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THE SYMPLECTIC GROUP OVER A RING WITH ONE IN ITS STABLE RANGE

B. KIRKWOOD AND B. R. MCDONALD

In this paper we determine the transitivity properties, generators and commutator subgroups of the symplectic group and its congruence subgroups over a commutative ring having one in its stable range and 2 a unit.

1. Introduction. Much of the classical theory of the symplectic group over a field has been generalized to symplectic groups where the scalar ring is local. The papers [6], [8], [11], [12], [17], [18] and [19] contain much of this literature and an introduction is provided in the monograph [13].

When using the local ring the technique is often to either "lift" results from the symplectic group over the residue class field or to utilize the abundance of units in local ring and mimic the arguments over a field.

However, the key to much of this theory is the ability to write units in the ring in a linear or polynomial fashion. This idea was exploited in [14] and several subsequent papers on the orthogonal group. In this paper, we show that the basic theory of the symplectic group over a commutative ring is available if the ring has "one in its stable range". This stable range condition is defined and discussed in (II). Examples of rings with one in their stable range include local rings, semilocal rings, von Neumann regular rings and zero dimensional rings.

The approach which allows this generalization is the extensive use of the "Eichler-Siegel-Dickson transvections" rather than the more traditional "symplectic transvections". In a sense, this "linearizes" the theory, allowing arguments which resemble the general linear group and elementary transvections. If R has one in its stable range, then utilizing repeatedly the formulas (*) and (**) we create units in desired locations. Once these units are available, the standard results easily follow. It should be emphasized that the theory we present is a consequence of formulas (*) and (**).

2. The symplectic group. Let R denote a commutative ring. We let V be a free R-module of R-dimension n where $n \geqq 2$. We assume V has a nonsingular symplectic form $\beta: V \times V \to R$. That is, β is R-bilinear, $\beta(x, x) = 0$ for all x in V and the R-module morphism from V to $V^* = \operatorname{Hom}_R(V, R)$ given by $x \to \beta(\ , x)$ is an isomorphism. We call the pair (V, β) a *symplectic space*. When the context is clear, (V, β) will be denoted by V.

Recall that an element x in V is *unimodular* if there is an f in V^* with $f(x) = 1$; equivalently, if $x = \alpha_1 b_1 + \cdots + \alpha_n b_n$ where $\{b_1, \cdots, b_n\}$ is a basis for V, then x is unimodular if $(\alpha_1, \cdots, \alpha_n) = R$. If x is unimodular then Rx is a free R-direct summand of dimension one. We call Rx a *line*. If x is unimodular and $V = Rx \oplus W$, we call the projective module W a *hyperplane*. Locally, W will have dimension $n - 1$, but W need not be free.

A *hyperbolic pair* $\{x, y\}$ is a pair of unimodular vectors in V with the property that $\beta(x, y) = 1$. The module $H = Rx \oplus Ry$ is called a *hyperbolic plane* and it is easy to see (for example, see [13], pp. 150–151) that V splits as an orthogonal direct sum $V = H \perp H^\perp$ where $H^\perp$ denotes the orthogonal complement of H.

Any unimodular vector u may be complemented to a hyperbolic pair as follows: By the above comments, there is an f in V^* with $f(u) = 1$. Since β is nonsingular, there is a v in V with $1 = f(u) = \beta(u, v)$. Then $\{u, v\}$ is a hyperbolic pair.

A ring R is *stably free* if whenever $V = V_1 \oplus P$ where V and V_1 are free R-modules, then P is a free R-module. Combining this with the above remarks on hyperbolic pairs, we have the following proposition.

PROPOSITION 2.1. *Let R be a stably free commutative ring and V be a symplectic space over R. Then V is an orthogonal direct sum $V = H_1 \perp H_2 \perp \cdots \perp H_m$ of hyperbolic planes $H_1, H_2, \cdots, H_m$. In particular, the dimension of V is even.*

Let (V, β) and $(\bar{V}, \bar{\beta})$ be symplectic spaces of the same dimension. An R-module isomorphism $\sigma\colon V \to \bar{V}$ is an *isometry* if for all x and y in V, we have $\bar{\beta}(\sigma(x), \sigma(y)) = \beta(x, y)$. In this case we say V and $\bar{V}$ are *isometric*, denoted $V \simeq \bar{V}$. The group of isometries $\sigma\colon (V, \beta) \to (V, \beta)$ is called the *symplectic group of* V and denoted by $\operatorname{Sp}(V)$.

Suppose $V = H \perp W$ where $H = Ru \oplus Rv$ is a hyperbolic plane in V. We next define several standard isometries with respect to H.

(a) If x is in V with $\beta(x, u) = 0$, then the *Eichler-Siegel-Dickson transvection* (denoted ESD-transvection) $\sigma_{u,x}$ is given by

$$\sigma_{u,x}(y) = y + \beta(u, y)x + \beta(x, y)u .$$

If $\beta(x, v) = 0$, then $\sigma_{v,x}$ is defined in a similar fashion.

(b) If ε is a unit in R, then define the isometry Φ_ε by

$$\Phi_\varepsilon(u) = \varepsilon u , \qquad \Phi_\varepsilon(v) = \varepsilon^{-1} v ,$$

and $\Phi_\varepsilon(w) = w$ for all w in W.

