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THE FACTORIZATION OF H^p ON THE SPACE OF HOMOGENEOUS TYPE

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Let K be a Calderon-Zygmund singular integral operator with smooth kernel. That is, there is an $\Omega(x)$ defined on $R^n\backslash\{0\}$ which satisfies

$$\int_{|x|=1}\Omega = 0\,, \quad \Omega \not\equiv 0\,,$$
$$(*) \qquad \Omega(rx) = \Omega(x) \quad \text{when} \quad r > 0 \quad \text{and} \quad x \in R^n\backslash\{0\}\,,$$
$$|\Omega(x) - \Omega(y)| \leq |x - y| \quad \text{when} \quad |x| = |y| = 1\,,$$

and that

$$Kf(x) = P.\,V.\int_{R^n}\Omega(x-y)\,|x - y|^{-n}f(y)dy\,.$$

Let

$$K'f(x) = P.\,V.\int_{R^n}\Omega(y - x)|y - x|^{-n}f(y)dy\,.$$

R. Coifman, R. Rochberg and G. Weiss showed the weak version of the factorization theorem of $H^1(R^n)$ and that was refined by Uchiyama in the following form.

THEOREM A. *If $1 < q < \infty$ and $1/q + 1/r = 1$, then*

$$c_{K,q}\|f\|_{H^1(R^n)} \leqq \inf\left\{\sum_{i=1}^{\infty}\|g_i\|_{L^q}\|h_i\|_{L^r}: \right.$$
$$\left. f = \sum_{i=1}^{\infty}(h_iKg_i - g_iK'h_i)\right\} \leqq c'_{K,q}\|f\|_{H^1(R^n)}\,.$$

In this note, we extend Theorem A to $H^p(X)$, where $p \in (1 - \varepsilon_X, 1]$ and X is a space of homogeneous type with certain assumptions.

1. Preliminaries. In the following, $A > 1$ and $\gamma \leqq 1$ are positive constants depending only on the space X.

Let X be a topological space endowed with a Borel measure μ and a quasi-distance d such that

$$(1) \qquad d(x, y) \geqq 0$$

$$(2) \qquad d(x, y) > 0 \quad \text{iff} \quad x \neq y$$

$$(3) \qquad d(x, y) = d(y, x)$$

$$(4) \qquad d(x, z) \leqq A(d(x, y) + d(y, z))$$

$$(5)\quad |d(x, z) - d(x, y)|/d(x, y) \leqq A(d(z, y)/d(x, y))^{\gamma} \quad \text{if } d(z, y) < d(x, y)/(2A)$$

$$(6)\quad t/A \leqq \mu(B(x, t)) \leqq t$$

for all x, y, z in X and all $t \in (0, A\mu(X))$, where $B(x, t) = \{y \in X: d(x, y) < t\}$. We postulate that $\{B(x, t)\}_{t \in (0, A\mu(X))}$ form a basis of open neighborhoods of the point x.

Let $\varphi(t) \in C^{\infty}(0, \infty)$ be a fixed nonnegative function such that $\varphi(t) = 0$ on $(0, 1/2)$, $\varphi(t) = 1$ on $(1, \infty)$ and $|d\varphi/dt| < 3$.

Further, we assume that X is endowed with a function $k(x, y)$ defined on $X \times X$ such that

$$(7)\quad |k(x, y)| \leqq 1/d(x, y) \quad \text{for all} \quad x, y \in X$$

$$(8)\quad \begin{aligned} &\sup\{|k(x, y)|: y \in X \text{ satisfying } A^{-2}t \leqq d(x, y) \leqq t\} \geqq 1/(At) \\ &\sup\{|k(y, x)|: y \in X \text{ satisfying } A^{-2}t \leqq d(x, y) \leqq t\} \geqq 1/(At) \\ &\qquad \text{for all } x \in X \text{ and all } t \in (0, A\mu(X)) \end{aligned}$$

$$(9)\quad |k(x, y) - k(x, z)|, |k(y, x) - k(z, x)| \leqq (d(y, z)/d(x, y))^{\gamma}/d(x, y) \quad \text{if } d(y, z) < d(x, y)/(2A)$$

and that for any $f \in L^2(X)$

$$Kf(x) = \lim_{t \to +0} \int k(x, y, t) f(y) d\mu(y)$$

$$K'f(x) = \lim_{t \to +0} \int k(y, x, t) f(y) d\mu(y)$$

exist almost everywhere and

$$(10)\quad \|Kf\|_2 \leqq \|f\|_2, \ \|K'f\|_2 \leqq \|f\|_2 ,$$

where

$$k(x, y, t) = k(x, y)\varphi(d(x, y)/t)$$

and $\|g\|_p$ denotes $\left(\int_X |g(y)|^p d\mu(y)\right)^{1/p}$.

