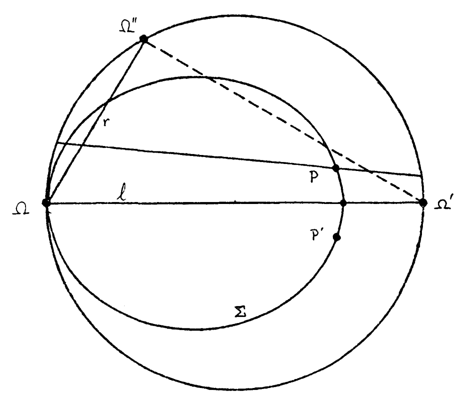

FIGURE 5

plane of Π and some orisphere through Ω. The hyperplane M equidistant from Q and Q' passes through Ω and is perpendicular to the line QQ'; therefore M is perpendicular to every hyperplane of Π. Also, Q is mapped to Q' by some element of G, of the form $\sigma_M\sigma_H$, where H passes through Q and is parallel or hyperparallel to M. If $\Omega \notin H$, there would be a unique hyperplane $L \in \Pi$ perpendicular to H, and some other hyperplane $R \in \Pi$ meeting H nonperpendicularly. Then we would have $\sigma_R\sigma_L \in G$ (because R, $L \in \Pi$), $\sigma_M\sigma_H \in G$ (by the hypothesis of transitivity), and thus $\sigma_R\sigma_L\sigma_H\sigma_M\sigma_L\sigma_R\sigma_M\sigma_H = \sigma_R\sigma_H\sigma_L\sigma_L\sigma_M\sigma_M\sigma_R\sigma_H = (\sigma_R\sigma_H)^2 \in G$, a proper rotation. We conclude therefore that $\Omega \in H$, that is, H and M are parallel at Ω, determining another pencil of asymptotic rotations about Ω, all of them elements of G.

If now P and P' are any two points on a common orisphere through Ω, let R be the hyperplane of Π through P, $L \in \Pi$ such that $P' \in \sigma_L(R)$, $Q = \sigma_L(P)$, M the perpendicular bisector of segment $P'Q$, and hyperplane H through Q and parallel at Ω to M. The above paragraph shows that $\sigma_M\sigma_H \in G$; but $\sigma_L\sigma_R \in G$ also, so $\sigma_M\sigma_H\sigma_L\sigma_R \in G$. This is an asymptotic rotation about Ω, mapping P to P'. Therefore G contains all asymptotic rotations about Ω, as required.

Proof of Lemma 4. Let $\sigma_N\sigma_L$ be either a nontrivial asymptotic rotation about $\Omega' \neq \Omega$, or a nontrivial translation along a line l not passing through Ω. In either case the hyperplanes N and L may be chosen so that Ω lies on N but not on L. Let P be any ordinary point of L, and let M be the hyperplane through P which is parallel to N at Ω. Now $\sigma_M\sigma_N$ is an asymptotic rotation about Ω, and the product $(\sigma_M\sigma_N)(\sigma_N\sigma_L) = \sigma_M\sigma_L$ is a proper rotation.

5. Other transitive groups on H^n. If $n = 2$, all direct isometries are expressible as products of two reflections, and so the theorem of § 4 characterizes all transitive groups, without fixed points, of isometries of H^2. For $n \geqq 3$ other groups are possible.

For example, let $n = 3$, choose some ideal point Ω, and consider the collection G consisting of all asymptotic rotations about Ω together with screw displacements along lines through Ω, such that the angle of rotation is some function $\theta(t)$ of the length t of translation. In order for this collection to be a group we require only that $\theta(t_1 + t_2) = \theta(t_1) + \theta(t_2)$; but if G is to exhibit many of the properties of H^3, more conditions are called for. For example, G will not be a Lie subgroup of the group of isometries unless $\theta(t)$ is continuous, that is, unless $\theta(t) = ct$ for some constant c. Thus there is at least a one-parameter family of groups for each ideal point Ω.

If $n \geqq 4$ there are still more possibilities. The orispheres through Ω are isometric to euclidean spaces of dimension at least 3, on which there are numerous transitive groups of isometries without fixed points. Any one of these, combined with screw displacements or pure translations along lines through Ω, gives a transitive group, without fixed points, on H^n.

It is not accidental that each of these examples has an ideal point fixed by each element of the group. A theorem of Chen [3, Theorem 4.1 (2)] implies that this is necessary, if G is to be a Lie group. The necessity of a fixed ideal point is also suggested by the fact that Brownian motion in hyperbolic space has an ideal limit point; see Kifer [8]. Both of these results require negative curvature, but hold in more generality, not just in hyperbolic spaces.

Among transitive groups without fixed points on E^n, the translation group is clearly preferred. One may ask whether the groups G_Ω have some analogous properties which justify preferring them over other groups on H^n. The fact that any element of G_Ω can be expressed as the product of two reflections is one such property. Two less obvious reasons for preferring G_Ω follow.