(c) Define the isometry Δ by $\Delta(u) = v$, $\Delta(v) = -u$ and $\Delta(w) = w$ for all w in W.

It is straightforward to check that the above are isometries. (Note that $\beta(\sigma(x), \sigma(y)) = \beta(x, y)$ may be checked locally since each of the above localize nicely at prime ideals of R and here one may use ([13], pp. 159–161).)

A *symplectic transvection* τ is an isometry satisfying any of the following equivalent statements:

(a) There is a unimodular vector a in V and a scalar λ in R such that for all x in V,

$$\tau(x) = x + \lambda\beta(a, x)a .$$

(b) There is a line $L = Ra$ satisfying $\tau(x) - x$ is in L for all x in V.

(c) There is a hyperplane P with $\tau|_P =$ identity. (See Theorem 4.1, p. 191 of [13].)

The above symplectic transvection is denoted by $\tau_{a,\lambda}$ and we call $L = Ra$ the *line of* $\tau_{a,\lambda}$ and P the *hyperplane of* $\tau_{a,\lambda}$.

The basic calculational properties of the above isometries are summarized in the next lemma.

LEMMA 2.2. *Let R be a commutative ring. Let V be a symplectic space over R with $V = H \perp W$ where $H = Ru \oplus Rv$ is a hyperbolic plane. Then*

(a) $\sigma_{u,x}\sigma_{u,y} = \sigma_{u,x+y}$, $(\sigma_{u,x})^{-1} = \sigma_{u,-x}$, *and* $\sigma_{u,0} = I$.

(b) *If θ is in* $\mathrm{Sp}(V)$ *then* $\theta\sigma_{u,x}\theta^{-1} = \sigma_{\theta(u),\theta(x)}$.

(c) $\Delta^{-1}\Phi_\varepsilon\Delta = \Phi_{\varepsilon^{-1}} = \Phi_\varepsilon^{-1}$, $\Phi_\varepsilon\Delta\Phi_\varepsilon = \Delta$, $\Delta^4 = I$, *and* $\Phi_\varepsilon\sigma_{u,x}\Phi_\varepsilon^{-1} = \sigma_{u,\varepsilon x}$ *when x is in W.*

(d) *If $x = \alpha u + \bar{x}$ and $y = \delta u + \eta v + \bar{y}$ where α, δ, η are in R and $\bar{x}$, $\bar{y}$ are in W, then*

$$\sigma_{u,x}(y) = [\delta + 2\alpha\eta + \beta(\bar{x}, \bar{y})]u + \eta v + (\bar{y} + \eta\bar{x}) .$$

If $x = \alpha v + \bar{x}$ *and* $y = \delta u + \eta v + \bar{y}$ *with the same hypothesis as above, then*

$$\sigma_{v,x}(y) = \delta u + [\eta - 2\alpha\delta + \beta(\bar{x}, \bar{y})]v + (\bar{y} - \delta\bar{x}) .$$

(e) $\tau_{a,\lambda}^{-1} = \tau_{a,-\lambda}$, $\tau_{a,\lambda}\tau_{a,\mu} = \tau_{a,\lambda+\mu}$ *and, more generally,* $\tau_{a,\lambda}\tau_{b,\mu}(x) = x + [\lambda\beta(a, x)a + \mu\beta(b, x)b] + \lambda\mu\beta(a, b)\beta(b, x)a$.

(f) *If* θ *is in* $\mathrm{Sp}(V)$, *then* $\theta\tau_{a,\lambda}\theta^{-1} = \tau_{\theta(a),\lambda}$.

(g) *The above isometries may be written as symplectic transvections and* ESD-*transvections as follows*:

$$\begin{aligned} \Phi_\varepsilon &= \tau_{u,\varepsilon(\varepsilon-1)}\tau_{v,\varepsilon^{-1}(\varepsilon-1)}\tau_{u+v,1-\varepsilon} \\ \Delta &= \tau_{v,-2}\tau_{u-v,-1} \\ \tau_{a,\lambda} &= \sigma_{a,(\lambda/2)a} \quad (\textit{if } 2 \textit{ is a unit}) . \end{aligned}$$

Thus, each of the above isometries may be written as products of ESD-transvections if 2 is a unit in R. Also, an argument analogous to the discussion in ([13], pp. 193–197) shows that each ESD-transvection may be written as a product of symplectic transvections.

THEOREM 2.3. *Let* R *be a commutative ring having* 2 *a unit. Let* V *be a symplectic space over* R *with* $V = H \perp W$ *where* $H = Ru \oplus Rv$ *is a hyperbolic plane. Suppose* σ *is in* $\mathrm{Sp}(V)$. *If* $\sigma(v) = \alpha u + \delta v + t$ (t *in* W) *and* δ *is a unit, then* σ *may be written as*

$$\sigma = \sigma_{u,x}\sigma_{v,y}\Phi_\varepsilon\bar{\sigma}$$

where $\bar{\sigma}$ *is in* $\mathrm{Sp}(W)$. *Further,* x *is in* $(Ru)^\perp$, y *is in* $(Rv)^\perp$, *and* x, y, ε *and* $\bar{\sigma}$ *are uniquely determined by* σ.