For $x \in X$ and $t \in (0, A\mu(X))$, let

$$T(x, t) = \{\Psi \in C(X):$$

$$(11)\quad \operatorname{supp} \Psi \subset B(x, t)$$

$$(12)\quad |\Psi(y)| \leqq 1/t$$

$$(13)\quad |\Psi(y) - \Psi(z)| \leqq (d(y, z)/t)^{\gamma}/t \quad \text{for any} \quad y, z \in X\} .$$

For $f \in L^1(X)$ and $p > 1/(1 + \gamma)$, let

$$f^*(x) = \sup_{t \in (0, A\mu(X))} \sup_{\Psi \in T(x,t)} \left|\int f(y)\Psi(y) d\mu(y)\right| \quad \|f\|_{H^p} = \|f^*\|_p .$$

If $p > 1$, then $\|f\|_{H^p} \approx \|f\|_p$ by the Hardy-Littlewood maximal theorem and we define $H^p(X) = L^p(X)$. If $1/(1+\gamma) < p \leqq 1$, then we define $H^p(X)$ to be the completion of $\{f \in L^1(X): \|f\|_{H^p} < \infty\}$ by the metric $\|f - g\|_{H^p}^p$.

A comment on notation: The letter C denotes a positive constant depending only on A and γ. The various uses of C do not all denote the same constant. All the functions considered are real valued functions.

2. The results.

Our results are the following.

THEOREM 1. *If* $1/p = 1/q + 1/r < 1 + \gamma$, $0 < 1/q < 1 + \gamma$, $0 < 1/r < 1 + \gamma$, $g \in H^q \cap L^2$ *and* $h \in H^r \cap L^2$, *then*

$$\|hKg - gK'h\|_{H^p} \leqq c_{q,r} \|g\|_{H^q} \|h\|_{H^r} ,$$

where $c_{q,r}$ *is a positive constant depending on* q, r *and* X.

REMARK 1. As a consequence of this theorem, for any $g \in H^q$ and any $h \in H^r$ we can define $hKg - gK'h$ as the limit of $\{h_i K g_j - g_j K' h_j\}_{j=1}^{\infty}$ in H^p, where $\{g_j\}_{j=1}^{\infty} \subset H^q \cap L^2$ converges to g in H^q and $\{h_j\}_{j=1}^{\infty} \subset H^r \cap L^2$ converges to h in H^r.

THEOREM 2. *If* $\mu(X) = \infty$, $1 \leqq 1/p = 1/q + 1/r < 1 + \gamma$, $0 \leqq 1/q < 1 + \gamma$, $0 \leqq 1/r < 1 + \gamma$ *and* $f \in H^p$, *then there exist* $\{g_j\}_{j=1}^{\infty} \subset H^q$ *and* $\{h_j\}_{j=1}^{\infty} \subset H^r$ *such that*

$$f = \sum_{j=1}^{\infty} (h_j K g_j - g_j K' h_j) ,$$
$$\left(\sum (\|g_j\|_{H^q} \|h_j\|_{H^r})^p\right)^{1/p} \leqq c_{q,r} \|f\|_{H^p} .$$

As a result of these theorems, we get

COROLLARY 1. *If* $\mu(X) = \infty$, $1 \leqq 1/p = 1/q + 1/r < 1 + \gamma$, $0 < 1/q < 1 + \gamma$, $0 < 1/r < 1 + \gamma$ *and* $f \in H^p$, *then*

$$c_{q,r} \|f\|_{H^p} \leqq \inf\left\{\left(\sum_{j=1}^{\infty} (\|g_j\|_{H^q} \|h_j\|_{H^r})^p\right)^{1/p}: \right.$$
$$\left. f = \sum_{j=1}^{\infty} (h_j K g_j - g_j K' h_j)\right\} \leqq c'_{q,r} \|f\|_{H^p} .$$

EXAMPLE 1. Let $X = R^n$, $d(x, y) = |x - y|^n \omega_n = (\sum_{j=1}^{n} (x_j - y_j)^2)^{n/2} \omega_n$, μ be the Lebesgue measure and let $k(x, y) = \Omega(x - y) |x - y|^{-n}$, where ω_n is the volume of the unit ball of R^n and Ω satisfies (*). Then, by taking $\gamma = 1/n$ and by taking A sufficiently large depending on

Ω, the conditions (1) $\sim$ (10) can be satisfied. In this case, the above definition of H^1 coincides with the definition of $H^1(R^n)$ given by Fefferman-Stein [5]. Thus, Corollary 1 is an extension of Theorem A.

EXAMPLE 2. Let $A_t = t^P (0 < t < \infty)$ be a group of linear transformation on R^n with infinitesimal generator P satisfying $(Px, x) \geqq (x, x)$, where $(\ , \ )$ is the usual inner product in R^n. For each $x \in R^n$ let $\rho(x)$ denote the unique t such that $|A_t^{-1}x| = 1$. Let $\Omega(x)$ be such that

$$\int_{|x|=1} \Omega(x)(Px, x) = 0 \ , \quad \Omega \not\equiv 0 \ ,$$
$$\Omega(A_t x) = \Omega(x) \text{ when } t > 0 \text{ and } x \in R^n \backslash \{0\}$$
$$|\Omega(x) - \Omega(y)| \leqq |x - y| \quad \text{when} \quad |x| = |y| = 1 \ .$$

Let $X = R^n$, $d(x, y) = \rho(x - y)^\nu \omega_n$, μ be the Lebesgue measure and let $k(x, y) = \Omega(x - y)/d(x, y)$, where $\nu = \operatorname{tr} P$. Then, by taking $\gamma = 1/\nu$ and by taking A sufficiently large depending on P and Ω, the conditions (1) $\sim$ (10) can be satisfied. [See Riviere [12].]

