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MAPS ON SIMPLE ALGEBRAS PRESERVING ZERO PRODUCTS. II: LIE ALGEBRAS OF LINEAR TYPE

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The study of maps on an algebra which preserve zero products is suggested by recent studies on linear transformations of various types on the space of $n \times n$ matrices over a field, particularly Watkins' work on maps preserving commuting pairs of matrices. This article generalizes the result of Watkins by determining the bijective semilinear maps f on a Lie algebra L with the property that

$$[x, y] = 0 \Longrightarrow [f(x), f(y)] = 0,$$

where $x, y \in L$, for a class of Lie algebras constructed from finite-dimensional simple associative algebras.

Introduction. In [8] we began the study of the semilinear maps on an algebra over a field k which preserve zero products, a problem arising from recent investigations characterizing the linear transformations on the $n \times n$ matrix algebra $M_n(k)$ over k which preserve various properties, particularly the work of Watkins on maps preserving commuting pairs of matrices [7]. If L is a Lie algebra, this means that we are concerned with the bijective semilinear maps f on L such that $[f(x), f(y)] = 0$ for all pairs of elements x, y of L such that $[x, y] = 0$. We say that f *preserves zero Lie products*.

If L is finite-dimensional, these maps f form a group $G(L)$ [8]. Clearly $G(L)$ contains the group G_1 of all semilinear automorphisms and anti-automorphisms (semilinear maps which are automorphisms or anti-automorphisms of the multiplicative structure of L), the group of units G_2 of the centroid of L (the algebra of linear transformations which commute with left multiplications in L), and the group G_3 of all bijective transformations f of the form $f(x) = x + g(x)$, where g is a linear map of L into its center $Z(L)$. Let $G_0(L) = G_1G_2G_3$.

In this paper we determine $G(L)$, for a class of simple Lie algebras L. These are obtained by taking finite-dimensional simple associative algebras A over a field k and forming the Lie algebra $L = [A, A]/[A, A] \cap Z(A)$, where $[A, A]$ is the subspace spanned by all the commutators $[x, y] = xy - yx$, and $Z(A)$ is the center of A. If A is noncommutative, then L is a simple Lie algebra, except when A has characteristic 2 and is 4-dimensional over $Z(A)$ [1, p. 17]. Except for cases of "small length," we show that $G(L) = G_0(L)$ for such a Lie algebra L. In fact, we can deal with a wider class of

Lie algebras, including the case $L = A$ (Theorem II and Corollary 9). We say that these are Lie algebras of linear type, since A is isomorphic with the algebra of all linear transformations on a vector space over a division algebra. Our result includes that of Watkins [7], and its extension to nonalgebraically closed ground fields by Pierce and Watkins [6].

The proof of Theorem II uses a result about maps between tensor products of vector spaces, preserving rank 1 elements, modulo a certain subspace (Theorem I), and a knowledge of the structure of linear transformations on a finite-dimensional vector space over a finite-dimensional division algebra, generalizing the usual elementary divisor theory for fields (§ 2).

It would be interesting to find $G(L)$ for the cases not covered by Theorem II. A particularly intriguing case occurs with the simple Lie algebra L obtained from a simple associative algebra of characteristic 2 having dimension 16 over its center, where it may be possible that not all elements of $G(L)$ come from the generalized rank 1 preservers classified in Theorem I.

1. Generalized rank 1 preservers.

In [8] we studied maps

$$f\colon U \otimes V \longrightarrow U_1 \otimes V_1$$

of tensor products of vector spaces over an associative division algebra D, preserving elements of rank 1. Here we shall need to consider the case when V and U form a pair of dual vector spaces, f may not be defined on the whole of $U \otimes V$, and the image of a rank 1 element under f is only assumed to be of rank 1 modulo a certain subspace S of $U_1 \otimes V_1$.

We assume familiarity with the notations and facts concerning tensor products contained in [8]. If U and V are a right and a left vector space over a division ring D, then $U \otimes V$ is an additive abelian group, which is a k-vector space if D is a division algebra over a field k. In particular, $U \otimes V$ is a vector space over the center $Z(D)$. The rank of an element x of $U \otimes V$ is the least number r such that x has an expression in the form

$$x = \sum_{i=1}^{r} u_i \otimes v_i\,, \qquad (u_i \in U,\ v_i \in V)\,.$$

In this case, the sets $\{u_1, \cdots, u_r\}$ and $\{v_1, \cdots, v_r\}$ are linearly independent, and span subspaces $U(x)$ and $V(x)$ of U and V, which are uniquely determined by x [2, Lemma 3.1].

If also U_1, V_1 are a right and a left vector space over a division ring D_1, we can speak of semilinear maps

$$g\colon U \longrightarrow U_1\,, \qquad h\colon V \longrightarrow V_1\,,$$

with respect to an isomorphism $\sigma\colon D \to D_1$. Such a pair of maps gives rise to a map

$$g \otimes h\colon U \otimes V \longrightarrow U_1 \otimes V_1\,,$$

such that $(g \otimes h)(u \otimes v) = g(u) \otimes h(v)$. Similarly, if $\sigma_1\colon D \to D_1$ is an anti-isomorphism, we can speak of semilinear maps

$$g_1\colon V \longrightarrow U_1\,, \qquad h_1\colon U \longrightarrow V_1\,,$$

with respect to σ_1, and these give rise to a map

$$g_1 \otimes h_1\colon U \otimes V \longrightarrow U_1 \otimes V_1\,,$$

such that $(g_1 \otimes h_1)(u \otimes v) = g_1(v) \otimes h_1(u)$.

If V, U form a pair of dual vector spaces over D with respect to a nondegenerate bilinear form $\langle\ ,\ \rangle$, and $[D, D]$ denotes the additive subgroup generated by the commutators $\alpha\beta - \beta\alpha$, where $\alpha, \beta \in D$, then the map taking the pair u, v to the coset of $\langle v, u\rangle$ (mod $[D, D]$) is a balanced map of $U \times V$ into $D/[D, D]$, so that there is a homomorphism

$$\mathrm{tr}\colon U \otimes V \longrightarrow D/[D, D]\,.$$

This is surjective, and is called the trace map. If D is a division algebra over a field k, then $[D, D]$ is a k-subspace of D, and tr is a k-linear map. If $U \otimes V$ is identified with the algebra A of finite-valued linear maps of V into itself which are continuous with respect to a certain topology defined by U (discrete if V is finite-dimensional), then the kernel of the trace map is the commutator $[A, A]$, so that the subspaces of A containing $[A, A]$ are in one-one correspondence with the subspaces of D containing $[D, D]$.

Because of the possibility of further applications, the result which we shall prove in this section is placed in a more general setting than that which is needed for the main purpose of this paper. The ingredients of the situation are as follows.

HYPOTHESES. (1) D and D_1 are associative division algebras of finite dimension over a field k, with $\dim_k D \geqq \dim_k D_1$.

(2) (V, U) is a pair of dual vector spaces over D.

(3) C is a k-subspace of D containing $[D, D]$, and L is the k-subspace of $U \otimes V$ consisting of all elements whose traces lie in $C/[D, D]$.

