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MONODROMY AND INVARIANTS OF ELLIPTIC SURFACES

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The purpose of this research is to analyze and compute the monodromy representation of the Gauss Manin connection associated with an elliptic surface and to relate properties of the monodromy to geometric properties of the surface. The results utilize the general theory of elliptic surfaces due to Kodaira.

Let E be an elliptic surface having a global section over its base curve X. We assume throughout that the functional invariant $\mathscr{J}$ is nonconstant and that E has no exceptional curves of the first kind in the fibres. We denote by G the homological invariant of E/X. On a Zariski open subset $X_0 \subset X$, G can be viewed as either a locally constant $\boldsymbol{Z} \oplus \boldsymbol{Z}$ sheaf or as a representation $\pi_1(X_0) \rightarrow \mathrm{SL}_2(\boldsymbol{Z})$. This representation corresponds to an algebraic vector bundle of rank two on X together with an integrable algebraic connection having regular singular points (Deligne [1], Griffiths [2]), which is known as the Gauss-Manin connection (Katz and Oda [4]). It can be expressed as a second order algebraic differential equation on X having regular singular points. The explicit form of this equation that we shall make use of appears in Stiller [12].

We begin with a brief section of preliminaries, recalling some previous results which relate the geometry of the elliptic surfaces over X to properties of the corresponding differential equations (K-equations, see Stiller [12]).

The first section describes a period mapping from the base curve X to the modular curve M_Γ where $\Gamma \subset \mathrm{SL}_2(\boldsymbol{Z})$ is the global monodromy group of both E/X and the differential equation. Also we give a number of conditions under which $\Gamma = \mathrm{SL}_2(\boldsymbol{Z})$ (see also § 3). When $\Gamma = \mathrm{SL}_2(\boldsymbol{Z})$ the group of $K(X)$-rational division points on the generic fibre (which is an elliptic curve over $K(X)$ the function field of the base curve X) is zero.

In section two we examine a number of invariants of E/X such as the Picard number, the valence of the functional invariant $\mathscr{J}$, the index of the monodromy group Γ in $\mathrm{SL}_2(\boldsymbol{Z})$, and other numerical invariants to determine their behavior when we pass to a generically isogeneous surface over X. The main results are that all of these invariants remain unchanged under generic isogeny! We will utilize the fact that in this case the differential equation does not change (Stiller [12]).

We finish by giving a method for determining the monodromy representation and computing several examples which illustrate the results.

0. **Preliminaries.** Let X be a complete smooth connected curve over $\boldsymbol{C}$ with function field denoted by $K(X)$. After fixing a parameter $x \in K(X)$, consider an algebraic differential equation on X

$$\Lambda f = \frac{d^2 f}{dx^2} + P\frac{df}{dx} + Qf = 0$$

with P and Q in $K(X)$ and f an unknown function.

DEFINITION 0.1. $\Lambda f = 0$ is called a K-equation if it possesses two solutions, ω_1 and ω_2, which are holomorphic nonvanishing multivalued functions on some Zariski open subset X_0 of X, satisfying:

(i) ω_1 and ω_2 form a basis of solutions,

(ii) for every closed path $\gamma \in \pi_1(X_0)$ the analytic continuation of $\binom{\omega_1}{\omega_2}$ around γ is $M_\gamma\binom{\omega_1}{\omega_2}$ with $M_\gamma \in \mathrm{SL}_2(\boldsymbol{Z})$ (the monodromy representation),

(iii) $\mathrm{Im}\,(\omega_1/\omega_2) > 0$ on X_0 (positivity).

Such a pair of solutions is called a K-basis. In addition, since the monodromy is in $\mathrm{SL}_2(\boldsymbol{Z})$, the Wronskian $W = \exp\left(-\int P\,dx\right)$ is single-valued. We assume as part of our definition:

(iv) $W \in K(X)$.

Let $\Lambda f = 0$ be a K-equation with K-basis ω_1 and ω_2. Consider the function $\mathscr{J} = J \circ \omega_1/\omega_2$,

$$X_0 \xrightarrow{\omega_1/\omega_2} \mathfrak{H} \xrightarrow{J} \boldsymbol{C}$$

where J is the elliptic modular function on the upper half plane $\mathfrak{H}$. This $\mathscr{J}$ is a single-valued holomorphic function on $X_0 \subset X$.