(*Note*: We identify $\mathrm{Sp}(W)$ as a subgroup of $\mathrm{Sp}(V)$ by $\bar{\sigma} \to I \perp \bar{\sigma}$.)

Proof. Suppose $\sigma(v) = \alpha u + \delta v + t$ where δ is a unit. Then

$$\begin{aligned} \sigma_{u,\delta^{-1}t+\delta^{-1}(\alpha/2)u}(v) &= v + \delta^{-1}t + \delta^{-1}\alpha u \\ &= \delta^{-1}[\alpha u + \delta v + t] \\ &= \delta^{-1}\sigma(v) . \end{aligned}$$

Thus, set

$$\hat{\sigma} = \Phi_\delta\sigma^{-1}_{u,\delta^{-1}t+\delta^{-1}(\alpha/2)u}$$

and we have $\hat{\sigma}(v) = v$. Suppose $\hat{\sigma}(u) = \gamma(u) + \mu v + s$ (s in W). Since $\beta(u, v) = 1$, we have $\hat{\sigma}(u) = u + \mu v + s$. Then

$$\sigma_{v,(\mu/2)v+s}(\hat{\sigma}(u)) = u$$
$$\sigma_{v,(\mu/2)v+s}(\hat{\sigma}(v)) = v\ .$$

That is,

$$\sigma_{v,(\mu/2)v+s}\Phi_\delta\sigma^{-1}_{u,\delta^{-1}t+\delta^{-1}(\alpha/2)u} = \bar{\sigma}$$

where $\bar{\sigma}|_H =$ identity, i.e., $\bar{\sigma}$ is in $\mathrm{Sp}(W)$. Thus, using 2.2, $\sigma = \sigma_{u,x}\sigma_{v,y}\Phi_\varepsilon\bar{\sigma}$ for suitable x, y and ε.

It remains to check uniqueness. Assume

$$\sigma_{u,x}\sigma_{v,y}\Phi_\varepsilon\bar{\sigma} = \sigma_{u,x_1}\sigma_{v,y_1}\Phi_{\varepsilon_1}\bar{\sigma}_1\ .$$

Apply both sides of the above equality to v. We obtain

$$\varepsilon^{-1}\beta(x, v)u + \varepsilon^{-1}v + \varepsilon^{-1}x = \varepsilon_1^{-1}\beta(x_1, v)u + \varepsilon_1^{-1}v + \varepsilon_1^{-1}x_1$$

which implies $\varepsilon = \varepsilon_1$. If $x = \alpha u + \bar{x}$ and $x_1 = \alpha_1 u + \bar{x}_1$ as in 2.2 (d), then

$$\alpha u + v + (\alpha u + \bar{x}) = \alpha_1 u + v + (\alpha_1 u + \bar{x}_1)\ .$$

Then, since 2 is a unit we have $\alpha = \alpha_1$ and subsequently $\bar{x} = \bar{x}_1$. A similar argument utilizing u will show $y = y_1$.

Suppose that V splits as a direct sum of hyperbolic planes $V = H_1 \perp H_2 \perp \cdots \perp H_m$ where $H_i = Ru_i \oplus Rv_i$ for $1 \leqq i \leqq m$. The basis $\{u_1, v_1, u_2, v_2, \cdots, u_m, v_m\}$ is called a *hyperbolic basis* of V.

Suppose we have the above hyperbolic basis for V. For x and z in V, let

$$x = \sum_{i=1}^m \alpha_i u_i + \sum_{i=1}^m \gamma_i v_i\ ,$$

and

$$z = \sum_{i=1}^m \delta_i u_i + \sum_{i=1}^m \eta_i v_i\ .$$

If $\beta(x, u_1) = 0$, i.e., $\gamma_1 = 0$, then

$$(*)\qquad \begin{aligned}\sigma_{u_1,x}(z) = \Big[\delta_1 + 2\alpha_1\eta_1 + \sum_{i=2}^m (\alpha_i\eta_i - \delta_i\gamma_i)\Big]u_1 \\ + \eta_1 v_1 + \sum_{i=2}^m [(\delta_i + \eta_1\alpha_i)u_i + (\eta_i + \eta_1\gamma_i)v_i]\ .\end{aligned}$$

If $\beta(x, v_1) = 0$, i.e., $\alpha_1 = 0$, then

$$(**)\qquad \begin{aligned}\sigma_{v_1,x}(z) = \delta_1 u_1 + \left[\eta_1 - 2\delta_1\gamma_1 + \sum_{i=2}^{m}(\alpha_i\eta_i + \gamma_i\delta_i)\right]v_1 \\ + \sum_{i=2}^{m}[(\delta_i - \delta_1\alpha_i)u_i + (\eta_i - \delta_1\gamma_i)v_i]\ .\end{aligned}$$

A commutative ring R is said to *have one in its stable range* or have *stable range one* if whenever α and β are in R with $(\alpha, \beta) = R$ then there is a δ in R with $\alpha + \delta\beta =$ unit. Stable range one rings (both commutative and noncommutative) have been examined from a ring theoretic viewpoint in [3], [4], [5], and [20]. The role that stable range one rings play in linear algebra and the general linear group is discussed in [1], [4], [7], [15], [21], [22], [23], and [24]. This definition was extended in [9], [10], [14], and [16] to examine the structure theory of quadratic forms, Witt rings and the orthogonal group. In particular, in ([14], 3.1) it was noted that a ring having stable range one was stably free and, hence, for our purposes their symplectic spaces are direct sums of hyperbolic planes. Examples of rings with one in their stable range are local rings, semilocal rings, von Neumann regular rings, and zero dimensional rings.