If we remove the condition $\mu(X) = \infty$, we can show the following a little weaker result.

THEOREM 2′. *If $\mu(X) < \infty$, X is connected, $1 \leqq 1/p = 1/q + 1/r < 1 + \gamma$, $1 < q$, $1 < r$, $f \in H^p$ and $\int f d\mu = 0$, then there exist $\{g_j\}_{j=1}^\infty \subset L^q$ and $\{h_j\}_{j=1}^\infty \subset L^r$ such that*

$$f = \sum_{j=1}^\infty (h_j K g_j - g_j K' h_j)$$
$$(\sum (\|g_j\|_q \|h_j\|_r)^p)^{1/p} \leqq c_{q,r} \|f\|_{H^p} \ .$$

COROLLARY 1′. *If $\mu(X) < \infty$, X is connected, $1 \leqq 1/p = 1/q + 1/r < 1 + \gamma$, $1 < q < \infty$, $1 < r < \infty$, $f \in H^p$ and $\int f d\mu = 0$, then*

$$c_{q,r} \|f\|_{H^p} \leqq \inf \left\{ \left(\sum_{j=1}^\infty (\|g_j\|_q \|h_j\|_r)^p \right)^{1/p} : \right.$$
$$\left. f = \sum_{j=1}^\infty (h_j K g_j - g_j K' h_j) \right\} \leqq c'_{q,r} \|f\|_{H^p} \ .$$

REMARK 2. When $\mu(X) < \infty$, for $f \in L^1(X)$ we can easily show

$$\left| \int f d\mu \right| \leqq C \inf_{x \in X} f^*(x) \ .$$

Thus, for any $f \in H^p$ we can define $\int f d\mu$ by $\lim_{n \to \infty} \int f_n d\mu$, where $\{f_n\} \subset$

$L^1 \cap H^p$ and $\lim f_n = f$ in H^p. And it follows easily that

$$\left|\int f d\mu\right| \leqq c_p \|f\|_{H^p} .$$

3. The basic lemmas.

DEFINITION 1. If $1/(1+\gamma) < p \leqq 1$, we say that a function $a(y)$ is a p-atom if there exists a ball $B(x, t)$ such that

$$\text{supp}\, a \subset B(x, t),\ \|a\|_\infty \leqq t^{-1/p},\ \int_X a d\mu = 0 . \tag{20}$$

We can show easily that $\|a\|_{H^p} \leqq c_p$.

DEFINITION 2. For $f \in L^1 + L^2$, $q > 0$ and $\alpha > 0$, let

$$M_q f(x) = \sup_{t>0} \left(\int_{B(x,t)} |f|^q d\mu/t\right)^{1/q}$$

$$K^* f(x) = \sup_{t>0} \left|\int k(x, y, t) f(y) d\mu(y)\right|$$

$$K'^* f(x) = \sup_{t>0} \left|\int k(y, x, t) f(y) d\mu(y)\right|$$

$$f^{*[\alpha]}(x) = \sup_{t>0} \sup_{\Psi \in T_\alpha(x,t)} \left|\int f \Psi d\mu\right|$$

where

$$\begin{aligned} T_\alpha(x, t) = \{&\Psi \in C(X) \colon |\Psi(z)| \leqq t^\gamma (t + d(x, z))^{-1-\gamma} \\ &|\Psi(z) - \Psi(y)| \leqq d(z, y)^\gamma d(x, z)^{-1-\gamma} \quad \text{if} \quad d(z, y) < \alpha d(x, z)\} . \end{aligned} \tag{21}$$

LEMMA 1. *If $p > q$, then*

$$\|M_q f\|_p \leqq c_{p,q} \|f\|_p .$$

This is an immediate consequence of the Hardy-Littlewood maximal theorem. We omit the proof.

LEMMA 2. *If $d(y, z) \leqq d(x, y)/(2A)$, then*

$$d(x, y)/(2A) \leqq d(x, z) \leqq 2A d(x, y) .$$

This follows easily from (4). We omit the proof.

LEMMA 3. *If $t > 0$ and if $d(y, z) \leqq d(x, y)/(2A)$, then*

$$\begin{aligned} |\varphi(d(x, y)/t) - \varphi(d(x, z)/t)| &= 0 \quad \textit{if} \quad d(x, y) \notin (t/(4A), 2At) , \\ &\leqq C(d(y, z)/d(x, y))^\gamma \end{aligned}$$

otherwise.