(4) U_1, V_1 are a right and a left vector space over D_1, and S is a k-subspace of $U_1 \otimes V_1$ containing no elements of rank 1 or 2.

(5) If S has an element of rank 3, then S is a 1-dimensional $Z(D_1)$-subspace of $U_1 \otimes V_1$.

(6) $\dim V \geqq 3$, $C \neq 0$ if $\dim V = 3$, and $\dim_k C > \dim_k [D_1, D_1]$ if $\dim V = 3$ and S has an element of rank 3.

The technical conditions (5), (6) will be used in the proof of Theorem I. It is not clear to what extent they are really necessary. In Hypothesis (6), $\dim V$ indicates the D-dimension of V. We make the convention that the unqualified words dimension, linear, subspace, $\cdots$ will always be taken to be with respect to D or D_1, as appropriate, and not with respect to k.

THEOREM I. *Assume Hypotheses* (1)-(6), *and suppose* $f: L \to U_1 \otimes V_1$ *is a semilinear map with respect to an automorphism* μ *of* k, *such that, whenever* x *is an element of rank* 1 *in* L, $f(x)$ *is the sum of an element of* S *and an element of rank* 1 *in* $U_1 \otimes V_1$. *Then one of the following holds.*

(i) *There exists an element* u_1 *of* U_1, *such that*

$$f(L) \subseteqq S + (u_1 \otimes V_1) .$$

(ii) *There exists an element* v_1 *of* V_1, *such that*

$$f(L) \subseteqq S + (U_1 \otimes v_1) .$$

(iii) μ *can be extended to an isomorphism* $\sigma: D \to D_1$, *and there exist injective* σ-*semilinear maps* $g: U \to U_1$, $h: V \to V_1$, *such that*

$$f = (g \otimes h)_L + r ,$$

where $(g \otimes h)_L$ *is the restriction of* $g \otimes h$ *to* L, *and* $r: L \to S$ *is a* μ-*semilinear map.*

(iv) μ *can be extended to an anti-isomorphism* $\sigma: D \to D_1$, *and there exist injective* σ-*semilinear maps* $g: V \to U_1$, $h: U \to V_1$, *such that*

$$f = (g \otimes h)_L + r ,$$

where $(g \otimes h)_L$ *is the restriction of* $g \otimes h$ *to* L, *and* $r: L \to S$ *is a* μ-*semilinear map.*

The proof of this theorem resembles that of [8, Theorem A], except that we can use the fundamental theorem of projective geometry, because of Hypothesis (6).

We shall call a k-subspace of L a rank 1 k-subspace if its nonzero elements all have rank 1. Such a k-subspace can be obtained by taking an element u of $U^\# = U - \{0\}$ and forming $u \otimes u^0$, where

$$u^0 = \{v \in V \mid \langle v, u \rangle \in C\} ,$$

a k-subspace of V containing the annihilator $u^\perp$ of u. If $d = \dim_k D$, $c = \dim_k C$, then $\dim_k V/u^0 = d - c$, $\dim_k V/u^\perp = d$. Since $2d + (d-c) <$ $\dim_k V$, by Hypothesis (6), we have

$$u^\perp \cap (u')^\perp \cap (u'')^0 \neq 0 ,$$

for all u, u', u'' in U. Similarly, if $v \in V^\#$, then $v^0 \otimes v$ is a rank 1 k-subspace of L, where $v^0 = \{u \in U \mid \langle v, u\rangle \in C\}$.

If an element w of $U_1 \otimes V_1$ has the form

$$w = s + t ,$$

where $s \in S$ and t has rank 1, then the rank 1 component t is uniquely determined by w, since $w = s + t = s' + t'$ gives $s - s' = t' - t$, and the only element of rank at most 2 in S is 0. We shall write w_0 for the rank 1 component t of an element w of this form. It will be convenient also to write $w_0 = 0$ for an element w of S. Clearly, if w_0 is defined and $\alpha \in k$, then $(\alpha w)_0$ is defined, and is equal to αw_0.

LEMMA 1. *If $u \in U^\#$, then the map $x \to f(x)_0$ is an injective μ-semilinear map of $u \otimes u^0$ into $U_1 \otimes V_1$.*

Proof. Our assumption on f shows that $f(x)_0$ is defined and nonzero if x has rank 1, and we have $f(\alpha x)_0 = (\alpha^\mu f(x))_0 = \alpha^\mu f(x)_0$, for $\alpha \in k$. Form $W = f(u \otimes u^0)$. It is enough to prove that if $w, w' \in W$, $w + w' = w''$, then $w_0 + w'_0 = w''_0$. Suppose this is not so. Then $w_0 + w'_0 - w''_0$ is an element s of S having rank 3, by Hypothesis (4). If $w_0 = u_1 \otimes v_1$, $w'_0 = u'_1 \otimes v''_1$, $w''_0 = u'_1 \otimes v''_1$, then $\{u_1, u'_1, u''_1\}$, $\{v_1, v'_1, v''_1\}$ are linearly independent subsets of U_1, V_1, generating the subspaces $U_1(s)$, $V_1(s)$.

If $\alpha \in Z(D_1)$, we see that $w_0 - \alpha s$ has rank 1 or 3 unless $\alpha = 1$. By Hypothesis (5), s is determined by w as the only element of S such that $w_0 - s$ has rank 2, and $U_1(w_0 - s) = u'_1 D_1 + u''_1 D_1$, $V_1(w_0 - s) =$ $D_1 v'_1 + D_1 v''_1$. If t is any element of W for which $(w + t)_0 \neq w_0 + t_0$, then the same argument with t in place of w' shows that

$$U_1(t_0) \subset U_1(w_0 - s) = u'_1 D_1 + u''_1 D_1 ,$$
$$V_1(t_0) \subset V_1(w_0 - s) = D_1 v'_1 + D_1 v''_1 .$$

Similarly, if $(w' + t)_0 \neq w'_0 + t_0$, then $U_1(t_0) \subset u_1 D_1 + u''_1 D_1$, $V_1(t_0) \subset$ $D_1 v_1 + D_1 v''_1$.

Now let Y be the set of all elements of $U_1 \otimes V_1$ of the form $u_1 \otimes \alpha v_1 + u'_1 \otimes \beta v'_1 + u''_1 \otimes \gamma v''_1$, where $\alpha, \beta, \gamma \in D_1$. Then Y is a k-subspace of $U_1 \otimes V_1$ containing S. Suppose that W is not contained in Y, and let $t \in W$, $t \notin Y$.

If $(w + t)_0 = w_0 + t_0$, then t_0 has form $u_1 \otimes v'''_1$ or $u'''_1 \otimes v_1$ [8, Lemma 1]. If $(w + t)_0 \neq w_0 + t_0$, then $t_0 = u_1''' \otimes v_1'''$, where $u_1''' \in u_1'D_1 + u_1''D_1$, $v_1''' \in D_1v_1' + D_1v_1''$. Considering $w' + t$, we similarly have the possibilities that $t_0 = u_1' \otimes v_1'''$, $u_1''' \otimes v_1'$, or $u_1''' \otimes v_1'''$, where in the last case $u_1''' \in u_1D_1 + u_1''D_1$, $v_1''' \in D_1v_1 + D_1v_1''$. Combining these, and using the linear independence of $\{u_1, u_1', u_1''\}$ and $\{v_1, v_1', v_1''\}$, we see that there are seven possibilities for t_0:

(i)	$u_1 \otimes \alpha v_1'$,	(ii)	$u_1' \otimes \alpha v_1$,
(iii)	$u_1 \otimes (\alpha v_1 + \beta v_1'')$,	(iv)	$(u_1\alpha + u_1''\beta) \otimes v_1$,
(v)	$u_1' \otimes (\alpha v_1' + \beta v_1'')$,	(vi)	$(u_1'\alpha + u_1''\beta) \otimes v_1'$,
(vii)	$u_1'' \otimes \alpha v_1''$.		