Suppose R has 2 a unit and stable range one. It is straightforward to show that if $(x, y_1, y_2, \cdots, y_n) = R$ then there are α_1, $\alpha_2, \cdots, \alpha_n$ in R with

$$x + \alpha_1 y_1 + \cdots + \alpha_n y_n = \text{unit}\ .$$

Returning to the above calcutation (*) (or (**)) suppose that $z = \sum_i \delta_i u_i + \sum_i \eta_i v_i$ is unimodular. Since 2 is a unit, we then have

$$(\delta_1, 2\eta_1, -\delta_2, \eta_2, \cdots, -\delta_m, \eta_m) = R\ .$$

Thus, there exist $\alpha_1, \alpha_2, \cdots, \alpha_m, \gamma_2, \gamma_3, \cdots, \gamma_m$ with

$$\delta_1 + 2\alpha_1\eta_1 + \sum_i (\alpha_i\eta_i - \gamma_i\delta_i) = \text{unit}\ .$$

That is, if $x = \alpha_1 u_1 + \cdots + \alpha_m u_m + \gamma_2 v_2 + \cdots + \gamma_m v_m$ then

$$\sigma_{u_1,x}(z) = \delta u_1 + \eta v_1 + \bar{z}$$

where δ is a unit.

We will now develop a number of consequences of the above observation. First, is a sharpening of 2.3.

THEOREM 2.4. *Let R be a commutative ring with stable range one and 2 a unit. Let $V = H \perp W$ be a symplectic space over R where $H = Ru \oplus Rv$ is a hyperbolic plane. Let σ be in* $\mathrm{Sp}(V)$.

Then there is a z in V such that

$$\sigma = \sigma_{u,z}\sigma_{u,x}\sigma_{v,y}\Phi_\varepsilon\bar{\sigma}$$

where x is in $(Ru)^\perp$, y is in $(Rv)^\perp$, ε is a unit, and $\bar{\sigma}$ is in $\mathrm{Sp}(W)$ *and each is uniquely determined by σ and z.*

Proof. Consider the unimodular vector $\sigma(v)$. By the discussion before 2.4, there is an ESD-transvection $\sigma_{u,z}$ with

$$(\sigma_{u,z}\sigma)(v) = \sigma_{u,z}(\sigma(v)) = \alpha u + \delta v + t \ (t \text{ in } W)$$

where δ is a unit. The result now follows from 2.3.

Let $H = Ru \oplus Rv$ be a hyperbolic plane. Let $E(u, v)$ denote the subgroup of $\mathrm{Sp}(V)$ generated by the isometries of the form $\sigma_{u,x}$ and $\sigma_{v,y}$ for suitable x and y. Let $P(u, v)$ denote the group generated by the Φ_ε for ε a unit. Finally, let $E(H)$ denote the subgroup generated by all ESD-transvections $\sigma_{a,x}$ and $\sigma_{b,y}$ where $H = Ra \oplus Rb$ and $\beta(a, b) = 1$.

COROLLARY 2.5 (*under the hypothesis of* 2.4).
(a) $\mathrm{Sp}(V) = E(u, v)P(u, v)\,\mathrm{Sp}(W)$.
(b) $\mathrm{Sp}(V) = E(H)\,\mathrm{Sp}(W)$.

Proof. Part (a) follows immediately from 2.4. Part (b) follows from 2.4 and part (g) of 2.2 which shows each Φ_ε may be written as a product of elements in $E(H)$.

COROLLARY 2.6 (*under the hypothesis of* 2.4). *The group $E(H)$ is a normal subgroup of* $\mathrm{Sp}(V)$.

Proof. Let $\tau = \sigma_{u,x}$ be in $E(H)$. Let ρ be in $\mathrm{Sp}(V)$. By 2.5, $\rho = \sigma\bar{\rho}$ where σ is in $E(H)$ and $\bar{\rho}$ is in $\mathrm{Sp}(W)$. Then

$$\begin{aligned}\rho\tau\rho^{-1} &= (\sigma\bar{\rho})(\sigma_{a,x})(\sigma\bar{\rho})^{-1}\\ &= \sigma(\bar{\rho}\sigma_{u,x}\bar{\rho}^{-1})\sigma^{-1}\\ &= \sigma\sigma_{u,\bar{\rho}(x)}\sigma^{-1} \quad (\text{by } 2.2(\text{b}))\end{aligned}$$

and this final product is in $E(H)$.