Proof. Set $w = \varphi(d(x, y)/t) - \varphi(d(x, z)/t)$. If $d(x, y) \leqq t/(4A)$, then, by Lemma 2, $d(x, z) \leqq t/2$. Thus, $w = 0 - 0 = 0$. If $d(x, y) \geqq 2At$, then, by Lemma 2, $d(x, z) \geqq t$. Thus, $w = 1 - 1 = 0$. If $t/(4A) < d(x, y) < 2At$, then, by (5),

$$|w| \leqq C|d(x, y) - d(x, z)|/t \leqq C(d(y, z)/d(x, y))^{\gamma} .$$

LEMMA 4. *If $t > 0$ and if $d(y, z) \leqq d(x, y)/(2A)$, then*

$$|k(x, y, t) - k(x, z, t)| \leqq Cd(y, z)^{\gamma} d(x, y)^{-1-\gamma}$$
$$|k(y, x, t) - k(z, x, t)| \leqq Cd(y, z)^{\gamma} d(x, y)^{-1-\gamma} .$$

Proof. We show only the first inequality. Note that

$$\begin{aligned} |k(x, y, t) - k(x, z, t)| \leqq &|k(x, y) - k(x, z)|\varphi(d(x, y)/t) \\ &+ |k(x, z)|\,|\varphi(d(x, y)/t) - \varphi(d(x, z)/t)| . \end{aligned} \tag{22}$$

By (9), the first term of (22) is dominated by $d(y, z)^{\gamma} d(x, y)^{-1-\gamma}$. By Lemma 2, Lemma 3 and (7), the second term of (22) is also dominated by $Cd(y, z)^{\gamma} d(x, y)^{-1-\gamma}$.

LEMMA 5. *Let $1/(1 + \gamma) < p \leqq 1$ and $u \in H^p$. Then, there exist a sequence of real numbers $\{\lambda_j\}_{j=1}^{\infty}$ and a sequence of p-atoms $\{a_j\}_{j=1}$ such that*

$$u(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) \quad in \quad H^p \quad when \quad \mu(X) = \infty , \tag{23}$$
$$u(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) + \int u d\mu/\mu(X) \ in \ H^p \ when \ \mu(X) < \infty ,$$
$$\Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p} \leqq c_p \|u\|_{H^p} .$$

This is the atomic decomposition of $H^p(X)$ which was shown by Macias-Segovia [10].

LEMMA 6. *Let $1/(1 + \gamma) < p \leqq 1$, $u \in L^1$, $\operatorname{supp} u \subset B(x_0, t)$ and $t \in (0, A\mu(X))$. Then, there exists a sequence of real numbers $\{\lambda_j\}_{j=1}^{\infty}$ and a sequence of p-atoms $\{a_j\}_{j=1}^{\infty}$ such that*

$$\begin{aligned} &u(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x) + \lambda_0 a_0(x) \\ &\Big(\sum_{j=0}^{\infty} |\lambda_j|^p\Big)^{1/p} \leqq c_p\Big(\int_{B(x_0, 2At)} u^{*p} d\mu\Big)^{1/p} , \end{aligned} \tag{24}$$

where

$$\lambda_0 = \int u d\mu t^{1/p} / \mu(B(x_0, t)) , \quad a_0(x) = t^{-1/p} \chi_{B(x_0, t)}(x)$$

and χ_E denotes the characteristic function of a measurable set E.

Note that $\left|\int u d\mu\right|/t \leqq C \inf_{x \in B(x_0, 2At)} u^*(x)$. Then, applying Lemma 5 to $u - \lambda_0 a_0$, we get Lemma 6.

LEMMA 7. *Let* $1/(1+\gamma) < p \leqq \infty$. *Then,*

$$\|f^{*[\alpha]}\|_p \leqq c_{p,\alpha} \|f\|_{H^p} . \tag{25}$$

Proof. It can be shown easily that

$$f^{*[\alpha]}(x) \leqq c_\alpha M_1 f(x) .$$

Thus, if $p > 1$, (25) follows from the Hardy-Littlewood maximal theorem.

Let $1/(1+\gamma) < p \leqq 1$. Note that if $\mu(X) < \infty$, then it is trivial that $\|\chi_X^{*[\alpha]}\|_p \leqq c_p \|\chi_X^{*[\alpha]}\|_\infty \leqq c_{p,\alpha}$. Thus, by Lemma 5, it suffices to show (25) for a p-atom $a(y)$ satisfying (20). If $y \in B(x, t/\alpha)^c$, $s > 0$ and $\Psi \in T_\alpha(y, s)$,

$$\begin{aligned} &\left|\int a(z)\Psi(z) d\mu(z)\right| \\ &\quad = \left|\int a(z)(\Psi(z) - \Psi(x)) d\mu(z)\right| \\ &\quad \leqq \int_{B(x,t)} t^{-1/p} d(z, x)^\gamma d(x, y)^{-1-\gamma} d\mu \text{ by (21)} \\ &\quad \leqq t^{1-1p+\gamma} d(x, y)^{-1-\gamma} . \end{aligned}$$

Thus,

$$a^{*[\alpha]}(y) \leqq t^{1-1/p+\gamma} d(x, y)^{-1-\gamma} . \tag{26}$$

If $y \in B(x, t/\alpha)$, then

$$a^{*[\alpha]}(y) \leqq c_\alpha t^{-1/p} . \tag{27}$$

Hence, by (26) and (27),

$$\|a^{*[\alpha]}\|_p^p \leqq c_{p,\alpha} .$$

LEMMA 8. *Let* $1/(1+\gamma) < p < \infty$. *Then,*

$$\|K^* f\|_p \leqq c_p \|f\|_{H^p} \tag{28}$$

$$\|K'^* f\|_p \leqq c_p \|f\|_{H^p} . \tag{29}$$

Proof. If $p > 1$, then these follow from the well known argument about the maximal singular integral.