Suppose case (i) holds. Since $w + w' + t$ lies in W, it is the sum of an element of S and an element of rank 1 or 0. By Hypothesis (5), there exists $\gamma \in Z(D_1)$ such that $w_0 + w_0' + t_0 + \gamma s$ has rank 1 or 0. However, this element is equal to

$$u_1 \otimes ((\gamma + 1)v_1 + \alpha v_1') + u_1' \otimes (\gamma + 1)v_1' - u_1'' \otimes \gamma v_1'' .$$

Since $\{u_1, u_1', u_1''\}$ is linearly independent, it follows that $\{(\gamma + 1)v_1 + \alpha v_1', (\gamma + 1)v_1', \gamma v_1''\}$ must span a space of dimension 1 or 0. Since $\alpha \neq 0$, it is easily seen that this is not so. Case (ii) may be eliminated in the same way.

Suppose case (iii) holds. Then $\beta \neq 0$, since $t \notin Y$. For $\gamma \in Z(D_1)$, we find that $w_0' + t_0 + \gamma s$ is equal to

$$u_1 \otimes ((\alpha + \gamma)v_1 + \beta v_1'') + u_1' \otimes (\gamma + 1)v_1' - u_1'' \otimes \gamma v_1'' .$$

For this to have rank 1 or 0, we must have $\gamma = -1$, $\alpha = 1$. Now, for $\delta \in Z(D_1)$, we find that $w_0 + w_0' + t_0 + \delta s$ is equal to

$$u_1 \otimes ((\delta + 2)v_1 + \beta v_1'') + u_1' \otimes (\delta + 1)v_1' - u_1'' \otimes \delta v_1'' ,$$

which cannot have rank 1 or 0. Cases (iv), (v), (vi) may be eliminated in a similar way.

Case (vii) cannot hold, since $t \notin Y$. Thus, W must be contained in Y.

Since $\dim_k Y = 3 \dim_k D_1$, $W \cap S = 0$ by our hypothesis on f, and $\dim_k S = \dim_k Z(D_1)$, we have

$$\dim_k W \leqq 3 \dim_k D_1 - \dim_k Z(D_1) .$$

Since W and u^0 are isomorphic k-vector spaces, and $\dim_k V/u^0 = \dim_k D - \dim_k C$, we see that V has finite dimension n, and

$$\dim_k W = (n - 1) \dim_k D + \dim_k C \geqq (n - 1) \dim_k D_1 + \dim_k C ,$$

by Hypothesis (1). Since $n \geqq 3$, we must have $n = 3$, and $\dim_k C \leqq \dim_k D_1 - \dim_k Z(D_1)$. This is a contradiction to Hypothesis (6), and completes the proof of Lemma 1, because of the following result.

LEMMA 2. *If B is a finite-dimensional simple associative algebra over a field k, then*

$$\dim_k B = \dim_k Z(B) + \dim_k [B, B] .$$

Proof. Replacement of k by $Z(B)$ divides all three dimensions by $\dim_k Z(B)$. Thus we may suppose $k = Z(B)$. If K is an algebraic closure of k, then $B_K = B \otimes_k K$ is a full matrix algebra over K, and

$$\dim_K B_K = 1 + \dim_K [B_K, B_K] ,$$

since $[B_K, B_K]$ is just the subspace of matrices of trace 0. Since $[B_K, B_K] = [B, B] \otimes_k K$, we have $\dim_K [B_K, B_K] = \dim_k [B, B]$, $\dim_K B_K = \dim_k B$, and the result follows.

LEMMA 3. *If $u \in U^{\#}$, then $f(u \otimes u^0) \subseteqq S + (u_1 \otimes V_1)$, for some $u_1 \in U_1^{\#}$, or $f(u \otimes u^0) \subseteqq S + (U_1 \otimes v_1)$, for some $v_1 \in V_1^{\#}$.*

Proof. By Lemma 1, $\{f(x)_0 \mid x \in u \otimes u^0\}$ is a rank 1 subgroup of $U_1 \otimes V_1$, and so is contained in $u_1 \otimes V_1$ for some u_1, or in $U_1 \otimes v_1$ for some v_1 [8, Lemma 2].

LEMMA 4. *It is impossible to have*

$$f(u \otimes u^0) \subseteqq S + (u_1 \otimes V_1) ,$$
$$f(u' \otimes u'^0) \subseteqq S + (U_1 \otimes v_1) ,$$

where $u, u' \in U^{\#}$, $u_1 \in U_1^{\#}$, $v_1 \in V_1^{\#}$.

Proof. We use two injective semilinear maps of k-vector spaces, $u^0 \cap u'^0 \to u_1 \otimes V_1$, $u^0 \cap u'^0 \to U_1 \otimes v_1$, given by $v \to f(u \otimes v)_0$, $v \to f(u' \otimes v)_0$ respectively. The inverse images A, A' of $u_1 \otimes D_1 v_1$ under these two maps are k-subspaces of $u^0 \cap u'^0$, each having k-dimension $\dim_k D_1$. If $d = \dim_k D$, $c = \dim_k C$, then $\dim_k V > 3d - 2c$, by Hypothesis (6), while $\dim_k V/u^0 = \dim_k V/u'^0 = d - c$. Hence $\dim_k u^0 \cap u'^0 > d \geqq \dim_k D_1$, by Hypothesis (1), and so A, A' are proper k-subspaces of $u^0 \cap u'^0$.

If $0 \neq v \in u^0 \cap u'^0$, then, by symmetry, Lemma 1 applies to $v^0 \otimes v$ in place of $u \otimes u^0$, and we see that the sum of the rank 1 elements $f(u \otimes v)_0$ and $f(u' \otimes v)_0$ has rank 1 or 0. It follows [8, Lemma 1] that either $U_1(f(u' \otimes v)_0) = U_1(f(u \otimes v)_0) = u_1 D_1$, or $V_1(f(u \otimes v)_0) = V_1(f(u' \otimes v)_0) = D_1 v_1$, so that $v \in A'$ or $v \in A$. Since a vector space cannot be the union of two proper subspaces, this is a contradiction.

