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SOME EXACT SOLUTIONS OF THE NONLINEAR PROBLEM OF WATER WAVES

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Our purpose here is to present two related, strictly constructive methods for proving existence and uniqueness of certain steady problems of water waves. By "water waves", we mean free-surface flows, under gravity, of inviscid, irrotational, incompressible fluids.

Our results are related to those of Gerber [3], Moiseev [5], Krasovskii [4], Beckert [2] and others, and, in fact, include all the earlier works but Krasovskii's. Moreover, unlike any other work we know of, ours makes no use of complex function theory. Thus, our methods also apply, at least in principle, to three-dimensional flows. However, there appear to be serious technical difficulties with the generalization to three dimensions, and we reserve all discussion of this for later work.

Before discussing our results, we state the problem precisely. For this, choose a coordinate system with the Y-axis pointing up. Then, since the flow is assumed steady, we suppose the fluid occupies a domain $-B(X) < Y < T(X)$, independent of time, and we seek [10] a velocity potential Φ satisfying Laplace's equation

$$\Phi_{XX} + \Phi_{YY} = 0 \text{ for } -B(X) < Y < T(X)\,, \tag{1.1}$$

with two conditions on the (unknown) free surface $Y = T(X)$ and one on the (given) bottom $Y = -B(X)$. When $Y = T(X)$, the boundary conditions are

$$\Phi_Y = T_X \Phi_X\,, \tag{1.2}$$

and

$$2gT + \Phi_X^2 + \Phi_Y^2 = \text{constant}; \tag{1.3}$$

here, g denotes the acceleration due to gravity. Subscripts denote partial differentiation. On the bottom, $Y = -B(X)$, we require

$$\Phi_Y + B_X \Phi_X = 0\,. \tag{1.4}$$

In addition, two parameters are given, say, the mean depth and a mean speed.

Our main hypothesis is that the bottom is not too far from being flat and horizontal. We impose this condition by supposing that B has the form

$$B(X) = B_\varepsilon(X) = d_0(1 + \varepsilon H(X)) , \tag{1.5}$$

where ε is a real parameter, varying in a neighborhood of zero. d_0 is the mean depth we just spoke of, and, to specify it precisely, we always suppose that H has mean value zero. This means that d_0 is the mean depth when there is no flow ($T = \Phi_X = \Phi_Y = 0$).

We also need the parameter U_0, which is the mean value of the horizontal velocity on the free surface $Y = T(X)$. The Froude number F is defined by

$$F = \frac{U_0^2}{gd_0} . \tag{1.6}$$

We begin by studying periodic flows. Let B be periodic, with period Ld_0. Then, we show that the problem (1.1)-(1.4) has a unique solution if ε is small enough and $F \neq F_n$, $n = 1, 2, \cdots$, where

$$F_n = \frac{L}{2n\pi} \tanh \frac{2n\pi}{L} . \tag{1.7}$$

This result is close to that of Moiseev [5], who also assumes the bottom periodic and nearly flat, as well as the hypothesis $F \neq F_n$. On the other hand, Moiseev supposes that the bottom is symmetric about vertical lines through its crests and troughs, while we need no symmetry hypothesis at all.

We derive our result by mapping the domain of the fluid (*not* conformally) onto a strip Σ. In Σ, the transformed equations are solved in a natural way by a contraction mapping argument, if ε is small enough. Using this fact, it is also easy to show that the solution is analytic in ε and so can be found by expanding everything in a series of powers of ε. Note that, to do this, we use the fact that Σ — unlike the physical domain of the water—is *fixed*, known *a priori*, and independent of ε.

The periodic case occupies us from § 2 through § 5. Then, in § 6, we drop the hypothesis that B be periodic and replace it by the assumption

$$\lim_{X \to \infty} B(X) = d_0 . \tag{1.8}$$

Here, we show that existence, at least, follows from our earlier results if ε is small enough and $F \geqq 1$. This fact is related to one proved by Krasovskii [4]. Krasovskii showed the problem has a unique solution if the hypothesis (1.8) is strengthened to read: $B(X) = d_0$ for all X large enough. He did not require anything like our hypothesis that ε be small, but he did only prove his result for *large enough* F. As mentioned above, we require that ε be small, but we do prove the theorem for all Froude numbers down to the critical one of unity.

Using the method of §§ 2–6 it seems impossible to prove a uniqueness theorem in the aperiodic case (1.8). Also, in this case, the method is no longer constructive. Therefore, in §§ 7–8, we take up a new method, based on the hint that, in the periodic case, the solution is analytic in ε. We show directly that, under a strengthened version of the hypothesis (1.8), the problem has a solution analytic in ε for ε small enough. It is easy to see that the analytic solution is unique (within the class of analytic solutions), and, moreover, we show how this analyticity can be used to construct the solution to any desired degree of accuracy by simply expanding everything in series of powers of ε. In § 8, we also state a slightly different uniqueness theorem in the case of periodic H.

The main difficulty in the water waves problem is that the function T describing the free surface is not known in advance. We overcome this difficulty in all cases by mapping the fluid domain onto the strip Σ discussed earlier. In Σ, the problem takes on a form in which the invertibility of the mapping from the fluid domain to Σ has no relevance. (It can be seen easily from § 2 that the invertibility is entirely a question of whether the top of the fluid remains above the bottom.) Therefore, since all our arguments are in Σ, we ignore the question of invertibility here and state all our results in Σ. (Except in one place: see the remark following Theorem 5.2.) If it turns out that the top remains above the bottom, then our solutions have physical significance, and, in this case, the transformation back to the physical variables is entirely trivial. Sufficiant conditions for invertibility are not hard to find, and we leave these to the interested reader. On the other hand, the interesting question of finding truly precise conditions for invertibility is open.

A few further remarks are in order before we proceed. First, we note that, in all that follows, we assume neither d_0 nor H depends on ε. However, both of them can be allowed to do so. What is usually needed is sufficient smoothness of d_0 and H as functions of ε (analyticity in §§ 7–8) and boundedness (as a function of ε) of the norms of H that appear in the hypotheses. We leave the details to the interested reader.

Next, we note that we may always assume $\varepsilon > 0$, since ε appears in the problem only in the definition (1.5), while H can always be replaced by $-H$.

Finally, we point out explicitly that, without further mention, we always use the letter c to represent a positive constant. The appearance of a c in a formula means that the formula is correct for some positive constant c.

2. A preliminary transformation. It is convenient to consider the problem directly in terms of the velocity field $(U, V) = \nabla \Phi$, instead of the velocity potential Φ. The fact that (U, V) is a gradient and Laplace's equation (1.1) give the equations

$$U_Y - V_X = 0, \; U_X + V_Y = 0, \text{ for } -B(X) < Y < T(X) , \tag{2.1}$$

with the boundary conditions

$$V = UT_X, \; 2gT + U^2 + V^2 = \text{constant, when } Y = T(X) \tag{2.2}$$

and

$$V + UB_X = 0, \text{ when } Y = -B(X) . \tag{2.3}$$

We attack the problem by transforming it to dimensionless variables, at the same time mapping the domain[1] $-B(X) < Y < T(X)$ into a strip, as follows. Define x and y by

$$xd_0 = X , \qquad y = \frac{Y - T(X)}{B(X) + T(X)} . \tag{2.4}$$

If the bottom were flat ($\varepsilon = 0$ in (1.5)), one solution would be the uniform flow defined by $T = 0$, $U = U_0$, $V = 0$. We look for a perturbation of this solution, and, accordingly, we write

$$T(X) = \varepsilon d_0 \eta(x) , \tag{2.5}$$

$$U(X, Y) = U_0[1 + \varepsilon u(x, y)] , \tag{2.6}$$

$$V(X, Y) = U_0 \varepsilon v(x, y) . \tag{2.7}$$

The dependent variables η, u, and v, as well as the independent variables, x and y, are dimensionless.