Let $1/(1+\gamma) < p \leqq 1$. We show only (28). Note that if $\mu(X) < \infty$, then it follows easily that

$$\|K^*\chi_X\|_p \leqq c_p\|K^*\chi_X\|_2 \leqq c_p\|\chi_X\|_2 \leqq c_p .$$

Thus, by Lemma 5, it suffices to show (28) for a p-atom $a(y)$ satisfying (20). If $d(x, y) > 2At$ and $s > 0$, then

$$\begin{aligned}&\left|\int k(y, z, s)a(z)d\mu(z)\right|\\ &\quad= \left|\int (k(y, z, s) - k(y, x, s))a(z)d\mu(z)\right|\\ &\quad\leqq C\int_{B(x,t)} d(x, z)^{\gamma} d(x, y)^{-1-\gamma} t^{-1/p} d\mu \text{ by Lemma 4}\\ &\quad\leqq Ct^{1-1/p+\gamma} d(x, y)^{-1-\gamma} .\end{aligned}$$

Thus,

$$K^*a(y) \leqq Ct^{1-1/p+\gamma} d(x, y)^{-1-\gamma} . \tag{30}$$

On the other hand, since (28) has been known for $p = 2$, we get

$$\int_{B(x,2At)} |K^*a|^p d\mu \leqq Ct^{1-p/2}\left(\int |K^*a|^2 d\mu\right)^{p/2} \leqq Ct^{1-p/2}\|a\|_2^p \leqq C . \tag{31}$$

Thus, by (30) and (31), we get

$$\int |K^*a|^p d\mu \leqq c_p . \tag{32}$$

LEMMA 9. *Let $\zeta(x, y)$ be a function defined on $X \times X$ such that*

$$\begin{aligned}&|\zeta(x, y)| \leqq d(x, y)^{\gamma-1}\\ &|\zeta(x, y) - \zeta(x, z)| \leqq d(y, z)^{\gamma}/d(x, y)\end{aligned} \tag{33}$$

if $d(y, z) < d(x, y)/(2A)$. Let $u \in L^2$, $\operatorname{supp} u \subset B(x_0, t)$, $t \in (0, A\mu(X))$

$$v(x) = \int \zeta(x, y)u(y)d\mu(y) \tag{34}$$

and $1 + \gamma > 1/s_1 > \gamma$. Then,

$$\left(\int_{B(x_0,t)} |v|^{s_2} d\mu\right)^{1/s_2} \leqq c_{s_1}\left(\int_{B(x_0,2At)} (u^*)^{s_1} d\mu\right)^{1/s_1} \tag{35}$$

where $1/s_2 = 1/s_1 - \gamma$.

Proof. If $s_1 > 1$, this can be shown in the same way as [13]

120. Let $1/(1+\gamma) < s_1 \leqq 1$. Applying Lemma 6 to $u(x)$ and $p = s_1$, we get $\{\lambda_j\}_{j=0}^\infty$ and $\{a_j(x)\}_{j=0}^\infty$ such that (24). For $j = 1, 2, 3 \cdots$, let

$$B(x_j, t_j) \supset \operatorname{supp} a_j,\ t_j^{-1/s_1} \geqq \|a_j\|_\infty . \tag{36}$$

For $j = 0, 1, 2, \cdots$, let

$$v_j(x) = \int \zeta(x, y) a_j(y) d\mu(y) . \tag{37}$$

Then,

$$\begin{aligned} &|v_0(x)| \leqq Ct^{\gamma-1/s_1} \\ &|v_j(x)| \leqq C \min(t_j^{\gamma-1/s_1}, t_j^{1+\gamma-1/s_1}/d(x, x_j)) \quad \text{for} \quad j \geqq 1 . \end{aligned} \tag{38}$$

Thus, by (24) and $s_1 \leqq 1 < s_2$,

$$\begin{aligned} \left(\int_{B(x_0,t)} |v|^{s_2} d\mu\right)^{1/s_2} &\leqq \sum_{j=0}^\infty |\lambda_j| \left(\int_{B(x_0,t)} |v_j|^{s_2} d\mu\right)^{1/s_2} \\ &\leqq c_{s_1} \sum |\lambda_j| \leqq c_{s_1} (\sum |\lambda_j|^{s_1})^{1/s_1} \\ &\leqq c_{s_1} \left(\int_{B(x_0,2At)} (u^*)^{s_1} d\mu\right)^{1/s_1} . \end{aligned} \tag{39}$$

4. **Proof of Theorem 1.** We may assume $q \leqq r$. Then $r > 1$. Let $x \in X$ be fixed. Let $t \in (0, A\mu(X))$ and $\Psi \in T(x, t)$. Then

$$\begin{aligned} &\int \Psi(y)(h(y)Kg(y) - g(y)K'h(y))d\mu(y) \\ &\quad = \int (\Psi(y)Kg(y) - K(\Psi g)(y))h(y)d\mu(y) . \end{aligned} \tag{40}$$