LEMMA 5. *Every element of L is a sum of elements of rank 1 in L.*

Proof. Let $x = \sum_{i=1}^{m} u_i \otimes v_i$ $(u_i \in U,\ v_i \in V)$ be an element of L. Since the result is trivial when $m = 1$, we assume $m > 1$ and use induction. If $\langle v_1, u_1\rangle = 0$, then $u_1 \otimes v_1 \in L$, and we apply induction to the remaining $m - 1$ terms. If $\langle v_j, u_i\rangle \neq 0$, for some $j \neq i$, say $\langle v_2, u_1\rangle \neq 0$, then $u_1 \otimes (v_1 - \alpha v_2) \in L$, where $\alpha = \langle v_1, u_1\rangle\langle v_2, u_1\rangle^{-1}$,

$$x = u_1 \otimes (v_1 - \alpha v_2) + (u_2 + u_1\alpha) \otimes v_2 + \sum_{i=3}^{m} u_i \otimes v_i ,$$

and induction can be applied. Finally, if $\langle v_1, u_1\rangle \neq 0$, and $\langle v_j, u_i\rangle = 0$ for all $j \neq i$, then $(u_1 + u_2) \otimes (\beta v_1 + v_2)$, $u_2 \otimes \beta v_1$ and $u_1 \otimes v_2$ all lie in L, where $\beta = -\langle v_2, u_2\rangle\langle v_1, u_1\rangle^{-1}$; since $x - (u_1 + u_2) \otimes (\beta v_1 + v_2) + (u_2 \otimes \beta v_1) + u_1 \otimes v_2$ is equal to

$$u_1 \otimes (1 - \beta)v_1 + \sum_{i=3}^{m} u_i \otimes v_i ,$$

induction can again be applied.

Proceeding with the proof of Theorem I, we have two possibilities, by Lemmas 3 and 4:

(A) For every $u \in U$, there exists $u_1 \in U_1$, such that

$$f(u \otimes u^0) \subseteqq S + (u_1 \otimes V_1) .$$

(B) For every $u \in U$, there exists $v_1 \in V_1$, such that

$$f(u \otimes u^0) \subseteqq S + (U_1 \otimes v_1) .$$

Similarly, we also have two possibilities for the $f(v^0 \otimes v)$:

(a) For every $v \in V$, there exists $v_1 \in V_1$, such that

$$f(v^0 \otimes v) \subseteqq S + (U_1 \otimes v_1) .$$

(b) For every $v \in V$, there exists $u_1 \in U_1$, such that

$$f(v^0 \otimes v) \subseteqq S + (u_1 \otimes V_1) .$$

In all we have four cases, (Aa), (Ab), (Ba), (Bb), where (Aa) means that (A) and (a) hold, etc. We consider these one at a time.

Case (Ab). Suppose $u, u' \in U^\#$. Since $\dim V \geqq 3$, we can choose a nonzero element v of $u^0 \cap u'^0$. We have

$$f(u \otimes u^0) \subseteqq S + (u_1 \otimes V_1) ,$$
$$f(u' \otimes u'^0) \subseteqq S + (u'_1 \otimes V_1) ,$$
$$f(v^0 \otimes v) \subseteqq S + (u_2 \otimes V_1) ,$$

where $u_1, u'_1, u_2 \in U_1$. Since $f(u \otimes v)$ lies in both $f(u \otimes u^0)$ and $f(v^0 \otimes v)$, we have

$$u_1D_1 = U_1(f(u \otimes v)) = u_2D_1 .$$

Similarly, $u_1'D_1 = u_2D_1$, so that $u_1D_1 = u_1'D_1$. Thus the same u_1 may be used in (A), for every $u \in U$. From Lemma 5, it follows that $f(L) \subseteqq S + (u_1 \otimes V_1)$, and case (i) of Theorem I holds.

Case (Ba). An exactly similar argument shows that case (ii) of Theorem I holds.

Case (Aa). Let $P(U)$, $P(U_1)$, $P(V_1)$ denote the sets of one-dimensional subspaces of U, U_1, V_1, respectively. We have maps $L: U^\# \to P(U_1)$, $R: V^\# \to P(V_1)$, such that

$$f(u \otimes u^0) \subseteqq S + (L(u) \otimes V_1) ,$$
$$f(v^0 \otimes v) \subseteqq S + (U_1 \otimes R(v)) ,$$

where $u \in U^\#$, $v \in V^\#$. If α is a nonzero element of D, and $u' = u\alpha$, let v be a nonzero element of $u^\perp$. Then

$$L(u) = U_1(f(u \otimes \alpha v)) = U_1(f(u' \otimes v)) = L(u') .$$

Thus, $uD \to L(u)$ is a well-defined map of $P(U)$ into $P(U_1)$.

If $u \in U^\#$, $v \in V^\#$, and $\langle v, u \rangle = 0$, then, for $\alpha \in D^\#$, $f(u\alpha \otimes v)_0 \in L(u\alpha) \otimes R(v) = L(u) \otimes R(v)$. Since $\alpha \to f(u\alpha \otimes v)_0$ is an injective semilinear map of D into $L(u) \otimes R(v)$, which is isomorphic to D_1 as a k-vector space, it follows from Hypothesis (1) that

$$L(u) \otimes R(v) = \{f(u\alpha \otimes v)_0 \mid \alpha \in D\} .$$

If $u, u' \in U^\#$, and $L(u) = L(u')$, choose a nonzero element v of $u^\perp \cap u'^\perp$. Then $f(u' \otimes v)_0 \in L(u') \otimes R(v) = L(u) \otimes R(v)$. Thus, $f(u' \otimes v)_0 = f(u\alpha \otimes v)_0$, for some $\alpha \in D$, so that $f((u' - u\alpha) \otimes v)_0 = 0$. Hence, $u' = u\alpha$. Thus, $uD \to L(u)$ is an injective map of $P(U)$ into $P(U_1)$.

Next, suppose uD, $u'D$, $u''D$ are three coplanar elements of $P(U)$, that is, $u'' = u\alpha + u'\beta$, where $\alpha, \beta \in D$. Let v be a nonzero element of $u^\perp \cap u'^\perp$. Then,

$$\begin{aligned} L(u'') \otimes R(v) &= \{f(u''\gamma \otimes v)_0 \mid \gamma \in D\} \\ &= \{f(u\alpha\gamma \otimes v)_0 + f(u'\beta\gamma \otimes v)_0 \mid \gamma \in D\} \\ &\subseteqq L(u) \otimes R(v) + L(u') \otimes R(v) . \end{aligned}$$

Since $R(v)$ is one-dimensional, it follows that $L(u'') \subseteqq L(u) + L(u')$, so that $L(u)$, $L(u')$, $L(u'')$ are coplanar.

If, conversely, $L(u'') \subseteqq L(u) + L(u')$, choose a nonzero element v of $u^\perp \cap u'^\perp \cap u''^0$. Then,

$$f(u'' \otimes v)_0 \in L(u'') \otimes R(v) \subseteqq L(u) \otimes R(v) + L(u') \otimes R(v) ,$$

so that there exist $\alpha, \beta \in D$, such that

$$f(u'' \otimes v)_0 = f(u\alpha \otimes v)_0 + f(u'\beta \otimes v)_0 .$$

Since the map $x \to f(x)_0$ is an injective semilinear map on $v^0 \otimes v$, it follows that $u'' = u\alpha + u'\beta$.