Setting

$$H(X) = h(x) , \tag{2.8}$$

we find that equations (2.1) go over into the form

$$u_y - v_x = -\varepsilon\phi(u, v), \; u_x + v_y = \varepsilon\psi(u, v), \; -1 < y < 0 , \tag{2.9}$$

where, because of (1.5), as well as (2.4) – (2.8),

$$\phi(u, v) = [yh_x + (1 + y)\eta_x]v_y - (h + \eta)v_x , \tag{2.10}$$

and

[1] For now, we suppress the dependence of B and T, as well as U and V, on ε.

(2.11) $$\psi(u, v) = [yh_x + (1 + y)\eta_x]u_y - (h + \eta)u_x \,.$$

Here, h is given (since, through (2.8) and (1.5), it defines the bottom B), while because of (2.2b) and (2.5)-(2.7), the function η is defined by

(2.12) $$\eta + F\left[u + \frac{\varepsilon}{2}(u^2 + v^2)\right]_{y=0} = 0 \,,$$

where F is the Froude number (1.6), and the constant in (2.2b) has been chosen appropriately.

We also have the boundary conditions (2.2a) and (2.3):

(2.13) $$v = \eta_x(1 + \varepsilon u) \text{ when } y = 0$$

and

(2.14) $$v + h_x(1 + \varepsilon u) = 0 \text{ when } y = -1 \,.$$

η can be eliminated from the free surface boundary conditions. Differentiating (2.12) and substituting into (2.13), we find

(2.15) $$v + Fu_x = \varepsilon\sigma(u, v) \text{ when } y = 0 \,,$$

where

(2.16) $$\sigma(u, v) = \eta_x u - \frac{F}{2}\frac{\partial}{\partial x}(u^2 + v^2) \,.$$

The problem we want to solve is now reduced to (2.9), with the boundary conditions (2.14) and (2.15), and with ϕ, ψ, η, and σ defined by (2.10), (2.11), (2.12), and (2.16). F is a given constant. But there is another side condition to be satisfied by the solution. Since, by definition, U_0 is the mean value of U on the surface (see § 1), we also have, for any solution of the original problem with period Ld_0,

$$U_0 = \frac{1}{Ld_0}\int_{-Ld_0/2}^{Ld_0/2} U(X, T(X))dX = \frac{1}{L}\int_{-L/2}^{L/2} U_0[1 + \varepsilon u(x, 0)]dx \,,$$

which entails

(2.17) $$\int_{-L/2}^{L/2} u(x, 0)dx = 0 \,.$$

Thus, we also require that the solutions satisfy (2.17).

3. A related linear problem. In § 5, we show that the problem we have posed has a unique solution by proving a certain operator A is a contraction. To construct A, we begin by discussing a simple linear problem. It is to find when there is a periodic (in x) solution of the equations

(3.1) $$u_y - v_x = f \text{ and } u_x + v_y = g \text{ in the strip } -1 < y < 0\,,$$

with the boundary conditions

(3.2) $$v(x, 0) + Fu_x(x, 0) = s(x) \text{ and } v(x, -1) = b(x)\,.$$

Here, f, g, s, and b are given functions, periodic in x, with period L, say. In addition to (3.1) and (3.2), we impose the side condition

(3.3) $$\int_{-L/2}^{L/2} u(x, 0)dx = 0\,.$$

Let Σ be the strip $\{(x, y) \in R^2\colon\ -1 < y < 0\}$. We say that a function $f\colon \Sigma \to R^1$ is *periodic* if it is periodic in x alone: there exists an $L > 0$ such that $f(x + L, y) = f(x, y)$ for all $(x, y) \in \Sigma$. If f is periodic and has period L, for brevity, we say that f is periodic (L). We always denote by R_L the rectangle

$$R_L = \{(x, y)\colon\ -L/2 < x < L/2,\ -1 < y < 0\}\,,$$

and by I_L the interval $(-L/2, L/2)$.

Let $f\colon \Sigma \to R^1$ be continuous and periodic (L). Then, it has an associated Fourier series $\sum_n f_n(y)e^{i\Omega nx}$, where

(3.4) $$\Omega = 2\pi/L\,.$$

We write

$$||f||_0 = \sum_n \sup_{-1<y<0} |f_n(y)|$$

if the sum converges, and we denote by $\Lambda_L^0(\Sigma)$ the set of all continuous, periodic (L) functions for which $||\cdot||_0$ is finite. We define another norm by

$$||f||_k = \sum_n (|n\Omega|^k + 1) \sup_{-1<y<0} |f_n(y)|$$

and denote the set of continuous, periodic (L) functions for which this norm is finite by $\Lambda_L^k(\Sigma)$.

If $f\colon\ R^1 \to R^1$ is periodic (L), we say that $f \in \Lambda_L^k(R^1)$ if the extension $f^*\colon \Sigma \to R^1$ defined by $f^*(x, y) = f(x)$ is in $\Lambda_L^k(\Sigma)$, $k = 0, 1, \cdots$.

Now, recall the formula (1.7) defining F_n. We prove

LEMMA 3.1. *Let $f, g \in \Lambda_L^0(\Sigma)$, $s \in \Lambda_L^0(R^1)$, $b \in \Lambda_L^1(R^1)$. If $F \neq F_n$, $n = 1, 2, \cdots$, then a necessary and sufficient condition that the pro-*

blem (3.1)-(3.3) *have a periodic* (*L*) *solution is*

$$\int_{I_L} [s(x) - b(x)]dx = \iint_{R_L} g(x, y)dxdy . \tag{3.5}$$

Furthermore, if (3.5) *is satisfied, the solution is unique, both functions* u *and* v *lie in* $\Lambda^1_L(\Sigma)$, *the derivatives* u_x, v_x, u_y, *and* v_y *lie in* $\Lambda^0_L(\Sigma)$, *and*

$$\|u_y\|_0 + \|v_y\|_0 + \|u\|_1 + \|v\|_1 \leqq c(\|f\|_0 + \|g\|_0 + \|s\|_0 + \|b\|_1) . \tag{3.6}$$

Proof. The condition is necessary, as integrating (3.1b) gives

$$\begin{aligned}\iint_{R_L} g(x, y)dxdy &= \int_{-1}^{0}\int_{-L/2}^{L/2} u_x dxdy + \int_{-L/2}^{L/2}\int_{-1}^{0} v_y dydx \\ &= \int_{-L/2}^{L/2} [v(x, 0) - v(x, -1)]dx ,\end{aligned}$$

since u is periodic. Also, subtracting and integrating the boundary conditions (3.2) gives

$$\int_{-L/2}^{L/2} [v(x, 0) - v(x, -1)]dx = \int_{I_L} [s(x) - b(x)]dx ,$$

again by the periodicity of u.

To prove the sufficiency, expand all functions into Fourier series. Thus, we define Ω by (3.4) and write $u(x, y) \sim \sum_{-\infty}^{\infty} u_n(y)e^{i\Omega nx}$, with a similar notation for the functions v, f, g, s, and b. Then, equations (3.1) become

$$\begin{aligned} u_n'(y) &= in\Omega v_n(y) + f_n(y) , \\ v_n'(y) &= -in\Omega u_n(y) + g_n(y) ,\end{aligned} \tag{3.7}$$

with boundary conditions

$$\begin{aligned} v_n(0) + in\Omega F u_n(0) &= s_n , \\ v_n(-1) &= b_n ,\end{aligned} \tag{3.8}$$

for $n = 0, \pm 1, \pm 2, \cdots$. In addition, because of (3.3), we require

$$u_0(0) = 0 . \tag{3.9}$$

The ordinary differential equations (3.7) are readily solved to give

$$\begin{aligned} u_n(y) = \alpha_n \operatorname{ch} n\Omega y + i\beta_n \operatorname{sh} n\Omega y - \int_y^0 [f_n(\eta) \operatorname{ch} n\Omega(y - \eta) \\ + i g_n(\eta) \operatorname{sh} n\Omega(y - \eta)]d\eta , \\ v_n(y) = -i\alpha_n \operatorname{sh} n\Omega y + \beta_n \operatorname{ch} n\Omega y - \int_y^0 [g_n(\eta) \operatorname{ch} n\Omega(y - \eta) \\ - i f_n(\eta) \operatorname{sh} n\Omega(y - \eta)]d\eta ,\end{aligned} \tag{3.10}$$

where α_n and β_n are constants. The boundary conditions (3.8) then give

$$\beta_n + in\Omega F\alpha_n = s_n \tag{3.11}$$

$$\begin{aligned}\beta_n \operatorname{ch} n\Omega + i\alpha_n \operatorname{sh} n\Omega = b_n + \int_{-1}^{0}[g_n(\eta) \operatorname{ch} n\Omega(1+\eta) \\ + if_n(\eta) \operatorname{sh} n\Omega(1+\eta)]d\eta\ .\end{aligned} \tag{3.12}$$