PROPOSITION 0.2. $\mathscr{J} \in K(X)$.

We now determine all K-equations. Fix a K-equation $\Lambda f = 0$ on X with K-basis ω_1, ω_2 such that $\mathscr{J} = J(\omega_1/\omega_2)$. Say

$$\Lambda f = \frac{d^2 f}{dx^2} + P\frac{df}{dx} + Qf = 0\ .$$

THEOREM 0.3. *There exists an algebraic function λ on X with $\lambda^2 \in K(X)$ such that*

$$P = \frac{\left(\frac{d\mathscr{J}}{dx}\right)^2 - \mathscr{J}\frac{d^2\mathscr{J}}{dx^2}}{\mathscr{J}\frac{d\mathscr{J}}{dx}} - \frac{d}{dx}\log\lambda^2$$

$$Q = \frac{\left(\frac{d\mathscr{J}}{dx}\right)^2\left(\frac{31}{144}\mathscr{J} - \frac{1}{36}\right)}{\mathscr{J}^2(\mathscr{J}-1)^2} - \left(\frac{\left(\frac{d\mathscr{J}}{dx}\right)^2 - \mathscr{J}\frac{d^2\mathscr{J}}{dx^2}}{\mathscr{J}\frac{d\mathscr{J}}{dx}}\right)\frac{d}{dx}\log\lambda$$
$$+\frac{-\frac{d^2\lambda}{dx^2}}{\lambda} + 2\left(\frac{\frac{d\lambda}{dx}}{\lambda}\right)^2 .$$

This is known as the K-equation $\Lambda_{(\mathscr{J},\lambda)}$.

It is shown in Stiller [12] that K-equations are precisely those differential equations which arise naturally as the Gauss-Manin connexions associated to elliptic surfaces.

One can directly compute the local behavior of the solutions at the singularities of the differential equation (Ince [3], Picard [7]). The local monodromy matrix corresponds with the marix associated to the particular type of singular fibre of the elliptic surface; see Kodaira's list in Kodaira [5]. The reader unfamiliar with the relation between representations and differential equations, and the local properties of differential equations with regular singularities can consult Poincaré [8], Deligne [1], or Griffiths [2]. The only terminology that we employ which is not standard is that we refer to a singular point as cosingular if the solutions are single-valued meromorphic functions in a neighborhood of the point.

Given a K-equation Λ on X with K-basis ω_1, ω_2 one can construct a basic elliptic surface E over X with functional invariant $\mathscr{J} = J(\omega_1/\omega_2)$ and homological invariant corresponding to the monodromy representation of Λ using the basis ω_1, ω_2. There is no unique K-equation associated to a given elliptic surface E/X and conversely a given K-equation may produce several surfaces for different choices of K-bases. However any other surface E'/X produced from the same K-equation Λ via a different K-basis will be generically isogeneous to E, i.e., there will exist a rational map $\phi: E \to E'$ over X which over a Zariski open set $X_0 \subset X$ will be a fibre by fibre isogeny. The converse also holds:

THEOREM 0.4. (*Stiller* [12].) *Let E, E' be basic elliptic surfaces over X which are generically isogeneous, then there is one K-equation Λ with two K-bases ω_1, ω_2 and ω_1', ω_2' such that E can be constructed from Λ, ω_1, ω_2, and E' can be constructed from Λ, ω_1', ω_2'. Moreover* *since both ω_1, ω_2 and ω_1', ω_2' form bases of solutions for Λ they are related by a constant matrix which is forced to be in* $\mathrm{GL}_2^+(\boldsymbol{Q})$ *as both are K-bases.*

Thus if E, E' are generically isogeneous we have

$$M\Gamma M^{-1} = \Gamma'$$

where $M \in \mathrm{GL}_2^+(\boldsymbol{Q})$ and Γ, $\Gamma' \in \mathrm{SL}_2(\boldsymbol{Z})$ are the global monodromy groups of E, E' respectively. Of course if $M \in \mathrm{SL}_2(\boldsymbol{Z})$ or scalar then $E \cong E'$.