Finally, suppose $u, u' \in U^\sharp$, $u_1 \in L(u)$, $u_1' \in L(u')$. Again let v be a nonzero element of $u^\perp \cap u'^\perp$, and take a nonzero element v_1 of $R(v)$. There exist $\alpha, \beta \in D$, such that

$$u_1 \otimes v_1 = f(u\alpha \otimes v)_0 , \qquad u_1' \otimes v_1 = f(u'\beta \otimes v)_0 ,$$

and so $(u_1 + u_1') \otimes v_1 = f((u\alpha + u'\beta) \otimes v)_0$. If $u\alpha + u'\beta = 0$, then $u_1 + u_1' = 0$. If $u\alpha + u'\beta \neq 0$, then $u_1 + u_1' \in L(u\alpha + u'\beta)$. Thus the union of all the $L(u)$, $u \in U^\sharp$, is a subspace U_2 of U_1.

We can now apply the fundamental theorem of projective geometry [4, p. 104]. There exist an isomorphism $\sigma: D \to D_1$ and an injective σ-semilinear map $g: U \to U_1$ (with image U_2), such that $L(u) = g(u)D_1$, for all $u \in U^\sharp$. Similarly, there exist an isomorphism $\tau: D \to D_1$ and an injective τ-semilinear map $h: V \to V_1$, such that $R(v) = D_1h(v)$, for all $v \in V^\sharp$. Thus, $u \in U$, $v \in V$, $\langle v, u\rangle \in C$, we have

$$f(u \otimes v)_0 = g(u)\alpha(u, v) \otimes h(v) ,$$

where $\alpha(u, v) \in D_1$ (and $\alpha(u, v) \neq 0$ if $u \neq 0$, $v \neq 0$).

Suppose $v \in V^\sharp$, and let u, u' be nonzero elements of v^0. The equation $f((u + u') \otimes v)_0 = f(u \otimes v)_0 + f(u' \otimes v)_0$ gives

$$g(u + u')\alpha(u + u', v) = g(u)\alpha(u, v) + g(u')\alpha(u', v) .$$

If u, u' are linearly independent, then $g(u)$, $g(u')$ are also linearly independent, since g is injective. Comparison of the above equation with the equation $g(u + u') = g(u) + g(u')$ shows that $\alpha(u, v) = \alpha(u', v)$. If u, u' are linearly dependent, we choose $u'' \in v^0$, such that u, u'' are linearly independent. Then we have $\alpha(u, v) = \alpha(u'', v) = \alpha(u', v)$. Thus, $\alpha(u, v)$ is the same for every nonzero $u \in v^0$. Similarly, if $u \in U^\sharp$, then $\alpha(u, v)$ is the same for every nonzero $v \in u^0$.

If u, u' are any elements of $U^\sharp$, $v, v' \in V^\sharp$, and $\langle v, u\rangle, \langle v', u'\rangle \in C$, we can choose a nonzero u'' in $v^0 \cap v'^0$. By what we have proved,

$$\alpha(u, v) = \alpha(u'', v) = \alpha(u'', v') = \alpha(u', v') .$$

Thus, $\alpha(u, v)$ is the same for all pairs u, v of nonzero elements such that $\langle v, u\rangle \in C$, say $\alpha(u, v) = \alpha$. We can also set $\alpha(u, v) = \alpha$ when u or v is zero. Since $u \to g(u)\alpha$ is an injective semilinear map, we can change notation appropriately and write

$$f(u \otimes v)_0 = g(u) \otimes h(v) ,$$

for all u, v such that $\langle v, u\rangle \in C$.

If $\langle v, u\rangle = 0$, we see from the equation $f(u\beta \otimes v) = f(u \otimes \beta v)$ that the isomorphism $\sigma: D \rightarrow D_1$ related to g is the same as the isomorphism related to h. If $\beta \in k$, we see from the equation $f(u\beta \otimes v) = \beta^{\mu} f(u \otimes v)$ that σ agrees with μ on elements of k.

The maps g, h give a map $g \otimes h: U \otimes V \rightarrow U_1 \otimes V_1$, whose restriction to L is denoted $(g \otimes h)_L$. Clearly $f - (g \otimes h)_L$ is a μ-semilinear map of L into $U_1 \otimes V_1$ which maps elements of rank 1 into elements of S. By Lemma 5, the whole of L is mapped into S. Thus, case (iii) of Theorem I holds.

Case (Bb). Take a division algebra D_2 anti-isomorphic to D_1, and make V_1 into a right D_2-vector space U_2, U_1 into a left D_2-vector space V_2 in the obvious way. There is an isomorphism of k-vector spaces

$$j: U_1 \otimes V_1 \longrightarrow U_2 \otimes V_2 ,$$

given by $j(u_1 \otimes v_1) = v_1 \otimes u_1$. If U_1, V_1, D_1, f, S are replaced by U_2, V_2, D_2, jf, $j(S)$, then Hypotheses (1)-(6) are still satisfied, and we are in Case (Aa). We can apply the result we have just proved for that case, and by the obvious translation obtain case (iv) of Theorem I. We omit the details.

This completes the proof of Theorem I.

2. Centralizer of a linear transformation. We need some facts concerning the structure of a linear transformation on a finite-dimensional vector space over a division algebra. These form a special case of a more general situation discussed by Jacobson [3].

Let D be a finite-dimensional associative division algebra over a field k, with center Z. We identify k with a subfield of Z. The polynomial ring $D[t]$ is defined in the usual way; in particular, the indeterminate t commutes with elements of D. If $a(t)$ is a nonzero element of $D[t]$, the right ideal $a(t)D[t]$ has finite codimension as a subspace of the right D-vector space $D[t]$. The codimension is still finite if $D[t]$ is regarded as a Z-vector space, since $\dim_Z D$ is finite. Thus, $Z[t] \cap a(t)D[t]$ is a nonzero ideal of $Z[t]$, since it has finite codimension over Z. Thus there is a nonzero multiple of $a(t)$ lying in $Z[t]$. In the terminology of [3], every nonzero polynomial in $D[t]$ is bounded. Because of this, the usual elementary divisor theory for linear transformations over a field will hold over D, with only minor modifications.

If $a(t), a_1(t) \in D[t]$, then $a(t)$ and $a_1(t)$ are *similar* if there exist monic polynomials $b(t)$, $b_1(t)$ in $D[t]$, such that $b(t)a(t) = a_1(t)b_1(t)$, where $b(t)$ and $a_1(t)$ have no nonconstant common left factor, and $a(t)$ and

$b_1(t)$ have no nonconstant common right factor. Similarity is an equivalence relation [5, p. 489], and similar polynomials have the same degree. In the case of degree 1, $t - \alpha$ and $t - \beta$ are similar if and only if α and β are conjugate under the inner automorphism group of D.

Let V be a finite-dimensional left D-vector space, and x a linear transformation on V. (As in § 1, the unqualified words linear, dimension, subspace will always be taken with respect to D.) If $a(t) = \sum_i \alpha_i t^i$ is a polynomial in $D[t]$, and $v \in V$, define

$$a(t)v = \sum_i \alpha_i x^i(v) .$$

Then V becomes a left $D[t]$-module, and a submodule is a subspace which is invariant under x. If W, X are submodules, the $D[t]$-homomorphisms of W into X form a k-vector space Hom (W, X). In particular, the endomorphism algebra Hom (V, V) is the centralizer $C(x)$, consisting of all linear transformations on V which commute with x. The *order* of a vector v is the monic polynomial $a(t)$ of least degree for which $a(t)v = 0$, and its degree is equal to the dimension of the submodule generated by v. The order of another vector generating the same cyclic submodule is similar to $a(t)$. Thus one can speak of the order of a cyclic submodule, defined to within similarity.