When $n \neq 0$, the determinant of the coefficients of β_n and α_n is not zero, since, by hypothesis, $F \neq F_n$. (Recall (3.4) and (1.7).) When $n = 0$, consistency of (3.11) and (3.12) requires

$$s_0 = b_0 + \int_{-1}^{0} g_0(\eta)d\eta\ ,$$

which is equivalent to (3.5). β_0 is now given by (3.11) or (3.12) with $n = 0$, while (3.9) and (3.10) give $\alpha_0 = 0$.

This argument shows the system (3.1)-(3.3) has a formal, unique solution. What remains is to prove (3.6). For this, solve (3.11)-(3.12) for α_n and β_n and substitute into (3.10). This gives an explicit formula for $u_n(y)$ which, for large $n > 0$, has for its dominant terms (recall $y \leqq 0$)

$$-\frac{is_n e^{n\Omega y}}{n\Omega F} - ib_n e^{-n\Omega(1+y)} - \frac{i}{2}\int_{-1}^{y}[g_n(\eta) + if_n(\eta)]e^{n\Omega(\eta - y)}d\eta\ ,$$

and this is bounded by

$$\left|\frac{s_n}{n\Omega F}\right| + |b_n| + \frac{1}{2|n\Omega|}[\max_{-1\leqq\eta\leqq 0}|g_n(\eta)| + \max_{-1\leqq\eta\leqq 0}|f_n(\eta)|]\ .$$

One can also show the dominant terms in $u_n(y)$ to be bounded by this same quantity when n is large and negative. Therefore,

$$\begin{aligned}|u_n(y)| \leqq c\Big[\frac{|s_n|}{|n\Omega|+1} + |b_n| + \frac{1}{|n\Omega|+1} \\ \times \{\max_\eta |g_n(\eta)| + \max_\eta |f_n(\eta)|\}\Big]\end{aligned} \tag{3.13}$$

for all n. There is a similar inequality for $|v_n(y)|$, proved in the same way. Multiplying (3.13) by $|n\Omega| + 1$, taking the maximum on y, and summing, we conclude that $||u||_1$ is bounded by a multiple of

$$||s||_0 + ||b||_1 + ||g||_0 + ||f||_0\ . \tag{3.14}$$

$||v||_1$ is similarly bounded. Thus, u and v lie in $\Lambda_L^1(\Sigma)$ and the sum of their norms is bounded by a multiple of (3.14). That u_x and v_x

lie in $\Lambda^0_L(\Sigma)$ follows immediately from the definition of the norms, for $||u_x||_0 + ||v_x||_0 \leqq ||u||_1 + ||v||_1$. Now, the fact that u_y and v_y lie in $\Lambda^0_L(\Sigma)$ and have their norms bounded by a multiple of (3.14) can be read off directly from equations (3.1). (3.6) and the lemma follow.

4. **The operator A.** Writing D for either Σ or R^1, we define ${}_0\Lambda^k_L(D)$ as the set of all functions in $\Lambda^k_L(D)$ satisfying (3.3). Let $u \in {}_0\Lambda^1_L(\Sigma)$, $v \in \Lambda^1_L(\Sigma)$, u_y, $v_y \in \Lambda^0_L(\Sigma)$. Define ϕ, ψ, and σ by (2.10), (2.11), (2.12), and (2.16). We show that, if $F \neq F_n$, $n = 1, 2, \cdots$, then there is a unique solution to the problem of finding functions $\boldsymbol{u} \in {}_0\Lambda^1_L(\Sigma)$, $\boldsymbol{v} \in \Lambda^1_L(\Sigma)$ with $\boldsymbol{u}_y$, $\boldsymbol{v}_y \in \Lambda^0_L(\Sigma)$, such that

$$\boldsymbol{u}_y - \boldsymbol{v}_x = -\varepsilon\phi(u, v) , \tag{4.1}$$

$$\boldsymbol{u}_x + \boldsymbol{v}_y = \varepsilon\psi(u, v) , \tag{4.2}$$

and satisfying

$$\boldsymbol{v} + F\boldsymbol{u}_x = \varepsilon\sigma(u, v) \text{ when } y = 0 \tag{4.3}$$

and

$$\boldsymbol{v} = -h_x(1 + \varepsilon u) \text{ when } y = -1 . \tag{4.4}$$

The system (4.1)-(4.4) has the same form as (3.1)-(3.3), and, in this case, we see, using (2.11) and (2.16),

$$\begin{aligned} s - b - \int_{-1}^{0} g dy &= h_x(1 + \varepsilon u)|_{y=-1} + \varepsilon\sigma(u, v)|_{y=0} - \varepsilon\int_{-1}^{0}\psi(u, v)dy \\ &= h_x(1 + \varepsilon u)|_{y=-1} + \varepsilon\Big[\eta_x u - \frac{F}{2}\frac{\partial}{\partial x}(u^2 + v^2)\Big]_{y=0} \\ &\quad - \varepsilon\int_{-1}^{0}\{[yh_x + (1 + y)\eta_x]u_y - (h + \eta)u_x\}dy . \end{aligned}$$

However,

$$\begin{aligned} &[yh_x + (1 + y)\eta_x]u_y - (h + \eta)u_x \\ &\qquad = h_x(yu)_y + \eta_x((1 + y)u)_y - \frac{\partial}{\partial x}[(h + \eta)u] . \end{aligned}$$

Thus, in the context of (4.1)-(4.4), we have, since h and η are independent of y,

$$s - b - \int_{-1}^{0} g dy = \frac{\partial}{\partial x}\Big[h - \frac{\varepsilon F}{2}(u^2 + v^2)\Big|_{y=0} + \varepsilon(h + \eta)\int_{-1}^{0} u dy\Big] .$$

Integrating with respect to x, we get zero, since the function in brackets is periodic. Thus the consistency condition (3.5) needed to be able to solve (4.1)-(4.4) is satisfied *automatically* when ψ and σ have the form (2.11) and (2.16)!

It is convenient at this point to introduce the space Λ_L consisting of pairs of functions $(u, v) \in {}_0\Lambda_L^1(\Sigma) \times \Lambda_L^1(\Sigma)$ with $(u_y, v_y) \in \Lambda_L^0(\Sigma) \times \Lambda_L^0(\Sigma)$. For $(u, v) \in \Lambda_L$, we define $||(u, v)|| = ||u||_1 + ||v||_1 + ||u_y||_0 + ||v_y||_0$. Because of Lemma 3.1 and what we have just proved, we have

LEMMA 4.1. *Let $h \in {}_0\Lambda_L^2(R^1)$. If $F \neq F_n$, $n = 1, 2, \cdots$, then, there exists a mapping A: $\Lambda_L \to \Lambda_L$ defined by $A(u, v) = (\boldsymbol{u}, \boldsymbol{v})$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the solutions of* (4.1)-(4.4). *This mapping satisfies*

$$(4.5) \qquad ||A(u, v)|| \leqq c_0[1 + \varepsilon(||(u, v)|| + ||(u, v)||^2 + ||(u, v)||^3)] ,$$

where c_0 is a constant, depending only on h and F.