Set

$$\eta(y, z) = k(y, z)(\Psi(y) - \Psi(z))g(z) .$$

Note that

$$\Psi(y)Kg(y) - K(\Psi g)(y) = \int \eta(y, z) d\mu(z) .$$

Let

$$d(x, y) > 16A^4 t . \tag{41}$$

Then $\Psi(y) = 0$. Set

$$\begin{aligned} \int \eta(y, z) d\mu(z) &= -k(y, x) \int \Psi(z) g(z) d\mu(z) \\ &\quad + \int (k(y, x) - k(y, z)) \Psi(z) g(z) d\mu(z) \end{aligned} \tag{42}$$

$$= \eta_1(y) + \int \zeta_2(y, z) g(z) d\mu(z) = \eta_1(y) + \eta_2(y) .$$

If $z \in \operatorname{supp} \Psi$, then, by (41),

$$d(x, z) < d(y, x)/(2A) .$$

Hence, by (9) and (12),

$$|\zeta_2(y, z)| \leqq d(x, y)^{-1-\gamma} t^{\gamma-1} . \tag{43}$$

If $z_1, z_2 \in B(x, 2At)$, then, by (41) and Lemma 2,

$$d(x, z_1) < d(y, x)/(2A) \quad \text{and} \quad d(z_1, z_2) < d(y, z_1)/(2A) .$$

Hence, by (9), (12) and (13),

$$\begin{aligned} &|\zeta_2(y, z_1) - \zeta_2(y, z_2)| \\ &\leqq |k(y, x) - k(y, z_1)|\, |\Psi(z_1) - \Psi(z_2)| + |k(y, z_1) - k(y, z_2)|\, |\Psi(z_2)| \\ &\leqq C(d(z_1, z_2)/t)^{\gamma} t^{\gamma-1} d(x, y)^{-1-\gamma} . \end{aligned} \tag{44}$$

Thus, by (43), (44) and $\operatorname{supp} \zeta_2(y, \cdot) \subset B(x, t)$,

$$Ct^{-\gamma} d(x, y)^{1+\gamma} \zeta_2(y, \cdot) \in T(x, t) .$$

So,

$$|\eta_2(y)| \leqq C d(x, y)^{-1-\gamma} t^{\gamma} g^*(x) . \tag{45}$$

Let

$$d(x, y) \leqq 16A^4 t . \tag{46}$$

Set

$$\begin{aligned} \int \eta(y, z) d\mu(z) &= \Psi(y) \int k(x, z) \varphi(d(x, z)/(\beta t)) g(z) d\mu(z) \\ &\quad + \Psi(y) \int (k(y, z) - k(x, z)) \varphi(d(x, z)/(\beta t)) g(z) d\mu(z) \\ &\quad + \int k(y, z)(\Psi(y) - \Psi(z)) \varphi'(d(x, z)/(\beta t)) g(z) d\mu(z) \chi_{B(x, 16A^4 t)}(y) \\ &= \Psi(y) \int k(x, z, \beta t) g(z) d\mu(z) + \Psi(y) \int \zeta_4(y, z) g(z) d\mu(z) \\ &\quad + \int \zeta_5(y, z) \varphi'(d(x, z)/(\beta t)) g(z) d\mu(z) \chi(y) \\ &= \eta_3(y) + \eta_4(y) + \eta_5(y) , \end{aligned} \tag{47}$$

where $\beta = 128A^6$ and $\varphi' = 1 - \varphi$.

Since β is sufficiently large, if $\varphi(d(x, z)/(\beta t)) \neq 0$, then

$$d(x, y) < d(x, z)/(2A) .$$

Hence, by (9),

$$|\zeta_4(y, z)| \leqq Ct^\gamma(t + d(x, z))^{-1-\gamma} . \tag{48}$$

Let

$$d(z_1, z_2) < d(x, z_1)/(2A)^2 . \tag{49}$$

Set

$$\begin{aligned} |\zeta_4(y, z_1) - \zeta_4(y, z_2)| &\leqq |k(x, z_1) - k(x, z_2)|\varphi(d(x, z_1)/(\beta t)) \\ &\quad + |k(y, z_1) - k(y, z_2)|\varphi(d(x, z_1)/(\beta t)) \\ &\quad + (|k(y, z_2)| + |k(x, z_2)|)|\varphi(d(x, z_1)/(\beta t)) - \varphi(d(x, z_2)/(\beta t))| \\ &= \zeta_{41} + \zeta_{42} + \zeta_{43} . \end{aligned} \tag{50}$$

By (49) and (9),

$$\zeta_{41} \leqq Cd(z_1, z_2)^\gamma d(x, z_1)^{-1-\gamma} . \tag{51}$$

Since β is sufficiently large, if $\varphi(d(x, z_1)/(\beta t)) \neq 0$, then, by (46) and Lemma 2,

$$d(x, z_1)/(2A) \leqq d(y, z_1) .$$

Hence, by (49) and (9),

$$\begin{aligned} \zeta_{42} &\leqq d(z_1, z_2)^\gamma d(y, z_1)^{-1-\gamma}\varphi(d(x, z_1)/(\beta t)) \\ &\leqq Cd(z_1, z_2)^\gamma d(x, z_1)^{-1-\gamma} . \end{aligned} \tag{52}$$