If V is indecomposable as a $D[t]$-module, then it has a unique composition series, and its composition factors are all isomorphic. The order of a composition factor of V is an irreducible polynomial $p(t)$ in $D[t]$, defined to within similarity, which we call the *irreducible divisor* of V. We call the length m of the composition series of V the *length* of V. To within isomorphism, V is determined by the similarity class of the irreducible divisor $p(t)$ and the length m. We note that $\dim V = m \deg p(t)$. Also, every submodule and every quotient module of V is indecomposable.

In general, V can be expressed as the direct sum of indecomposable $D[t]$-modules, each of which is as described above.

PROPOSITION 6. *Let D be a division algebra of finite dimension d over a field k, and x a linear transformation on an n-dimensional vector space V over D. Suppose that the corresponding structure of V as a $D[t]$-module is given by the decomposition*

$$V = \sum_{i,j} V_{ij} \qquad (j = 1, \cdots, n_i;\ i = 1, \cdots, r) ,$$

where each V_{ij} is indecomposable, V_{ij} has irreducible divisor $p_i(t)$ and length m_{ij}, and no two of the polynomials $p_1(t), \cdots, p_r(t)$ are *similar. Let $C(x)$ be the algebra of linear transformations on V which commute with x, and let $c(x) = \dim_k C(x)$.*

(i) *As a k-vector space, $C(x)$ is isomorphic with the direct sum*

$$\sum_{i,j,k} \mathrm{Hom}\,(V_{ij}, V_{ik}) \qquad (j, k = 1, \cdots, n_i;\ i = 1, \cdots, r)\ .$$

(ii) *For given i, j, k, if $m = \min\{m_{ij}, m_{ik}\}$, then $\mathrm{Hom}\,(V_{ij}, V_{ik})$ is isomorphic (as a k-vector space) to the space of all polynomials $a(t)$ in $D[t]$ for which $\deg a(t) < m \deg p_i(t)$ and $p_{im}(t)a(t)$ is a left multiple of $p_{im}(t)$, where $p_{im}(t)$ is the order of an indecomposable module with irreducible divisor $p_i(t)$ and length m.*

(iii) $c(x) \leqq d \sum_{i=1}^{r} (\sum_{j,k=1}^{n_i} \min\{m_{ij}, m_{ik}\}) \deg p_i(t)$.

(iv) $\sum_{i,j} m_{ij} \deg p_i(t) = n$.

Proof. Since $C(x) = \mathrm{Hom}\,(V, V)$, and two V_{ij} having different values of i do not have any irreducible constituents in common, assertion (i) is clear.

The image of any homomorphism h of V_{ij} into V_{ik} has length at most $m = \min\{m_{ij}, m_{ik}\}$. Thus h is essentially a homomorphism from the unique quotient module of V_{ij} having length m to the unique submodule of V_{ik} having length m. These modules are each isomorphic with the indecomposable module W with irreducible divisor $p_i(t)$ and length m, and so $\mathrm{Hom}\,(V_{ij}, V_{ik})$ is isomorphic with $\mathrm{Hom}\,(W, W)$. Let w_0 be a generator of W, with order $p_{im}(t)$. Each endomorphism of W is determined by the image of w_0, which can be any element w for which $p_{im}(t)w = 0$. Every element of W can be uniquely expressed in the form $w = a(t)w_0$, where $a(t)$ is an element of $D[t]$ for which $\deg a(t) < \deg p_{im}(t) = m \deg p_i(t)$. The condition that $p_{im}(t)w = 0$ is then equivalent with the condition that $p_{im}(t)a(t)$ is a left multiple of $p_{im}(t)$. This proves assertion (ii).

Assertion (iii) follows from (i) and (ii), since the space of all polynomials $a(t)$ in $D[t]$ for which $\deg a(t) < m \deg p_i(t)$ has k-dimension $dm \deg p_i(t)$, and assertion (iv) follows from the equation $\dim V_{ij} = m_{ij} \deg p_i(t)$. This proves Proposition 6.

A special case of statement (ii) of the proposition is easily calculated. Suppose $p_i(t) = t - \alpha$ and $m = \min\{m_{ij}, m_{ik}\} = 1$. Then we seek the elements β of D for which $(t - \alpha)\beta$ is a left multiple of $t - \alpha$. This means $(t - \alpha)\beta = \beta(t - \alpha)$, so that $\alpha\beta = \beta\alpha$. Thus, $\mathrm{Hom}\,(V_{ij}, V_{ik})$ is isomorphic with $C_D(\alpha)$, the centralizer of α in D.

COROLLARY 7. *In the situation of Proposition 6,*

$$c(x) \leqq dn \max\{n_1, n_2, \cdots, n_r\}\ .$$

If equality holds, then (a) $n_1 = n_2 = \cdots = n_r$, *and* (*b*) *for each* i, m_{ij} *is independent of* j.

Proof. Let $n_0 = \max\{n_1, n_2, \cdots, n_r\}$. By Proposition 6 (iii), since $\min\{m_{ij}, m_{ik}\} \leqq m_{ij}$,

$$c(x) \leqq d \sum_{i=1}^{r} n_i \left(\sum_{j=1}^{n_i} m_{ij}\right) \deg p_i(t) \ ,$$

and equality implies $m_{ij} = m_{ik}$, for all j, k. Since $n_i \leqq n_0$, this gives

$$c(x) \leqq dn_0 \sum_{i,j} m_{ij} \deg p_i(t) = dn_0 n \ ,$$

by Proposition 6 (iv), and equality implies $n_i = n_0$, all i. This proves Corollary 7.

If $\alpha \in Z(D)$, then α defines a linear transformation $s_\alpha: v \rightarrow \alpha v$ on V, called a *central homothety*. This corresponds to the case where V has n indecomposable components, each with length 1 and irreducible divisor $t - \alpha$, and then $c(s_\alpha) = dn^2$. If y is a linear transformation of rank 1 on V, then $x = s_\alpha + y$ corresponds either to the case that V has $n - 2$ indecomposable components of length 1 and one of length 2, all with irreducible divisor $t - \alpha$, and then $c(x) = d(n^2 - 2n + 2)$, or to the case that V has $n - 1$ indecomposable components of length 1 with irreducible divisor $t - \alpha$ and one of length 1 with irreducible divisor $t - \beta$, and then $c(x) = d(n - 1)^2 + \dim_k C_D(\beta)$. We show that these are essentially the only cases when $c(x) \geqq d(n^2 - 2n)$.

PROPOSITION 8. *In the situation of Proposition* 6, *assume that* $n \geqq 3$ *and* $c(x) \geqq d(n^2 - 2n)$. *Then one of the following holds.*

(1) x *is a central homothety*; $c(x) = dn^2$.

(2) x *is the sum of a central homothety and a linear transformation of rank* 1; $c(x) = d(n - 1)^2 + \dim_k C_D(\beta)$, *for some element* β *of* D.