Indeed, Lemma 3.1 actually gives

$$||A(u, v)|| \leqq c[||h||_2 + \varepsilon(||\phi||_0 + ||\psi||_0 + ||\sigma||_0 + ||h_x \underline{u}||_1)] ,$$

where we have written $\underline{u}(x) = u(x, -1)$. However, the two facts that, in Λ_L^1, the norm of a product is bounded by the product of the norms and $||\cdot||_0 \leqq ||\cdot||_1$ give (4.5) when taken together with the formulas (2.10), (2.11), (2.12), and (2.16).

5. The main theorem for periodic h. It is clear that any fixed point of A is a solution of our problem. To show that A has a fixed point, we begin with

LEMMA 5.1. *Let $h \in {}_0\Lambda_L^2(R^1)$. Let $F \neq F_n$, $n = 1, 2, \cdots$. Then, there exists an ε_0 and a corresponding ball $\mathscr{B}_{R_0} = \{(u, v) \in \Lambda_L: ||(u, v)|| \leqq R_0\}$ such that A maps $\mathscr{B}_{R_0}$ into itself for each $\varepsilon \leqq \varepsilon_0$.*

Proof. Let $||(u, v)|| \leqq R$, where $R > c_0$ and c_0 is the constant occurring in (4.5). Then (4.5) shows that $||A(u, v)|| \leqq R$ also, if only $\varepsilon c_0(R + R^2 + R^3) \leqq R - c_0$. Set $\varepsilon(R) = (R - c_0)c_0(R + R^2 + R^3)$. $\varepsilon(R)$ is a nonnegative, continuous function of R for $c_0 < R < \infty$ which tends to zero as R tends to either c_0 or infinity. Hence $\varepsilon(R)$ assumes its maximum at, say, R_0. We state the above lemma with $\varepsilon_0 = \varepsilon(R_0)$ since it is the maximal ε for which we can assert that A maps a ball into itself.

It is easy to show that A is a contraction on the ball $\mathscr{B}_{R_0}$. For this, let $||(u^i, v^i)|| \leqq R_0$, $i = 1, 2, \cdots$. Also, let $(\boldsymbol{u}^i, \boldsymbol{v}^i) = A(u^i, v^i)$, and let $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{v}^1 - \boldsymbol{v}^2)$. Then by the definition of A (cf. (4.1)-(4.4)) we have

$$\boldsymbol{u}_y - \boldsymbol{v}_x = \varepsilon\phi(u^2, v^2) - \varepsilon\phi(u^1, v^1) ,$$
$$\boldsymbol{u}_x + \boldsymbol{v}_y = \varepsilon\psi(u^1, v^1) - \varepsilon\psi(u^2, v^2) ,$$

while

$$\boldsymbol{v} + F\boldsymbol{u}_x = \varepsilon\sigma(u^1, v^1) - \varepsilon\sigma(u^2, v^2) \text{ when } y = 0 ,$$

and

$$\boldsymbol{v} = \varepsilon h_x(u^2 - u^1) \text{ when } y = -1 .$$

We need not verify the condition (3.5), as these equations have a solution by Lemma 4.1. Moreover, Lemma 3.1 gives

$$\begin{aligned}||(\boldsymbol{u}, \boldsymbol{v})|| &= ||A(u^1, v^1) - A(u^2, v^2)|| \\ &\leqq c\varepsilon[||\phi(u^2, v^2) - \phi(u^1, v^1)||_0 + ||\psi(u^2, v^2) - \psi(u^1, v^1)||_0 \\ &\quad + ||\sigma(u^2, v^2) - \sigma(u^1, v^1)||_0 + ||u^2 - u^1||_1] ,\end{aligned}$$

where the formula (4.4) for b has been used. As in the proof of (4.5), the polynomial character of ϕ, ψ, η, and σ shows that, if $||(u, v)|| \leqq R_0$, then the quantity in brackets is bounded by $c(R_0)||(u^1, v^1) - (u^2, v^2)||$, where, as the notation indicates, $c(R_0)$ is a constant, depending on R_0. Thus, we obtain

$$||A(u^1, v^1) - A(u^2, v^2)|| \leqq c\varepsilon||(u^1, v^1) - (u^2, v^2)|| . \tag{5.1}$$

Choosing ε_0 smaller, if necessary, we see that A is a contraction on $\mathscr{B}_{R_0}$ for each $\varepsilon < \varepsilon_0$. Consequently, A has a fixed point in any such ball. Since a fixed point of A is a solution of our problem, we have our first main result.

THEOREM 5.2. *Let* $h \in {}_0\Lambda_L^2(R^1)$, *and let* $F \neq F_n$, $n = 1, 2, \cdots$. *Then, there exists an* ε_0 *and a corresponding ball* $\mathscr{B}_{R_0}$ *of radius* R_0 *in* Λ_L, *centered at* (0, 0), *such that the problem* (2.9)-(2.17) *has a unique solution in* $\mathscr{B}_{R_0}$ *for each fixed* $\varepsilon < \varepsilon_0$.

In view of the formulas (2.6) and (2.7), relating (u, v) to (U, V), it might appear that the reference to $\mathscr{B}_{R_0}$ is unnecessary, since (u, v) is multiplied by ε (and so makes a small contribution if ε is small enough) in going from (u, v) to (U, V). The necessity for the statement in terms of $\mathscr{B}_{R_0}$ appears to be related to the question of the invertibility of the mapping (2.4) as discussed in the introduction. We leave the details of the transformation from (u, v) to (U, V) to the interested reader.

We now turn to some simple but interesting corollaries of the main Theorem 5.2. First, since $\{F_n\}$ is decreasing, and since

$$F_1 = \frac{L}{2\pi} \tanh \frac{2\pi}{L} < 1 ,$$

we have

COROLLARY 5.3. *Let* $h \in {}_0A_L^2(R^1)$. *Then, the problem* (2.9)-(2.17) *has a periodic* (L) *solution if* ε *is small enough and* $F > F_1$. *In particular, since* $F_1 < 1$, *the problem has a solution if* $F \geqq 1$.

Of course, as is the case whenever an object is a limit of iterates of a contraction mapping, beginning with zero, we have

$$(u, v) = \sum_{k=0}^{\infty} [A^{k+1}(0, 0) - A^k(0, 0)] .$$

Now, it is a triviality that each term in this series is a polynomial in ε (with coefficients in A_L), and it follows easily from (5.1) that

$$||A^{k+1}(0, 0) - A^k(0, 0)|| \leqq ||A(0, 0)|| c^k \varepsilon^k .$$

Therefore, the solution (u, v) is analytic in ε in a neighborhood of $\varepsilon = 0$, and also

$$\begin{aligned} ||(u, v) - A^n(0, 0)|| &\leqq \sum_{k=n}^{\infty} ||A^{k+1}(0, 0) - A^k(0, 0)|| \\ &\leqq ||A(0, 0)|| \frac{(c\varepsilon)^n}{1 - c\varepsilon} , \end{aligned}$$

which gives an estimate for the error in computing the solution. We restate these facts as

COROLLARY 5.4. *The solution of Theorem* 5.2 *is an analytic function of* ε *in a neighborhood of* $\varepsilon = 0$. *Moreover, if* (u, v) *denotes this solution, then* $||(u, v) - A^n(0, 0)|| = 0(\varepsilon^n)$ *as* $\varepsilon \to 0$.

Two remarks should be made here. First, the analyticity means that one can compute the solution by expanding u and v in a series of powers of ε, substituting into (2.9)-(2.17), equating like powers of ε on each side, and solving the resulting equations. This method for computing the solution is unquestionably more efficient than simply calculating $\{A^n(0, 0)\}$, since $A^n(0, 0)$ contains terms of order higher than ε^n, terms that change when one proceeds further to calculate $A^{n+1}(0, 0)$.