By Lemma 2,

$$d(x, z_2) \geqq d(x, z_1)/(2A) . \tag{53}$$

If $\zeta_{43} > 0$, then, by Lemma 3,

$$d(x, z_2) > \beta t/(4A) .$$

So

$$d(x, y) \leqq 16A^4 t \leqq 64A^5 d(x, z_2)/\beta = d(x, z_2)/(2A) .$$

Thus, by Lemma 2 and (53),

$$d(y, z_2) \geqq d(x, z_2)/(2A) \geqq d(x, z_1)/(2A)^2 . \tag{54}$$

Hence, by (7), Lemma 3, (53) and (54),

$$\begin{aligned} \zeta_{43} &\leqq (d(y, z_2)^{-1} + d(x, z_2)^{-1})C(d(z_1, z_2)/d(x, z_1))^\gamma \\ &\leqq Cd(z_1, z_2)^\gamma d(x, z_1)^{-1-\gamma} . \end{aligned} \tag{55}$$

So, by (48), (51), (52) and (55),

$$C\zeta_4(y, \cdot) \in T_{(2A)^{-2}}(x, t) .$$

Thus,

$$|\eta_4(y)| \leqq C|\Psi(y)|g^{*[(2A)^{-2}]}(x) . \tag{56}$$

By (7) and (13),

$$|\zeta_5(y, z)| \leqq t^{-1-\gamma}d(y, z)^{\gamma-1} . \tag{57}$$

If $d(z_1, z_2) < d(y, z_1)/(2A)$, then by (7), (9) and (13),

$$\begin{aligned}&|\zeta_5(y, z_1) - \zeta_5(y, z_2)|\\ &\leqq |k(y, z_1)(\Psi(z_1) - \Psi(z_2))| + |k(y, z_1) - k(y, z_2)|\,|\Psi(z_2) - \Psi(y)|\\ &\leqq d(y, z_1)^{-1}t^{-1-\gamma}d(z_1, z_2)^{\gamma} + d(y, z_1)^{-1-\gamma}d(z_1, z_2)^{\gamma}t^{-1-\gamma}d(z_2, y)^{\gamma}\\ &\leqq Cd(y, z_1)^{-1}t^{-1-\gamma}d(z_1, z_2)^{\gamma} .\end{aligned} \tag{58}$$

So, by (57) and (58), $Ct^{1+\gamma}\zeta_5(y, z)$ satisfies the hypothesis of Lemma 9. Note that if $z \in B(x, 2A\beta t)$,

$$(\varphi'(d(x, \cdot)/(\beta t))g(\cdot))^*(z) \leqq Cg^*(z) .$$

Thus, by Lemma 9, we get

$$\begin{aligned}&\left(\int_{B(x,16A^4t)}|\eta_5|^{s_2}d\mu\right)^{1/s_2}\\ &\qquad\leqq Cc_{s_1}t^{-1-\gamma}\left(\int_{B(x,2A\beta t)}(g^*)^{s_1}d\mu\right)^{1/s_1} ,\end{aligned} \tag{59}$$

where $\gamma < 1/s_1 < 1 + \gamma$ and $1/s_2 = 1/s_1 - \gamma$.

By (42), (45), (47) and (56),

$$\int\eta(y, z)d\mu(z) = -\int\Psi g d\mu k(y, x)\varphi(d(y, x)/t) + \eta_3(y) + \eta_5(y) + \eta_6(y)$$

where

$$|\eta_6(y)| \leqq Cg^{*[(2A)^{-2}]}(x)t^{\gamma}(t + d(x, y))^{-1-\gamma} .$$

Thus,

$$\begin{aligned}|(40)| &\leqq \left|\iint\eta(y, z)d\mu(z)h(y)d\mu(y)\right|\\ &\leqq C\Big\{g^*(x)K'^*h(x) + h^*(x)K^*g(x)\\ &\qquad + \int\eta_5(y)h(y)d\mu(y) + g^{*[(2A)^{-2}]}(x)M_1h(x)\Big\} .\end{aligned} \tag{60}$$

Since $1/p = 1/q + 1/r$ and $1/p < 1 + \gamma$, we can take s_1 such that

$$1 + \gamma > 1/s_1 > \max(1/q, \gamma),\ 1/s_2' = 1 - s_2 > 1/r . \tag{61}$$

Then, by (59),

$$\begin{aligned} &\int \eta_5(y)h(y)d\mu(y) \\ (62) \qquad &\leqq \left(\int_{B(x,16A^4t)} |\eta_5(y)|^{s_2}d\mu(y)\right)^{1/s_2}\left(\int_{B(x,16A^4t)} |h|^{s_2'}d\mu\right)^{1/s_2'} \\ &\leqq c_{s_1}M_{s_1}g^*(x)M_{s_2'}h(x) . \end{aligned}$$

By (40), (60) and (62), we get

$$\begin{aligned} (hKg - gK'h)^*(x) \leqq C\{&g^*(x)K'^*h(x) + h^*(x)K^*g(x) \\ &+ M_{s_1}g^*(x)M_{s_2'}h(x) + g^{*[(2A)^{-2}]}(x)M_1h(x)\} . \end{aligned}$$

All the terms on the right hand side belong to L^p by Lemma 1, Lemma 7, Lemma 8 and (61).