1. In E^n, a translation displaces each point the same distance. No isometry of H^n (except the identity) does this; but translations and asymptotic rotations displace the points of certain hypersurfaces by fixed distances. The situation is essentially analogous to that of simple rotations in E^n, which displace cylinders along themselves by a distance which increases as one moves away from the axis of rotation.

Note that an isometry without ordinary fixed points must have exactly one or two ideal fixed points: at least one, by the Brouwer fixed-point theorem, and not more than two else the ordinary plane they span would consist entirely of fixed points. Suppose g has exactly two ideal fixed points, Ω and Ω', determining a line l. Then g translates l along itself by some positive distance t. Let Q be any point, and H the hyperplane through Q, perpendicular to l. $g(H)$ is another hyperplane perpendicular to l, whose minimum distance from H is t. Thus the distance between Q and $g(Q)$ is at least t; indeed, if the distance between Q and l is $L_l(Q)$, then the distance between Q and $g(Q)$ is at least D, where

$$\sinh \frac{D}{2} = \cosh L_l(Q) \sinh \frac{t}{2} .$$

(This is a simple exercise in hyperbolic trigonometry; see Gans [6], pg. 162–166.) Moreover, this distance will be exceeded for at least some points not on l, unless g is the translation of length t along line l.

Alternatively, suppose g has only one ideal fixed point Ω. Let Σ be an orisphere through Ω, and $\Sigma' = g(\Sigma)$. Let π be projection from

Σ' to Σ along lines through Ω. Note that Σ and Σ' are each isometric to E^{n-1}, and π corresponds to a direct similarity, with the proportionality factor $\lambda \neq 1$ unless Σ and Σ' coincide (in which case of course $\pi = id_\Sigma$). Thus $\pi g|_\Sigma$ corresponds to a direct similarity of E^{n-1}; if $\lambda \neq 1$ this must have an ordinary fixed point. But if $P \in \Sigma$ were fixed under πg, then P, $g(P)$ and Ω would be collinear, and the line $P\Omega$ invariant under g; and then the other ideal point of this line would also be fixed under g, contradicting the hypothesis. We conclude that g maps each orisphere through Ω onto itself. Since each such orisphere is isometric to a euclidean space, g will displace each point on a given orisphere through Ω by the same distance, if and only if g restricts on each orisphere to a euclidean translation, that is, if and only if g is a (simple) asymptotic rotation about Ω. (Note that the distance of displacement varies from one orisphere to another.)

2. When an origin is chosen in E^n, each linear subspace through that origin defines a subgroup of the translation group; and the orbit of any point in E^n under such a linear subgroup is a linear subspace of the same dimension. Again, we cannot do quite so well in H^n; but orbits of points, under analogous subgroups of G_Ω, do lie in low-dimensional linear subspaces.

Let G be any transitive group of isometries of H^n, without ordinary fixed points but with a universally fixed ideal point Ω. First consider the subgroup G_l mapping an arbitrarily chosen point O to points on line $l = \mathrm{O}\Omega$; note that l is invariant under each element of G_l. The orbit $G_l(Q)$ of a point Q not on l contains exactly one point of each hyperplane perpendicular to l and lies on a "cylinder" of points equidistant from l. The only plane curve through Q which satisfies these conditions is that obtained when G_l is the collection of pure translations along l. If G_l involves any rotations about l, there are at least some points Q for which $G_l(Q)$ is not planar. In particular, if $n = 3$, G_Ω is the only transitive group of isometries of H^3, without fixed points, such that the orbits of points under G_l are plane curves.

For $n \geqq 4$, one additional condition characterizes G_Ω. Again let G be any transitive group without fixed points, with a universally fixed ideal point Ω. Let O be an arbitrarily chosen point of H^n, let $l = \mathrm{O}\Omega$, and let $\mathscr{P}$ be any 2-plane through l. Consider the collection $G_\mathscr{P}$ of elements of G mapping O to points of $\mathscr{P}$. $G_\mathscr{P}$ is a subgroup of G if and only if $\mathscr{P}$ is invariant under each element of $G_\mathscr{P}$. (This holds for any plane $\mathscr{P}$ if $G = G_\Omega$, and for some but not all of the other examples given earlier; for $n = 3$ this condition provides an alternate characterization of G_Ω.) If $G_\mathscr{P}$ is a subgroup of G, and Q is any point not on $\mathscr{P}$, then the orbit $G_\mathscr{P}(Q)$ contains exactly one point of each $(n-2)$-plane perpendicular to $\mathscr{P}$, and lies on a (connected) hypersurface equidistant from $\mathscr{P}$. The intersection of this hypersurface with the 3-plane of Q and $\mathscr{P}$ consists of two 2-dimensional surfaces, one of them containing Q. This surface coincides with $G_{\mathscr{P}}(Q)$ if $G = G_{\Omega}$; if $G \neq G_{\Omega}$ there are at least some planes $\mathscr{P}$ and some points Q such that $G_{\mathscr{P}}(Q)$ does not lie in any 3-plane.

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