The second remark is this. Corollary 5.4 shows that the error in u and v after n iterates is $O(\varepsilon^n)$. What one really wants, however, is not u, v, and η, but the actual flow quantities U, V, and T, that satisfy (2.1)-(2.3). However, as (2.5)-(2.7) show, in going from u, v, and η to U, V, and T, one always multiplies by ε. Therefore, although the error in calculating u, v, and η is $O(\varepsilon^n)$ after n iterations, the error in calculating U, V, and T is $O(\varepsilon^{n+1})$.

All the results so far are also valid in spaces other than Λ_L^k. The spaces Λ_L^k are very convenient, as precise estimates on the various constants that appear can be found, without reference to outside results. On the other hand, for some purposes, it is useful to have our results in other spaces.

For this, note that the entire basis for the argument is Lemma 3.1 and the inequality (3.6). On the other hand, the equations (3.1)-(3.3) are elliptic and one can use the Schauder-type estimates of Agmon, Douglis, and Nirenberg [1] to derive an estimate exactly like (3.6) in the spaces $C^{k+\alpha}$, where by $C^{k+\alpha}$ we denote the space obtained by completing the trigonometric polynomials in the usual $C^{k+\alpha}$ norm. One has merely to replace Λ^k by $C^{k+\alpha}$ and interpret $\|\cdot\|_k$ as the norm in $C^{k+\alpha}$. Thus, writing $C_L^{k+\alpha}$ for the periodic (L) functions in $C^{k+\alpha}$ and ${}_0C_L^{k+\alpha}$ for those functions in $C_L^{k+\alpha}$ with mean value zero on $y = 0$, it can be seen that the complete argument then goes through exactly as before to give, instead of Theorem 5.2,

THEOREM 5.2 bis. *Let $h \in {}_0C_L^{2+\alpha}(R^1)$, and let $F \neq F_n$, $n=1, 2, \cdots$. Let $\mathscr{B}_{R_0}^{1+\alpha}$ be the ball of radius R_0 in ${}_0C_L^{1+\alpha}(\Sigma) \times C_L^{1+\alpha}(\Sigma)$, centered at* (0, 0). *Then the problem* (2.9)-(2.17) *has a unique solution in $\mathscr{B}_0^{1+\alpha}$ for each fixed $\varepsilon < \varepsilon_0$.*

One remark should convince the reader that the proof of Theorem 5.2 bis is the same as that of Theorem 5.2. First, note that Theorem 9.3 of [1] gives, instead of (3.6),

$$\|u\|_{1+\alpha}^{\Sigma} + \|v\|_{1+\alpha}^{\Sigma} \leqq c(\|f\|_{\alpha}^{\Sigma} + \|g\|_{\alpha}^{\Sigma} + \|s\|_{\alpha}^{R^1} + \|b\|_{1+\alpha}^{R^1} + \|u\|_0^{\Sigma} + \|v\|_0^{\Sigma}) ,$$

where $\|\cdot\|_0$ denotes the supremum. To obtain the exact analog of (3.6), one has only to notice that (3.10) and the similar formula for $v_n(y)$ imply, via a tedious calculation, that

$$\|u\|_0^{\Sigma} + \|v\|_0^{\Sigma} \leqq c(\|f\|_{\alpha}^{\Sigma} + \|g\|_{\alpha}^{\Sigma} + \|s\|_{\alpha}^{R^1} + \|b\|_{1+\alpha}^{R^1}) .$$

In a similar way, all the results of this section have analogs with the spaces Λ_L^k replaced by $C_L^{k+\alpha}$. Given any result in this section, we refer to its analog in the spaces $C_L^{k+\alpha}$ by adding a "bis" to its title, as we did in the case of Theorem 5.2 bis above.

We complete this section with the comment that the solution (u, v) is as smooth as the datum h allows. For the proof of Lemma

3.1 shows that the solution of the related linear problem is as smooth as its data allows (cf. (3.13)). It follows that if $h \in {}_0P_L^{2+\alpha}(R^1)$, where P denotes either Λ or C, then $(u, v) \in {}_0P_L^{1+\alpha}(\Sigma) \times P_L^{1+\alpha}(\Sigma)$, $k = 0, 1, \cdots$. In particular, we have

COROLLARY 5.5. *If* $h \in {}_0C_L^{\infty}(R^1)$, $F \neq F_n$, $n = 1, 2, \cdots$, *and* $\varepsilon < \varepsilon_0$, *then the solution* (u, v) *of problem* (2.9)-(2.17) *lies in* ${}_0C_L^{\infty}(\Sigma) \times C_L^{\infty}(\Sigma)$.

Although we will make no later mention of it, we remark here that the solutions found in §§ 6 and 8 below are similarly as smooth as the datum h allows.

6. A theorem for nonperiodic h. In this section, we prove the existence of a steady flow over an arbitrary $C^{2+\alpha}$ bottom, assuming

$$\lim_{|x| \to \infty} h(x) = 0 \tag{6.1}$$

and that $F \geqq 1$. This last condition is needed to provide the assurance that $F \neq F_n$ for every $n > 0$ and every $L > 0$. (Cf. the proof of Corollary 5.3 bis.) Since we construct a solution in the nonperiodic case as a limit of periodic solutions whose period goes to infinity, the hypothesis $F \geqq 1$ is needed to use Theorem 5.2 bis.

To begin, we assume, along with Krasovskii [4], that, for some $L > 0$,

$$h(x) = 0 \text{ for } |x| > L/2 . \tag{6.2}$$

For every $n = 2, 3, \cdots$, we define a periodic (nL) function h_n as follows:

$$h_n(x) = h(x) - \frac{1}{nL}\int_{-L/2}^{L/2} h(x)dx \quad \text{for } |x| \leqq nL/2 ,$$

while, outside the interval $|x| \leqq nL/2$, we define h_n to be periodic (nL). With this definition, we have that $h_n \in {}_0C_{nL}^{2+\alpha}(R^1)$ if $h \in C^{2+\alpha}(R^1)$. For each n, then, Theorem 5.2 bis provides a periodic (nL) solution (u_n, v_n) associated with the bottom defined by $h_n(x)$, if ε is small enough. Moreover, the proof of Lemma 5.1 bis shows that we may choose $||u_n||_{C^{1+\alpha}} + ||v_n||_{C^{1+\alpha}} \leqq R_0$, where R_0 is independent of n, since $||h_n||_{C^{2+\alpha}}$ is bounded independent of n. Also, the bound on ε is independent of n, because of (5.1), which shows that we only need $c(R_0)\varepsilon < 1$. It follows from the Ascoli-Arzela theorem, then, that, given any compact subset K of the closure, $\bar{\Sigma}$, of Σ, there is a subsequence of $\{(u_n, v_n)\}$ converging in $C^1(K)$.

Let $\{K_m\}$ be any increasing sequence of compact subsets of $\bar{\Sigma}$ whose union is all of $\bar{\Sigma}$. Then, for every $m \geqq 1$, we can find a sequence $\{(u_n^{(m)}, v_n^{(m)})\}$ such that $\{(u_n^{(m)}, v_n^{(m)})\} \subset \{(u_n^{(m-1)}, v_n^{(m-1)})\}$ (if $m \geqq 2$), convering[2] in K_m. The usual diagonal argument then shows that the diagonal sequence $\{(u_n^{(n)}, v_n^{(n)})\}$ converges (to (u, v), say) in $C^1(\Sigma)$. It is an easy matter to prove that (u, v) is a solution of our problem, and even that $(u, v) \in C^{1+\alpha}(\Sigma) \times C^{1+\alpha}(\Sigma)$, since $\|u_n^{(n)}\|_{1+\alpha}^{\Sigma} + \|v_n^{(n)}\|_{1+\alpha}^{\Sigma} \leqq R$. Moreover, since $u_n^{(n)} \in {}_0C_{nL}^{1+\alpha}(\Sigma)$ (cf. Theorem 5.2 bis), we have

$$\int_{-nL/2}^{nL/2} u_n^{(n)}(x, 0)dx = 0 ,$$

and $u_n^{(n)}$ is periodic (nL), so that

$$\lim_{\zeta\to\infty} \frac{1}{2\zeta}\int_{-\zeta}^{\zeta} u_n^{(n)}(x, 0)dx = 0 .$$

Also, it is trivial that the linear functional defined by taking the mean value along $y = 0$ is continuous on $C^{1+\alpha}(\Sigma)$. Therefore, the solution satisfies

$$\lim_{\zeta\to\infty} \frac{1}{2\zeta}\int_{-\zeta}^{\zeta} u(x, 0)dx = 0 . \tag{6.3}$$

Now, let ${}_0C^{k+\alpha}(\Sigma)$ denote the set of all functions in $C^{k+\alpha}(\Sigma)$ satisfying (6.3). Notice that, since U_0 is defined as a mean value, we must require that any solution satisfy (6.3). Using [1] and what we have already proved, it is now easy to complete the proof of the following version of Krasovskii's theorem [4].