Under the hypothesis of 2.4, the symplectic space V splits as an orthogonal sum $V = H_1 \perp \cdots \perp H_m$ where the H_i are hyperbolic planes. On the other hand $V = H_1 \perp W$. Since orthogonal complements are unique, $W = H_2 \perp \cdots \perp H_m$. An induction argument will now give the first part of the following corollary.

COROLLARY 2.7 (*under the hypothesis of* 2.4). *Suppose* $V = H_1 \perp H_2 \perp \cdots \perp H_m$ *is a decomposition of* V *into hyperbolic planes. Then*

(a) $\mathrm{Sp}(V) = E(H_1)E(H_2)\cdots E(H_m)$. *Thus, the symplectic group* $\mathrm{Sp}(V)$ *is generated by* ESD-*transvections.*

(b) *Each element in* $\mathrm{Sp}(V)$ *is a product of* $\leqq 6m$ ESD-*transvections where* $m = (\dim V)/2$.

Proof. Part (a) is immediate. Part (b) follows from 2.4, 2.2(g) and induction.

We next develop some transitivity results related to the calculations (*) and (**).

Let R have stable range one and $V = H \perp W$ where $H = Ru \oplus Rv$ is a hyperbolic plane. If z is a unimodular vector in V then, by using (*) and the calculation prior to 2.4, there is an ESD-transvection $\sigma_{u,x}$ with

$$\sigma_{u,x}(z) = \delta u + \eta v + \bar{z} \quad (\bar{z} \text{ in } W)$$

where δ is a unit. Let $y = \delta^{-1}\bar{z}$. Then

$$\begin{aligned}\sigma_{v,y}\sigma_{u,x}(z) &= \sigma_{v,y}[\delta u + \eta v + \bar{z}] \\ &= \delta u + \eta v + (\bar{z} - \delta y) \\ &= \delta u + \eta v\,.\end{aligned}$$

That is, $\sigma_{v,y}\sigma_{u,x}(z)$ is in the hyperbolic plane H.

THEOREM 2.8. *Let R be a commutative ring with stable range one and 2 a unit. Let V be a symplectic space over R. Then,*

(a) $\mathrm{Sp}(V)$ *is transitive on unimodular vectors.*

(b) $\mathrm{Sp}(V)$ *is transitive on hyperbolic planes.*

Proof. (a) Suppose z and $\bar{z}$ are unimodular vectors in V. By the above discussion, there are products σ_1 and σ_2 of isometries in $\mathrm{Sp}(V)$ such that

$$\begin{aligned}\sigma_1(z) &= \delta u + \eta v \\ \sigma_2(\bar{z}) &= \bar{\delta} u + \bar{\eta} v\end{aligned}$$

where δ, $\bar{\delta}$ are units and $H = Ru \oplus Rv$ is a hyperbolic plane. Consider the vector $\sigma_1(z) = \delta u + \eta v$. Then $\Phi_{\delta^{-1}}(\sigma_1(z)) = u + \mu v$ where $\mu = \delta\eta$. Then, using transvection $\tau = \tau_{v+u+\mu v,1}$, we have

$$\tau\bar{\Phi}_{\delta^{-1}}(\sigma_1(z)) = -v\,.$$

Similarly, a product of two isometries will carry $\bar{\delta}u + \bar{\eta}v$ to $-v$. That is, there are products $\sum_1$ and $\sum_2$ of isometries with

$$\sum_1 (z) = -v = \sum_2 (\bar{z}) ,$$

i.e., $\sum_2^{-1} \sum_1 (z) = \bar{z}$. This completes part (a). To show (b), let $\{u, v\}$ and $\{x, y\}$ be hyperbolic pairs. Since u and x are unimodular, there is a suitable product σ of isometries so that $\sigma(u) = x$. Thus, without loss of generality, assume $u = x$. We need to carry v to y while fixing u. Let $y = \alpha u + \delta v + w$ where w is in $(Ru \oplus Rv)^{\perp}$. Since $1 = \beta(x, y) = \beta(u, y) = \delta$, we have $y = \alpha u + v + w$. Then

$$\{u, y\} \xrightarrow[\sigma_{u,-w}]{} \{u, \alpha u + v\} \xrightarrow[\tau_{u,-\alpha}]{} \{u, v\} ,$$

and we are done.

COROLLARY 2.9 (*cancellation*). *Let R be a commutative ring having stable range one and 2 a unit. If U, V and Y are symplectic spaces with $U \perp V \simeq U \perp Y$, then $V \simeq Y$.*

Proof. By an induction argument, it suffices to prove the result when $U = H$ is a hyperbolic plane. Let $\sigma: H \perp V \to H \perp Y$ be an isometry. Let $H_1 = \sigma(H)$ and $V_1 = \sigma(V)$. There is a product τ of elements in $\mathrm{Sp}(H \perp Y)$ with $\tau H_1 = H$. Then $\tau H_1^{\perp} = H^{\perp}$, i.e., $\tau V_1 = Y$. Thus $V \simeq Y$.

If V is a symplectic space of dimension 2 over a commutative ring with stable range one, then $\mathrm{Sp}(V)$ is precisely the special linear group $\mathrm{SL}(V)$ of V. In this case, the structure of $\mathrm{SL}(V)$ is given in [15].