(3) $n = 3$, *and* V *has a basis for which* x *has matrix* $\operatorname{diag}\{\alpha, \alpha, \alpha\}$, *where* $\dim_k C_D(\alpha) = (1/2)d$; $c(x) = 9d/2$.

(4) $n = 3$ *or* $n = 4$, *and* $c(x) = d(n^2 - 2n)$.

Proof. We may suppose $n_1 \geqq n_i$, all i. By Corollary 7, $n_1 \geqq n - 2$. Also, if $n_1 = n - 2$, then every n_i is $n - 2$, $m_{ij} = m_{i1}$ for all j, and $c(x) = d(n^2 - 2n)$. In this case, Proposition 6 (iv) shows that $n - 2$ divides n, so that $n = 3$ or $n = 4$, and case (4) holds. We may now suppose $n_1 \geqq n - 1$.

If $n_1 = n$, then V has n indecomposable components, each of length 1, with the same irreducible divisor $t - \alpha$. By the remark following Proposition 6, $c(x) = n^2 \dim_k C_D(\alpha)$. Since $C_D(\alpha)$ is a division subalgebra of D, $\dim_k C_D(\alpha)$ is a divisor of d. We must have $\dim_k C_D(\alpha) \geqq (1/3)d$, since $(1/4)\, dn^2 < d(n^2 - 2n)$ when $n \geqq 3$. If $\dim_k C_D(\alpha) = (1/3)d$, then the inequality $(1/3)\, dn^2 \geqq d(n^2 - 2n)$ gives $n = 3$, so that $c(x) = 3d = d(n^2 - 2n)$, and case (4) holds. If $\dim_k C_D(\alpha) = (1/2)d$, we similarly find that $n = 3$ and case (3) holds, or $n = 4$ and case (4) holds. If $\dim_k C_D(\alpha) = d$, then case (1) holds.

If $n_1 = n - 1$, then $r \leqq 2$. If $r = 2$, then V has $n - 1$ components of length 1 with irreducible divisor $t - \alpha$, and one of length 1 with irreducible divisor $t - \beta$, where β is not conjugate to α under the inner automorphism group of D. Now, $c(x) = (n - 1)^2 \dim_k C_D(\alpha) + \dim_k C_D(\beta)$. We must have $\dim_k C_D(\alpha) \geqq (1/2)d$, since $(1/3)d(n - 1)^2 + d < d(n^2 - 2n)$ when $n \geqq 3$. If $\dim_k C_D(\alpha) = (1/2)d$, the inequality $c(x) \geqq d(n^2 - 2n)$ gives $n = 3$, $\beta \in Z(D)$, and $c(x) = 3d = d(n^2 - 2n)$, so that case (4) holds. If $\dim_k C_D(\alpha) = d$, then case (2) holds.

If $r = 1$, then V has $n - 2$ components of length 1 and one of length 2, all with irreducible divisor $t - \alpha$. If W is the component of length 2, we find that

$$c(x) = (n^2 - 2n) \dim_k C_D(\alpha) + \dim_k \operatorname{Hom}(W, W) .$$

There is a submodule X of W such that X and W/X are each irreducible, with irreducible divisor $t - \alpha$. As in the remark following Proposition 6,

$$\operatorname{Hom}(W, X) \simeq \operatorname{Hom}(W, W/X) \simeq C_D(\alpha) .$$

Now the exact sequence

$$0 \longrightarrow \operatorname{Hom}(W, X) \longrightarrow \operatorname{Hom}(W, W) \longrightarrow \operatorname{Hom}(W, W/X)$$

shows that $\dim_k \operatorname{Hom}(W, W) \leqq 2 \dim_k C_D(\alpha)$, so that $c(x) \leqq (n^2 - 2n + 2) \dim_k C_D(\alpha)$. We must have $\alpha \in Z(D)$, since $(1/2)d(n^2 - 2n + 2) < d(n^2 - 2n)$ when $n \geqq 3$. Thus, $c(x) = d(n^2 - 2n + 2)$, and case (2) holds.

This proves Proposition 8.

3. Lie algebras of linear type. If A is an associative algebra over a field k, then A becomes a Lie algebra under the operation $[x, y] = xy - yx$. If A is noncommutative and simple, with center $Z(A)$, then $[A, A]/[A, A] \cap Z(A)$ is a simple Lie algebra, except when k has characteristic 2 and A is 4-dimensional over $Z(A)$ [1, p. 17]. By Wedderburn's theorem, if A is finite-dimensional over k, then A is isomorphic with the complete algebra $L(V)$ of linear transformations on an n-dimensional vector space V over an associative division algebra D of finite dimension over k. Here $Z(A)$ corresponds to the set of all central homotheties on V, and $[A, A]$ to the kernel of the trace map from $L(V)$ to $D/[D, D]$ mentioned in § 1.

We shall find the bijective semilinear maps preserving zero Lie products in a general situation having as special cases both the simple Lie algebra associated with $L(V)$ and the complete algebra $L(V)$ itself. Let L be any k-subspace of A containing $[A, A]$, where $A = L(V)$, and let E be any k-subspace of $L \cap Z(A)$. Then L is a Lie subalgebra of $L(V)$, E is a central ideal of L, and we can form the Lie algebra $\bar{L} = L/E$. We are interested in the group $G(\bar{L})$ of all bijective semilinear maps on $\bar{L}$ which preserve zero Lie products. Since every semilinear map on $\bar{L}$ lifts to one on L, we need to find all the bijective semilinear maps f on L with the properties that $f(E) = E$, and $[f(x), f(y)] \in E$ whenever $[x, y] \in E$. We say that such a map *preserves zero Lie products* (mod E).

THEOREM II. *Let D be a finite-dimensional associative division algebra over a field k, and V a left vector space of finite dimension n over D. Let A be the algebra $L(V)$ of all linear transformations on V, and S the set of all central homotheties on V. Suppose L is a k-subspace of A containing $[A, A]$, and E a k-subspace of $L \cap S$. Assume that $n \geqq 3$. If $n = 4$ and D is commutative of characteristic 2, assume that $E \neq S$ or $L \neq [A, A]$. If $n = 3$, assume that $L \neq [A, A]$, and that D does not contain an element α such that $\dim_k C_D(\alpha) = (1/2) \dim_k D$. If f is a bijective map on L which is semilinear with respect to an automorphism μ of k, such that f preserves zero Lie products* (mod E), *then one of the following holds.*

(1) *μ can be extended to an automorphism σ of D, and there exist a bijective σ-semilinear transformation h on V, a nonzero element s of S, and a μ-semilinear map $r: L \to S$, such that*

$$f(x) = hxsh^{-1} + r(x) ,$$

for all $x \in L$.

(2) *μ can be extended to an anti-automorphism σ of D, and there exist a bijective σ-semilinear map h of the dual space V' onto V, a nonzero element s of S, and a μ-semilinear map $r: L \to S$, such that*

$$f(x) = h(xs)'h^{-1} + r(x) ,$$

for all $x \in L$, where $(xs)'$ denotes the adjoint of xs.