LEMMA 6.1. *Let $h \in C^{2+\alpha}(R^1)$ satisfy* (6.2). *Choose $F \geqq 1$. Then, if ε is small enough, the problem* (2.9)-(2.16) *has a solution $(u, v) \in {}_0C^{1+\alpha}(\Sigma) \times C^{1+\alpha}(\Sigma)$.*

Now, suppose we replace (6.2) by the weaker condition (6.1). Then, for every $n = 1, 2, \cdots$, we can approximate h by a function h_n, equal to h for $|x| \leqq n$, equal to zero for $|x| > n + 1$, and lying uniformly in $C^{2+\alpha}(R^1)$. Lemma 6.1 gives a solution (u_n, v_n), corresponding to h_n, for every $n = 1, 2, \cdots$. The construction in Lemma 6.1 shows that $\{\|u_n\|_{1+\alpha}^{\Sigma} + \|v_n\|_{1+\alpha}^{\Sigma}\}$ is bounded, and each (u_n, v_n) exists for the same range of values of ε. Therefore, another use of the diagonal process gives

[2] Notice that we are again using the fact that Σ is a fixed domain, this time in spite of the fact that the functions h_n, which describe the bottoms in the physical domain, are varying with n.

THEOREM 6.2. *Let* $h \in C^{2+\alpha}(R^1)$ *satisfy* (6.1). *Choose* $F \geqq 1$. *Then, if* ε *is small enough, the problem* (2.9)-(2.16) *has a solution* $(u, v) \in {}_0C^{1+\alpha}(\Sigma) \times C^{1+\alpha}(\Sigma)$.

That (6.3) is true of the solution follows, as before, from the continuity of the functional defined by the left side of (6.3).

Notice that, for the aperiodic flows of Theorem 6.2, both the uniqueness of the solution and our ability to construct it have been lost in the process of taking subsequences. Also lost is any equivalent of Corollary 5.4 (analyticity in ε). On the other hand, we are able to recover all these properties in the aperiodic case by using another method of proof, to which we now turn. As we shall see, the main price to be paid for these gains is that we must restrict F to be strictly greater than unity; thus the single case $F = 1$ is lost in the following argument.

7. Recursion relations for an analytic solution. Our aim now is to solve the problem (2.9)-(2.16) in the aperiodic case when h satisfies a condition like (6.1), by looking directly for a solution that is analytic in ε. Therefore, we write, formally,

$$u(x, y, \varepsilon) = \sum_{n=0}^{\infty} u^n(x, y)\varepsilon^n, \quad v(x, y, \varepsilon) = \sum_{n=0}^{\infty} v^n(x, y)\varepsilon^n \,. \tag{7.1}$$

Substitution of these expressions into (2.9)-(2.16) gives the following recursion relations: for $n = 0, 1, 2, \cdots$

$$u_y^n - v_x^n = f^{n-1}, \quad u_x^n + v_y^n = g^{n-1} \text{ in the strip } -1 < y < 0 \tag{7.2}$$

with the boundary conditions

$$v^n + Fu_x^n = s^{n-1} \text{ when } y = 0, \text{ and } v^n = b^{n-1} \text{ when } y = -1 \,. \tag{7.3}$$

Here,

$$f^{-1} = g^{-1} = s^{-1} = 0, \; b^{-1} = -h_x \,, \tag{7.4}$$

and, for $n \geqq 1$,

$$f^{n-1} = -yh_x v_y^{n-1} + hv_x^{n-1} + \sum_{\nu=0}^{n-1} [\eta^\nu v_x^{n-1-\nu} - (1 + y)\eta_x^\nu v_y^{n-1-\nu}] \,, \tag{7.5}$$

$$g^{n-1} = yh_x u_y^{n-1} - hu_x^{n-1} - \sum_{\nu=0}^{n-1} [\eta^\nu u_x^{n-1-\nu} - (1 + y)\eta_x^\nu u_y^{n-1-\nu}] \,, \tag{7.6}$$

$$s^{n-1} = \sum_{\nu=0}^{n-1} \eta_x^\nu \tilde{u}^{n-1-\nu} - F \sum_{\nu=0}^{n-2} (\tilde{u}^\nu \tilde{u}_x^{n-2-\nu} + v^\nu v_x^{n-2-\nu}) \,, \tag{7.7}$$

$$b^{n-1} = -h_x \underset{\sim}{u}^{n-1} \,, \tag{7.8}$$

$$\eta^{n-1} = -F\tilde{u}^{n-1} - \frac{F}{2}\sum_{\nu=0}^{n-2}(\tilde{u}^{\nu}\tilde{u}^{n-2-\nu} + \tilde{v}^{\nu}\tilde{v}^{n-2-\nu}) . \tag{7.9}$$

In (7.5) — (7.9), we have used the following conventions: a tilde over a function indicates it is to be evaluated at $y = 0$; a tilde under a function indicates it is to be evaluated at $y = -1$; any sum whose upper limit is less than its lower is to be taken as zero.

If we assume $\{u_k\}_{k=0}^{n-1}$ and $\{v_k\}_{k=0}^{n-1}$ known, then f^{n-1}, g^{n-1}, s^{n-1}, and b^{n-1} can be calculated using (7.5)-(7.9). u^n and v^n must then be found by solving the boundary value problem (7.2)-(7.3). Notice that this problem has the same form as the problem (3.1)-(3.2) studied in § 3. Thus, because of Lemma 3.1, we expect a condition like

$$\int_{-\infty}^{\infty}[s^{n-1}(x) - b^{n-1}(x)]dx = \int_{-1}^{0}\int_{-\infty}^{\infty} g^{n-1}(x, y)dxdy \tag{7.10}$$

to be necessary in order to solve (7.2)-(7.3). Now, the straightforward generalization of the spaces $\Lambda_L^k(\Sigma)$ that we used in §§ 2-5 would be to a space with the norm

$$\int_{-\infty}^{\infty}(1 + |\xi|^k) \sup_{-1<y<0} |\varphi^{\wedge}(\xi, y)| d\xi ,$$

$\varphi^{\wedge}$ denoting the Fourier transform of φ. However, none of the functionals appearing in (7.10) is continuous in the topology defined by this norm. For this reason, we set, instead

$$||\varphi||_k = \sup_{\Sigma} (1 + |\xi|^k)|\varphi^{\wedge}(\xi, y)| + \int_{-\infty}^{\infty}(1 + |\xi|^k) \sup_{1<y<0} |\varphi^{\wedge}(\xi, y)| d\xi$$

when $\varphi \in C_0^\infty(\bar{\Sigma})$, and we denote by $\Lambda^k(\Sigma)$ the completion of $C_0^\infty(\bar{\Sigma})$ in this norm. As before, $\varphi^{\wedge}$ is the Fourier transform of φ:

$$\varphi^{\wedge}(\xi, y) = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x, y)dx .$$

$\Lambda^k(R^1)$ is defined as we defined $\Lambda_L^k(R^1)$: we say that $f: R^1 \to R^1$ is in $\Lambda^k(R^1)$ if the extension f^*, defined by $f^*(x, y) = f(x)$, lies in $\Lambda^k(\Sigma)$.