PROPOSITION 2.10 (*under the hypothesis of* 2.4). *The center of* $\mathrm{Sp}(V)$ *is precisely*

$$\{\alpha I \mid \alpha \textit{ is in } R \textit{ and } \alpha^2 = 1\} .$$

Proof. If $\dim(V) = 2$, this is given in [15]. If $\dim(V) \geqq 3$, the proof is analogous to ([13], Thm. 3.22).

Let A be an ideal of R. The natural ring morphism $\pi_A: R \to R/A$ induces a surjective morphism $\pi_A: V \to V/AV$ of symplectic spaces where if $V = (V, \beta)$ then $(V/AV, \bar{\beta})$ is given by

$$\bar{\beta}(\pi_A x, \pi_A y) = \pi_A \beta(x, y) .$$

It is easy to see that R/A has stable range one if R has stable range one, e.g., see (Prop. 2.6(a) of [14]). In turn, π_A induces a group morphism $\pi_A: \mathrm{Sp}(V) \to \mathrm{Sp}(V/AV)$ by

$$(\pi_A \sigma)(\pi_A x) = \pi_A(\sigma(x)) .$$

A splitting $V = H_1 \perp \cdots \perp H_m$ of V into hyperbolic planes induces a splitting of $V/AV = \bar{H}_1 \perp \cdots \perp \bar{H}_m$ into hyperbolic planes where $\pi_A H_i = \bar{H}_i$, $1 \leqq i \leqq m$. Each generator, i.e., ESD-transvection, in $E(\bar{H}_i)$ has a preimage under π_A in $E(H_i)$. By 2.7 these elements generate $\mathrm{Sp}(V/AV)$. Thus, we have the following proposition.

PROPOSITION 2.11 (*under the hypothesis of* 2.4). *The group morphism* $\pi_A: \mathrm{Sp}(V) \to \mathrm{Sp}(V/AV)$, *where A is an ideal, is surjective.*

We now study the commutator subgroup of $\mathrm{Sp}(V)$. To achieve the expected results we will see that we need units ε and η in R such that $\varepsilon - \eta = 1$. If 3 is a unit in R, then $3 - 2 = 1$ and 3 and 2 will do. More generally, we can always assure this will happen if R has "2-fold" stable range one. Precisely, R has *2-fold stable range one* if whenever $(a_1, b_1) = R$ and $(a_2, b_2) = R$ then there is an α with $a_1 + \alpha b_1 =$ unit and $a_2 + \alpha b_2 =$ unit. The concept of "k-fold stable range one" rings was introduced in [7]. Suppose R has 2-fold stable range one. Then, using $(1, 1) = R$ and $(0, 1) = R$ we can find η with $1 + \eta = \varepsilon$ (unit) and $\eta =$ unit. Thus, we have units ε and η with $\varepsilon - \eta = 1$.

If G is a group, denote the commutator subgroup of G by $[G, G]$.

THEOREM 2.12 (*under the hypothesis of* 2.4). *Suppose there exists units ε and η in R with $\varepsilon - \eta = 1$. Then*

$$\mathrm{Sp}(V) = [\mathrm{Sp}(V), \mathrm{Sp}(V)] .$$

Proof. It suffices to show that each generator $\sigma_{u,x}$ or $\sigma_{v,x}$ of $\mathrm{Sp}(V)$ can be written as a commutator. Consider $\sigma_{u,x}$. Select units ε and η in R with $\varepsilon - \eta = 1$. Let $\alpha = \eta^{-1}$. Then

$$\begin{aligned} [\Phi_\varepsilon, \sigma_{u,\alpha x}] &= \Phi_\varepsilon \sigma_{u,\alpha x} \Phi_\varepsilon^{-1} \sigma_{u,\alpha x}^{-1} \\ &= \sigma_{u,\varepsilon\alpha x} \sigma_{u,-\alpha x} \quad \text{(by 2.2)} \\ &= \sigma_{u,(\varepsilon\alpha - \alpha)x} \quad \text{(by 2.2)} \\ &= \sigma_{u,x} . \end{aligned}$$

The isometry $\sigma_{v,x}$ is handled similarly.

3. The congruence subgroups. Let A be an ideal of the ring R. As noted in the previous section, the ring morphism $\pi_A: R \to R/A$ induces a group morphism $\pi_A: \mathrm{Sp}(V) \to \mathrm{Sp}(V/AV)$. The group morphism is in general not surjective; however, if R has stable range one and 2 a unit then it is surjective by 2.11.