5. Proof of Theorem 2. By Lemma 5, we may assume that f is a p-atom such that

$$\operatorname{supp} f \subset B(x_0, t),\ \|f\|_\infty < t^{-1/p} \quad \text{and} \quad \int f d\mu = 0 .$$

Let $q \leqq r$. Then $r > 1$.

Let N be a large number depending only on X and p. Then, by (8), there exists y_0 such that

$$A^{-2}Nt \leqq d(x_0, y_0) \leqq Nt,\ |k(y_0, x_0)| > 1/(ANt) .$$

By (9),

$$\inf\{|k(y, x)|: d(x, x_0) < t,\ d(y, y_0) < t\} > 1/(2ANt) .$$

Let

$$h(x) = \chi_{B(y_0,t)}(x)N . \tag{70}$$

Then, $|K'h(x)| > C$ on $B(x_0, t)$. Let

$$g(x) = -f(x)/K'h(x_0) .$$

Then, $g \in H^q$, $h \in H^r$ and

$$\|g\|_{H^q}\|h\|_{H^r} \leqq C(t^{-1/p+1/q})Nt^{1/r} = CN .$$

Set

$$\begin{aligned} w(x) &= f(x) - (h(x)Kg(x) - g(x)K'h(x)) \\ &= f(x)(K'h(x_0) - K'h(x))/K'h(x_0) - h(x)Kg(x) \\ &= w_1(x) + w_2(x) . \end{aligned}$$

Since $\operatorname{supp} w_1 \subset B(x_0, t)$ and $\|w_1\|_\infty \leqq t^{-1/p}N^{-\gamma}$, we see that

$$\int_{B(x_0,4A^2Nt)} w_1^{*p}(x)d\mu(x)$$
$$\leqq \int_{B(x_0,4A^2Nt)} t^{-1}N^{-\gamma p}(1+d(x_0, x)/t)^{-p}d\mu(x)$$
$$\leqq c_p N^{-\gamma p+1-p}\log N .$$

A similar estimate holds for w_2. Thus,

$$\int_{B(x_0,4A^2Nt)} w^{*p}(x)d\mu(x) \leqq \int_{B(x_0,4A^2Nt)} w_1^{*p}+w_2^{*p}d\mu$$
$$\leqq c_p N^{-\gamma p+1-p}\log N \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty .$$

Since $\operatorname{supp} w \subset B(x_0, 2ANt)$ and $\int w d\mu = 0$, by taking N sufficiently large and applying Lemma 6 to $w(x)$, we get

$$w(x) = \sum_{j=1}^{\infty} \lambda_j f_j(x) ,$$

where $\{f_j\}_{j=1}^{\infty}$ are p-atoms and

$$\sum_{j=1}^{\infty} |\lambda_j|^p < 1/2 .$$

Hence,

$$f = (hKg - gK'h) + \sum_{j=1}^{\infty} \lambda_j f_j .$$

Applying the same argument to each f_j and repeating this process, we get the desired result.

6. **Proof of Theorem 2′.** Since $\mu(X) < \infty$ and X is connected, we can easily see that for any $\varepsilon > 0$ and any p-atom $a(x)$, there exist $\{a_j(x)\}_{j=1}^{c_{p,\varepsilon}}$ such that

$$a(x) = \sum_{j=1}^{c_{p,\varepsilon}} a_j(x)$$

and that each a_j is a p-atom supported on the ball with radius $<\varepsilon$.

Thus, for the proof of Theorem 2′, we may assume that f is a p-atom such that the radius of its support is less than $N^{-1}\mu(X)$, where N is a sufficiently large number, depending only on X and p, to be determined later.

Following the proof of Theorem 2, we define $h(x)$ by (70) and $g(x)$ by

$$g(x) = -f(x)/K'h(x) .$$

Then,

$$w(x) = f(x) - (h(x)Kg(x) - g(x)K'h(x)) = -h(x)Kg(x) .$$

Note that if $y \in B(y_0, t)$, then

$$\begin{aligned} |Kg(y)| &\leqq \left|\int k(y, z)f(z)d\mu(z)/K'h(x_0)\right| \\ &\quad + \left|\int k(y, z)f(z)(1/K'h(x_0) - 1/K'h(z))d\mu(z)\right| \\ &\leqq C\int |k(y, z) - k(y, x_0)|\,|f(z)|\,d\mu(z) \\ &\quad + \int |k(y, z)|\,|f(z)|\,N^{-\gamma}d\mu(z) \\ &\leqq C\int (Nt)^{-1}N^{-\gamma}|f(z)|\,d\mu(z) \leqq CN^{-1-\gamma}t^{-1/p} . \end{aligned}$$

Thus,

$$\begin{aligned} &\|g\|_q\|h\|_r \leqq C\|f\|_q\|h\|_r \leqq Ct^{-1/p+1/q}Nt^{1/r} = CN , \\ &\int w d\mu = 0 , \\ &\operatorname{supp} w \subset B(y_0, t) \\ &\|w\|_\infty \leqq \|h\|_\infty \sup_{y \in B(y_0,t)}|Kg(y)| \leqq NCN^{-1-\gamma}t^{-1/p} . \end{aligned}$$

If N is sufficiently large, then $2w$ is a p-atom and the radius of its support is less than $N^{-1}\mu(X)$. Iterating this process, we get desired result.

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