As in § 3, we begin our work by studying the system

$$u_y - v_x = f,\ u_x + v_y = g \text{ for } (x, y) \in \Sigma \tag{7.11}$$

with the boundary conditions

$$v + Fu_x = s \text{ when } y = 0,\ v = b \text{ when } y = -1 \tag{7.12}$$

and the additional side condition (cf. (6.3))

$$\lim_{\zeta\to\infty}\frac{1}{2\zeta}\int_{-\zeta}^{\zeta}u(x,0)dx = 0\ . \tag{7.13}$$

Notice, however, that, in the spaces $\Lambda^k(\Sigma)$, (7.13) is automatic, since, if $u \in \Lambda^k(\Sigma)$ with $k \geqq 0$, then $u(x, 0)$ is the Fourier transform of the L^1-function $u\hat{}(\xi, 0)$ and so $u(x, 0)$ is continuous and goes to zero at infinity. (7.13) follows from this. Thus, as long as we operate in the spaces Λ^k, we may ignore (7.13).

We write $\Lambda = \{(u, v) \in \Lambda^1(\Sigma) \times \Lambda^1(\Sigma)\colon\ u_y, v_y \in \Lambda^0(\Sigma)\}$. As in § 4, we use the norm $\|(u, v)\| = \|u\|_1 + \|v\|_1 + \|u_y\|_0 + \|v_y\|_0$ on Λ. Then, we have

LEMMA 7.1. *Let $f, g \in \Lambda^0(\Sigma)$, $s \in \Lambda^0(R^1)$, $b \in \Lambda^1(R^1)$. If $F > 1$, then a necessary and sufficient condition that the problem* (7.11)-(7.13) *have a solution $(u, v) \in \Lambda$ is that the function*

$$k(\xi) \equiv \frac{1}{\xi}\left[s\hat{}(\xi) - b\hat{}(\xi) - \int_{-1}^{0} g\hat{}(\xi, y)dy\right] \text{ be continuous at } \xi = 0. \tag{7.14}$$

In this case, the solution is unique, and if N is a bound for $|k(\xi)|$ in a neighborhood of zero, then the solution satisfies

$$\|(u, v)\| \leqq c(\|s\|_0 + \|b\|_1 + \|f\|_0 + \|g\|_0 + N)\ . \tag{7.15}$$

Proof. The proof is similar to that of Lemma 3.1. Therefore, we merely sketch it. We begin by taking the Fourier transform of (7.11). The result is two ordinary differential equations, the general solution of which is (cf. (3.10))

$$\begin{aligned} u\hat{}(\xi, y) &= \alpha(\xi)\operatorname{ch}\xi y + i\beta(\xi)\operatorname{sh}\xi y - \int_y^0 [f\hat{}(\xi, y')\operatorname{ch}\xi(y - y') \\ &\qquad + ig\hat{}(\xi, y')\operatorname{sh}\xi(y - y')]dy'\ , \\ v\hat{}(\xi, y) &= -i\alpha(\xi)\operatorname{sh}\xi y + \beta(\zeta)\operatorname{ch}\xi y - \int_y^0 [g\hat{}(\xi, y')\operatorname{ch}\xi(y - y') \\ &\qquad - \mathrm{i}f\hat{}(\xi, y')\operatorname{sh}\xi(y - y')]dy'\ . \end{aligned}$$

Imposing the boundary conditions (7.12), we find that α and β are determined by the equations

$$\begin{aligned} i\xi F\alpha(\xi) + \beta(\xi) &= s\hat{}(\xi)\ , \\ i\alpha(\xi)\operatorname{sh}\xi + \beta(\xi)\operatorname{ch}\xi &= b\hat{}(\xi) + \int_{-1}^{0}[g\hat{}(\xi, y)\operatorname{ch}\xi(1 + y) \\ &\qquad + \mathrm{i}f\hat{}(\xi, y)\operatorname{sh}\xi(1 + y)]dy\ . \end{aligned} \tag{7.16}$$

The determinant of this pair of equations is $\Delta = i\xi F\operatorname{ch}\xi - i\operatorname{sh}\xi$. Since $F > 1$, Δ vanishes only for $\xi = 0$. Thus, α and β are continuous for $\xi \neq 0$. Solving (7.16), we find that β is also continuous at $\xi = 0$, provided only that (7.16) is consistent there, and this is easily seen to be a consequence of even a weaker form of (7.14), namely, (7.10) without the superscripts. Next, α is continuous at $\xi = 0$ only if (7.14) is satisfied, while, if $u \in \Lambda^1(\Sigma)$, then $u^\wedge$ must be continuous, and the continuity of α is necessary for that of $u^\wedge$. Thus, (7.14) is necessary for a solution of (7.11)-(7.12) in Λ.

Conversely, if (7.14) is satisfied, one can solve (7.16) for the continuous functions α and β and substitute into the formulas for $u^\wedge$ and $v^\wedge$. It is then easy to show that $u^\wedge$ and $v^\wedge$ are continuous and bounded in Σ. Moreover, the behavior of $u^\wedge$ and $v^\wedge$ for large $|\xi|$ can be found just as in § 3, simply by replacing $n\Omega$ by ξ. (See the argument before (3.13).) Using these facts, as well as (7.11), we arrive at (7.15). This completes the proof of Lemma 7.1.

We now use Lemma 7.1 to show that equations (7.2)-(7.9) inductively define a sequence $\{(u^n, v^n)\} \subset \Lambda$. The condition for this is that $h \in \Lambda^2(R^1)$. Notice that, just as membership in Λ^k implies (6.3), so too the hypothesis $h \in \Lambda^2(R^1)$ implies (6.1). Thus, we are still working within the class of bottoms satisfying (6.1).

We begin by using (7.4) to find immediately that (u^0, v^0) exists, is an element of Λ, and, by (7.15), satisfies

$$||(u^0, v^0)|| \leqq c||b^{-1}||_1 \leqq c||h||_2 .$$

Now, write $w^\nu = (u^\nu, v^\nu)$. Suppose $w^\nu \in \Lambda$ and satisfies

$$\begin{aligned} ||w^\nu|| \leqq c\Big(&||s^{\nu-1}||_0 + ||b^{\nu-1}||_1 + ||f^{\nu-1}||_0 + ||g^{\nu-1}||_0 \\ &+ \sum_{j=1}^{2}\Big\{||w^{\nu-j}|| + \sum_{\mu=0}^{\nu-j}||w^{\nu-j-\mu}||\cdot||w^\mu|| + \sum_{\mu=0}^{\nu-1-j}||w^{\nu-1-j-\mu}||\sum_{\lambda=0}^{\mu}||w^{\mu-\lambda}||\cdot||w^\lambda||\Big\}\Big) \end{aligned}$$

for $\nu = 0, 1, \cdots, n-1$, where, when $n = 0, 1$, or 2, the terms involving w^γ with $\gamma < 0$ are taken to be zero.