The *general congruence subgroup of level A* is

$$\mathrm{GSp}(V, A) = \pi_A^{-1} \quad (\text{center } (\mathrm{Sp}(V/AV)))$$

where $0 \subsetneqq A \subsetneqq R$. The special cases are

$$\mathrm{GSp}(V, R) = \mathrm{Sp}(V) , \qquad \mathrm{GSp}(V, 0) = \text{Center}(\mathrm{Sp}(V)) .$$

If $0 \subsetneqq A \subsetneqq R$, then the *special congruence subgroup of level A* is

$$\begin{aligned}\mathrm{SSp}(V, A) &= \ker(\pi_A)\\ &= \{\sigma \text{ in } \mathrm{Sp}(V) \mid \pi_A\sigma = I\} .\end{aligned}$$

The special cases are

$$\mathrm{SSp}(V, R) = \mathrm{Sp}(V) , \qquad \mathrm{SSp}(V, 0) = \{I\} .$$

If α is in R, then the *order of* α, denoted $O(\alpha)$, is the ideal generated by α. If x is in V, then the *order* $O(x)$ *of* x is the smallest ideal A of R satisfying $\pi_A x = 0$. Note that if $x = \sum \alpha_i b_i$ relative to a basis $\{b_1, b_2, \cdots, b_n\}$, then $O(x) = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. If σ is in $\mathrm{Sp}(V)$, then the *order* $O(\sigma)$ *of* σ *is the smallest ideal A satisfying* $\pi_A\sigma$ is in Center $(\mathrm{Sp}(V/AV))$. That is, $O(\sigma)$ is the smallest ideal A with σ in $\mathrm{GSp}(V, A)$[1]. If G is a subgroup of $\mathrm{Sp}(V)$, then the *order* $O(G)$ *of* G is the smallest ideal A with $G \leqq \mathrm{GSp}(V, A)$.

LEMMA 3.1. *For the isometries in* (2),
(1) $O(\sigma_{u,x}) = O(x)$, $O(\sigma_{v,x}) = O(x)$.
(2) $O(\tau_{a,\lambda}) = O(\lambda)$.
(3) $O(\Delta) = R$.
(4) $O(\Phi_\varepsilon) = (\varepsilon - 1)$.

THEOREM 3.2. *Let R be a commutative ring and A be an ideal of R. Let* $V = H \perp W$ *be a symplectic space where* $H = Ru \oplus Rv$ *is a hyperbolic plane. Suppose*

$$\begin{aligned}x &= \delta u + \eta v + \bar{x}\\ y &= \bar{\delta} u + \bar{\eta} v + \bar{y}\end{aligned}$$

($\bar{x}$, $\bar{y}$ *in* W) *where* $x \equiv y$ *modulo* AV. *If* δ *and* $\bar{\delta}$ *are units, then there is an isometry* σ *in* $\mathrm{SSp}(V, A)$ *with* $\sigma(x) = y$.

Proof. Let $z = \bar{\delta}^{-1}\bar{y}$. Then

$$\begin{aligned}\sigma_{v,z}(z) &= \delta u + \eta_1 v + (\bar{x} - \delta\bar{\delta}^{-1}\bar{y})\\ \sigma_{v,z}(y) &= \bar{\delta} + \bar{\eta} v\end{aligned}$$

where $\eta_1 = \eta - \beta(\bar{\delta}^{-1}\bar{y}, \bar{x})$. Set $q = \bar{x} - \delta\bar{\delta}^{-1}\bar{y}$. Since $x \equiv y$ modulo

[1] This ideal exists due to 2.10.

AV, we have $\delta \equiv \bar{\delta}$ modulo A and $\bar{x} \equiv \bar{y}$ modulo AV, and consequently $O(q) \subset A$. Let $\sigma_1 = \sigma_{v,\delta^{-1}q}$. Then $O(\sigma_1) \subset A$, i.e., σ_1 is in $\mathrm{SSp}(V, A)$, and

$$\begin{aligned} \sigma_1\sigma_{v,z}(x) &= \delta u + \eta_1 v \\ \sigma_{v,z}(y) &= \bar{\delta} u + \bar{\eta} v \end{aligned}$$

where δ and $\bar{\delta}$ are units. Since $\delta \equiv \bar{\delta}$ modulo A, we have $\delta\bar{\delta}^{-1} \equiv 1$ modulo A. Then

$$\begin{aligned} \Phi_{\bar{\delta}}^{-1}\sigma_1\sigma_{v,z}(x) &= u + \lambda v \\ \Phi_{\bar{\delta}}^{-1}\sigma_{v,z}(y) &= (1 + a)u + \bar{\lambda} v \end{aligned}$$

where $1 + a$ is a unit and a is in A. Then, Φ_{1+a}^{-1} is in $\mathrm{SSp}(V, A)$ and

$$\Phi_{1+a}^{-1}\Phi_{\bar{\delta}}^{-1}\sigma_{v,z}(y) = u + \mu v$$

where $\lambda \equiv \mu$ modulo A. Let $\lambda = \mu - \alpha$ where α is in A. Then

$$\begin{aligned} \sigma_{v,\alpha v}(u + \mu v) &= u + [\mu - \alpha]v \\ &= u + \lambda v \end{aligned}$$

where $\sigma_{v,\alpha v}$ has order $\subset A$. That is, if $\sigma = \sigma_{v,z}^{-1}\Phi_{\delta}\Phi_{1+a}\sigma_{v\ \alpha v}^{-1}\Phi_{\bar{\delta}}^{-1}\sigma_1\sigma_{v,z}$ then $\sigma(x) = y$. Further, $\sigma \equiv I$ modulo A and thus σ is in $\mathrm{SSp}(V, A)$. This completes the proof.