Proof. Identifying A with $U \otimes V$, where U is the dual space V', we shall apply Theorem I. Since the trace map induces an isomorphism of $A/[A, A]$ with $D/[D, D]$, we have

$$L = \{x \in A \mid \operatorname{tr} x \in C/[D, D]\} ,$$

where C is a k-subspace of D containing $[D, D]$. We note that S is a 1-dimensional $Z(D)$-subspace of A containing no elements of rank 1 or 2, and that S contains an element of rank 3 only when $n = 3$, in which case C contains $[D, D]$ properly, since $L \neq [A, A]$. We need to determine the structure of $f(x)$, where x has rank 1.

For any element x of L, we let $C(x)$ denote the centralizer of x in A, and $c(x) = \dim_k C(x)$, as in § 2. Also set

$$C_L(x) = C(x) \cap L , \qquad c_L(x) = \dim_k C_L(x) ,$$
$$C^*(x) = \{y \in L \mid [x, y] \in E\} , \qquad c^*(x) = \dim_k C^*(x) .$$

Let $d = \dim_k D$, $c = \dim_k C$, $e = \dim_k E$. Then $c \leqq d$, and $e \leqq \dim_k Z(D) \leqq d$. Since A/L is isomorphic with D/C (as k-vector spaces), we have

$$c(x) - d + c \leqq c_L(x) \leqq c(x) .$$

The map $y \to [x, y]$ is a k-linear map of $C^*(x)$ into E, with kernel $C_L(x)$, so that

$$c_L(x) \leqq c^*(x) \leqq c_L(x) + e .$$

Thus, we have

$$c(x) - d + c \leqq c^*(x) \leqq c(x) + e .$$

The condition that f preserves zero Lie products $(\operatorname{mod} E)$ implies that $f(C^*(x)) \subseteqq C^*(f(x))$, so that $c^*(x) \leqq c^*(f(x))$. From the last inequalities we get

$$c(x) - d + c \leqq c(f(x)) + e .$$

Now suppose that $x \in L \cap S$, so that x is a central homothety, and $c(x) = dn^2$. Then,

$$c(f(x)) \geqq dn^2 - d + c - e \geqq d(n^2 - 2) .$$

By Proposition 8, $f(x) \in L \cap S$. Since f is injective and $L \cap S$ is a k-subspace of L, $f(L \cap S) = L \cap S$.

Next suppose x has rank 1. By what we have just proved, $f(x)$ is not a central homothety. Now, $c(x) = d(n-1)^2 + \dim_k C_D(\beta)$, for some element β of D. Note that $\dim_k C_D(\beta) \geqq \dim_k Z(D) \geqq e$. Then,

$$c(f(x)) \geqq d(n^2 - 2n) + c + (\dim_k C_D(\beta) - e) .$$

By Proposition 8, and our assumptions for the case $n = 3$, $f(x)$ is the sum of a central homothety and a linear transformation of rank 1, except possibly when $n = 4$, $c = 0$, and $\dim_k C_D(\beta) = e$. In the latter case, D is commutative, since C contains $[D, D]$, and so $C_D(\beta) = D$. Then $e = d$, so that $E = S$. Also, $L = [A, A]$, since $C = [D, D]$. Since $S \subseteqq L$, every homothety has trace 0, so that the characteristic of k must be 2. This case is ruled out by hypothesis.

We can now apply Theorem I, with $V = V_1$, and $U = U_1 = V'$, the dual space of V.

Cases (i) and (ii) of Theorem I do not hold, since f is assumed to be bijective. Suppose case (iii) of Theorem I holds. Then μ can be extended to an automorphism σ of D, and there exist a bijective σ-semilinear transformation h on V, a bijective σ^{-1}-semilinear transformation g on V, and a μ-semilinear map $r: L \to S$, such that

$$f(x) = hxg + r(x) ,$$

for all $x \in L$. (Here g is the adjoint of the map denoted g in Theorem I (iii).)

Let W be any 1-dimensional subspace of V, H any hyperplane of V containing W. Let X be a hyperplane of V containing both W and $g(h(W))$. Let x be a linear transformation on V with kernel X and image W, and y a linear transformation on V with kernel H and image W. Then x and y have rank 1, and lie in L since they have zero trace. Since $xy = yx = 0$, and f preserves zero Lie products (mod E),

$$[hxg, hyg] = [f(x), f(x)] \in E .$$

Since hxg and hyg have rank 1, the left side has rank at most 2. Since E contains no nonzero elements of rank less than n, we have $[hxg, hyg] = 0$. Since $g(h(W)) \subseteqq X$, $hxghyg = 0$. Thus $hyghxg = 0$, so that $yghx = 0$, since g and h are bijective. Hence, $g(h(W)) \subseteqq H$. Since W is the intersection of all hyperplanes H which contain it, $g(h(W)) = W$. Since gh is linear, this implies that $gh = s$, where s is a nonzero central homothety. Then $g = sh^{-1}$, so that case (1) of Theorem II holds.

Finally suppose case (iv) of Theorem I holds. Then μ can be extended to an anti-automorphism σ of D, and there exist a bijective σ-semilinear map h of the dual space V' onto V, a bijective σ^{-1}-semilinear map g of V onto V', and a μ-semilinear map $r: L \to S$ such that

$$f(x) = hx'g + r(x) ,$$

for all $x \in L$, where x' is the adjoint of x. (Here g is the adjoint of the map denoted g in Theorem I (iv).)

If x, y are elements of rank 1 in L such that $[x, y] = 0$, then, as in the previous case, we see that

$$[hx'g,\ hy'g] = [f(x),\ f(y)] = 0\ .$$

Taking adjoints, we see that $[g'xh', g'yh'] = 0$. The same method as before shows that $h'g' = s$, where s is a nonzero central homothety. Then $gh = s'$, $g = s'h^{-1}$, so that case (2) of Theorem II holds.

This completes the proof of Theorem II.

We remark that Pierce and Watkins obtained the case of Theorem II when $k = D$, f is linear, L is the whole algebra $L(V)$, and $E = 0$ [6], extending the earlier paper [7] of Watkins, in which the additional assumptions that k is algebraically closed and $n \geqq 4$ were made. Watkins also pointed out that the conclusion of the theorem does not hold when $n = 2$. When $n = 3, 4$, the cases not covered by the theorem remain open.

If s is a central homothety, then the map $x \rightarrow xs$ is an element of the centroid of the Lie algebra A. In case (1) of the theorem, the map $x \rightarrow hxh^{-1}$ is a semilinear automorphism of A; and in case (2), the map $x \rightarrow hx'h^{-1}$ is a semilinear anti-automorphism of A. It is not clear that L is always invariant under these maps. However, this is so in the case $L = [A, A]$. We obtain the following result.

COROLLARY 9. *Let A be a finite-dimensional simple associative algebra over a field k, such that A can be written as the direct sum of* 4 *nonzero right ideals. If k has characteristic* 2, *assume further that the dimension of A over its center $Z(A)$ is greater than* 16. *Let L be the simple Lie algebra $[A, A]/[A, A] \cap Z(A)$ associated with A. Then every bijective semilinear map on L which preserves zero Lie products is the product of an element of the unit group of the centroid of L with a semilinear automorphism or anti-automorphism of L.*

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