As in § 4, the consistency condition (7.14) is automatic in the context of (7.2)-(7.9). We use the notation that, if χ is an analytic function of ε, then $[\![\chi]\!]^n$ is the coefficient of ε^n in the Taylor series for χ. When s, b, and g are given by (7.7), (7.8), and (7.6), we have, with this notation,

$$\begin{aligned} s^{n-1} &= \Big[\!\!\Big[\eta_x\tilde{u} - \frac{F}{2}\frac{\partial}{\partial x}(\tilde{u}^2 + \tilde{v}^2)\Big]\!\!\Big]^{n-1} , \\ b^{n-1} &= -[\![h_x \underset{\sim}{u}]\!]^{n-1} , \\ g^{n-1} &= \Big[\!\!\Big[h_x(yu)_y + \eta_x((1+y)u)_y - \frac{\partial}{\partial x}[(h+\eta)u]\Big]\!\!\Big]^{n-1} . \end{aligned}$$

Therefore, exactly as in § 4,

$$
\begin{aligned}
s^{n-1} - b^{n-1} - \int_{-1}^{0} g^{n-1} dy &= \frac{\partial}{\partial x}\Big[\!\Big[h - \frac{\varepsilon F}{2}(\tilde{u}^2 + \tilde{v}^2) \\
&\qquad + \varepsilon(h + \eta)\int_{-1}^{0} u dy \Big]\!\Big]^{n-1} \\
&= \frac{\partial}{\partial x} K^{n-1} ,
\end{aligned} \tag{7.17}
$$

say. Since the transform of $\partial K^{n-1}/\partial x$ is $i\xi(K^{n-1})\hat{}(\xi)$, it follows from Lemma 7.1 that $w^n = (u^n, v^n) \in \Lambda$ and

$$
\|w^n\| \leqq c(\|s^{n-1}\|_0 + \|b^{n-1}\|_1 + \|f^{n-1}\|_0 + \|g^{n-1}\|_0 + N^{n-1}) ,
$$

where N^{n-1} is a bound for $|(K^{n-1})\hat{}(\xi)|$ in a neighborhood of zero. On the other hand, the definition of K^{n-1} gives (recall that h does not depend on ε)

$$
\begin{aligned}
N^{n-1} &\leqq \frac{F}{2}\sum_{\nu=0}^{n-2}(\|\tilde{u}^{n-2-\nu}\|_0 \cdot \|\tilde{u}^\nu\|_0 + \|\tilde{v}^{n-2-\nu}\|_0 \cdot \|\tilde{v}^\nu\|_0) + \|h\|_0\|u^{n-2}\|_0 \\
&\qquad + \sum_{\nu=0}^{n-2}\|\eta^{n-2-\nu}\|_0\|u^\nu\|_0 \\
&\leqq c'\Big(\|w^{n-2}\| + \sum_{\nu=0}^{n-2}\|w^{n-2-\nu}\|\cdot\|w^\nu\| + \sum_{\nu=0}^{n-3}\|w^{n-3-\nu}\|\sum_{\mu=0}^{\nu}\|w^{\nu-\mu}\|\cdot\|w^\mu\|\Big) ,
\end{aligned}
$$

where we have written $w^\nu = (u^\nu, v^\nu)$, and c' is a fixed constant, depending only on F and h. It follows, then, that

$$
\begin{aligned}
\|w^n\| \leqq c\Big(&\|s^{n-1}\|_0 + \|b^{n-1}\|_1 + \|f^{n-1}\|_0 + \|g^{n-1}\|_0 + \|w^{n-2}\| \\
&+ \sum_{\mu=0}^{n-2}\|w^{n-2-\mu}\|\cdot\|w^\mu\| + \sum_{\mu=0}^{n-3}\|w^{n-3-\mu}\|\sum_{\lambda=0}^{\mu}\|w^{\mu-\lambda}\|\cdot\|w^\lambda\|\Big) .
\end{aligned} \tag{7.18}
$$

Notice that the constant c here depends only on F and h, and not on n.

Finally, we return to the definitions (7.5)-(7.9) and use them in (7.18) to conclude

$$
\begin{aligned}
\|w^n\| \leqq c\sum_{j=1}^{2}\Big(&\|w^{n-j}\| + \sum_{\mu=0}^{n-j}\|w^{n-j-\mu}\|\cdot\|w^\mu\| \\
&+ \sum_{\mu=0}^{n-1-j}\|w^{n-1-j-\mu}\|\sum_{\lambda=0}^{\mu}\|w^{\mu-\lambda}\|\cdot\|w^\lambda\|\Big) ,
\end{aligned} \tag{7.19}
$$

$n = 0, 1, 2, \cdots$, where, again, a w having a negative superscript is understood to be zero. Thus, we have

LEMMA 7.2. *Let $h \in \Lambda^2(R^1)$. Then, if $F > 1$, the equations* (7.2)-(7.9) *define a unique sequence $\{(u^n, v^n)\} \subset \Lambda$ and, if we write $w^n = (u^n, v^n)$, w^n satisfies* (7.19).

8. Analytic solutions in the space Λ. We now use a method, first introduced in [8], and then used in [9, 6, 7], to show that the series (7.1) converge. The method is based on

LEMMA 8.1. *There exists a positive constant C such that*

$$\sum_{\nu=0}^{n} \frac{1}{(\nu+1)^2(n-\nu+1)^2} \leqq \frac{C}{(n+1)^2} .$$

The proof is easy and can be found in [8] and [9]. Hence, we omit it here.

A simple corollary is

LEMMA 8.2. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences satisfying*

$$|a_n| \leqq \frac{c_1 r^n}{(n+1)^2} \quad \textit{and} \quad |b_n| \leqq \frac{c_2 r^n}{(n+1)^2} .$$

Then,

$$\left| \sum_{\nu=0}^{n} a_{n-\nu} b_\nu \right| \leqq \frac{c_1 c_2 C r^n}{(n+1)^2} ,$$

where C is the constant of Lemma 8.1.

Now, let $\{w^n\} = \{(u^n, v^n)\}$ be the sequence of Lemma 7.2. Write $c_0 = ||w^0||$ and suppose, as an inductive hypothesis, that

$$||w^\nu|| \leqq \frac{c_0 R^\nu}{(\nu+1)^2} , \qquad \nu = 0, 1, \cdots, n-1 .$$

Using this inequality in (7.19) and applying Lemma 8.2, we find

$$||w^n|| \leqq \frac{c}{R}\left[\sum_{j=1}^{2} \left(\frac{c_0 R^{-j+1}}{(n-j+1)^2} + \frac{C c_0^2 R^{-j+1}}{(n-j+1)^2} + \frac{C^2 c_0^3 R^{-j}}{(n-j)^2} \right)(n+1)^2 \right] \frac{R^n}{(n+1)^2} ,$$

where, when $n = 1$ or 2, the terms with $(n-j+1)$ or $(n-j) \leqq 0$ in the denominator are understood to be zero. All of the constants are independent of n. Thus, the induction can be completed by choosing R large enough. It follows, then, that

$$||w^n|| \leqq \frac{c_0 R^n}{(n+1)^2} ,$$

for some constants c_0 and R. From this, it is a triviality that the series $w = \sum_{n=0}^{\infty} w^n \varepsilon^n$ converges in Λ if only ε is small enough. The limit, $w = (u, v)$, is a solution of our problem (2.9)-(2.16), by the way $\{(u^n, v^n)\}$ was constructed. It is also unique since $\{(u^n, v^n)\}$ is unique (Lemma 7.2). Thus, we have proved

THEOREM 8.3. *Let $h \in \Lambda^2(R^1)$. If $F > 1$ and ε is small enough, there exists a unique solution of* (2.9)-(2.16) *in Λ that is analytic in ε in a neighborhood of $\varepsilon = 0$. This solution can be calculated to any desired degree of accuracy by expanding u and v in a series of powers of ε and solving the resulting sequence of equations* (7.2)-(7.3).

This same method can also be applied to the case when h is periodic. One then reproduces the existence result of Corollary 5.4, and a slightly different uniqueness result from that of that part of Theorem 5.2. In Theorem 5.2, we proved the solution found there is unique if $||(u, v)||$ is small enough. Using the method of this section, we can conclude that there is a unique solution within the smaller class of functions which are *analytic in ε*, regardless of the size of $||(u, v)||$. The argument is the same as that which led to Theorem 8.3 and, as can be seen from that theorem, what is needed is $F \neq F_n$ and ε small enough. Nothing need be said about the size of $||(u, v)||$. We state these facts as

THEOREM 8.4. *Let $h \in \Lambda^2_L(R^1)$. Then, for every $F \neq F_n$, $n = 1, 2, \cdots$, the problem* (2.9)-(2.16) *has a periodic* (L) *solution in Λ_L if ε is small enough. This solution is analytic in ε in a neighborhood of $\varepsilon = 0$ and is unique in the closs of elements of Λ_L that are analytic in ε.*

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