We note that, using the calculations in 2.2, the above expression for σ may be rewritten as a product of ESD-transvections of orders $\subset A$.

Suppose R is a commutative ring with 2 a unit and 2-fold stable range one (see discussion before 2.12): Let $V = H_1 \perp \cdots \perp H_m$ be a decomposition into hyperbolic planes where $H_i = Ru_i \oplus Rv_i$ for $1 \leqq i \leqq m$. Let

$$\begin{aligned} &u = u_1\,, \qquad v = v_1\,, \\ &x = \sum_i \delta_i^{(1)}u_i + \sum_i \eta_i^{(1)}v_i\,, \quad \text{and} \\ &y = \sum_i \delta_i^{(2)}u_i + \sum_i \eta_i^{(2)}v_i \end{aligned}$$

be unimodular vectors in V with $x \equiv y$ modulo AV. Since 2 is a unit and x and y are unimodular,

$$(\delta_1^{(i)}, 2\eta_1^{(i)}, -\delta_2^{(i)}, \eta_2^{(i)}, \cdots, -\delta_m^{(i)}, \eta_m^{(i)}) = R$$

for $i = 1, 2$. Since R has 2-fold stable range one, it is straightforward to produce $\alpha_1, \alpha_2, \cdots, \alpha_m, \gamma_2, \cdots, \gamma_m$ with

$$\delta_1^{(i)} + 2\alpha_1\eta_1^{(i)} + \sum_j [\alpha_j\eta_j^{(i)} - \gamma_j\delta_j^{(i)}] = \mu_i$$

where μ_i is a unit for $i = 1, 2$. Thus, if $w = \alpha_1 u_1 + \cdots + \alpha_m u_m + \gamma_2 v_2 + \cdots + \gamma_m v_m$ then (in the notation of 3.2).

$$\sigma_{u,w}(x) = \delta u + \eta v + \bar{x}$$
$$\sigma_{u,w}(y) = \bar{\delta} u + \bar{\eta} v + \bar{y}$$

where $\sigma_{u,w}(x) \equiv \sigma_{u,w}(y)$ modulo A and δ and $\bar{\delta}$ are units. Combining this discussion with Theorem 3.2, we have the next result.

THEOREM 3.3. *Let R be commutative ring which has 2-fold stable range one and 2 a unit. Let $A \neq R$ be an ideal in R. Let V be a symplectic space over R and $E = \{x$ in $V \mid x$ is unimodular$\}$. Then* SSp(V, A) *acts as a transformation group on E and the* SSp(V, A)*-orbits of E are precisely the congruence classes of E modulo AV.*

Thus, under the hypothesis of 3.3, then for x and y in E we have $x \equiv y$ modulo AV if and only if there is a σ in SSp(V, A) with $\sigma x = y$.

COROLLARY 3.4 (*under the hypothesis of* 3.3). *The group* SSp(V, A) *acts as a transformation group on the family $\mathscr{H}$ of hyperbolic planes of V. Two hyperbolic planes H and $\bar{H}$ are in the same* SSp(V, A)*-orbit if and only if $\pi_A H = \pi_A \bar{H}$.*

Proof. The proof is similar to the proof of 2.8(b) or the proof in the local ring case given in ([13], Theorem 3.24).

COROLLARY 3.5 (*under the hypothesis of* 3.3). *The group* SSp(V, A) *is generated by* ESD*-transvections of order contained in A.*

Proof. The proof is analogous to the proof where R is a local ring given in ([13], Theorem 4.9). The key idea is that if $x \equiv y$ modulo AV then the σ in SSp(V, A) with $\sigma(x) = y$ may be written as a product of ESD-transvections as was noted after Theorem 3.2

COROLLARY 3.6 (*under the hypothesis of* 3.3).

$$\mathrm{SSp}(V, A) = [\mathrm{Sp}(V), \mathrm{GSp}(V, A)] = [\mathrm{Sp}(V), \mathrm{SSp}(V, A)]$$

where A is an ideal of R.

Proof. The case $A = R$ is given in the previous section. We may assume $A \neq R$. By reducing modulo A, it is easy to see that

$$[\mathrm{Sp}(V), \mathrm{SSp}(V, A)] \subset [\mathrm{Sp}(V), \mathrm{GSp}(V, A)] \subset \mathrm{SSp}(V, A) .$$

So it suffices to show that each generator of $\mathrm{SSp}(V, A)$ is in $[\mathrm{Sp}(V), \mathrm{SSp}(V, A)]$. Consider $\sigma_{u,x}$ where the $O(x) \subset A$. Select units ε and α as in the proof of 2.12. Then $O(\sigma_{u,\alpha x}) = O(\sigma_{u,x}) \subset A$ and, as in 2.12,

$$\sigma_{u,x} = [\Phi_{\varepsilon}, \sigma_{u,\alpha x}] ;$$

thus, $\sigma_{u,x}$ is in $[\mathrm{Sp}(V), \mathrm{SSp}(V, A)]$.

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SWEET BRIAR COLLEGE
SWEET BRIAR, VA 24595
AND
THE UNIVERSITY OF OKLAHOMA
NORMAN, OK 73019