In this way information about the entire isogeny class (of the generic fibre) is fixed in one differential equation. Any invariants which depend only on the differential equation are then the same for members of a given isogeny class. It is this idea that we shall pursue.

1. The monodromy. Let X be a complete smooth curve over the field of complex numbers $\boldsymbol{C}$, and let $\Lambda f = 0$ be a K-equation on X with a K-basis of solutions ω_1, ω_2. By definition ω_1, ω_2 are holomorphic nonvanishing multivalued functions on a Zariski open subset $X_0 \subset X$ with $\mathrm{Im}\,(\omega_1/\omega_2) > 0$ on X_0 and $\mathrm{SL}_2(\boldsymbol{Z})$ monodromy. From ω_1, ω_2 we obtain a commutative diagram:

$$\begin{tikzcd}
\tilde{X}_0 \arrow[rrrr, "\omega_1/\omega_2"] \arrow[drr, "\omega_1/\omega_2"'] \arrow[dd] & & & & \mathfrak{H}^- \arrow[dd] \arrow[dll, equal] \\
& & \mathfrak{H}^- \arrow[dd, "J"] & & \\
X_0 \arrow[rrrr] \arrow[drr, "\mathscr{J}"'] & & & & \mathfrak{H}^-/\bar{\Gamma} \arrow[dll] \\
& & \mathfrak{H}^-/PSL_2(\boldsymbol{Z}) & &
\end{tikzcd}$$

where

(i) $\tilde{X}_0$ is the universal cover of X_0.

(ii) $\mathfrak{H}^-$ is the upper-half-plane minus the $\mathrm{PSL}_2(\boldsymbol{Z})$ orbits of a finite set of points.

(iii) J is the elliptic modular function.

(iv) $\mathscr{J} = J \circ \omega_1/\omega_2$ (see § 0, Proposition 0.2).

(v) $\bar{\Gamma} \subset \mathrm{PSL}_2(\boldsymbol{Z})$ is the projective monodromy of Λ, ω_1, ω_2. Note that $\bar{\Gamma}$ has finite index in $\mathrm{PSL}_2(\boldsymbol{Z})$ (Stiller [12]).

(vi) $\mathfrak{H}^-/\mathrm{PSL}_2(\boldsymbol{Z})$ is $\boldsymbol{P}_{\boldsymbol{C}}^1$ minus a finite set of points.

Two remarks are in order. First, we will take X_0 small enough to insure that Λ will be holomorphic on X_0. Then neither ω_1 nor ω_2

vanish on X_0 and also the Wronskian $W = \omega_1\omega_2' - \omega_2\omega_1' = \exp\left(-\int P\,dx\right)$ will be nonvanishing. Thus the map

$$\tilde{X}_0 \xrightarrow{\omega_1/\omega_2} \mathfrak{H}^- \tag{1.1}$$

will be locally biholomorphic. Moreover it will be onto $\mathfrak{H}^-$ where we have removed the $\mathrm{PSL}_2(\boldsymbol{Z})$ orbits of a finite set of points (Stiller [12]). In fact it will be enough to remove the points where $\mathscr{J} = 0, 1, \infty$, $\lambda^2 = 0, \infty$, or $\operatorname{ord} d\mathscr{J} \neq 0$. From the explicit form of the equation (see § 0, Theorem 0.3) one sees that choice of derivation d/dx does not effect the map (1.1). Our second remark is that the triple Λ, ω_1, ω_2 corresponds to a unique basic elliptic surface E over X with functional invariant $\mathscr{J}$ and homological invariant G given by the monodromy representation of Λ for the basis ω_1, ω_2. Moreover, if Λ', ω_1', ω_2' also gives rise to E over X then there is a $g \in K(X)$ such that $\Lambda' = \Lambda_g = \Lambda_{(\mathscr{J}, g\lambda)}$ where $\Lambda = \Lambda_{(\mathscr{J}, \lambda)}$ (see § 0, Theorem 0.3) and $\omega_i' = g\omega_i$ up to the action of $\mathrm{SL}_2(\boldsymbol{Z})$ changing bases. Thus the map (1.1) depends only on E/X.

DEFINITION 1.1. The map $X_0 \to \mathfrak{H}^-/\bar{\Gamma}$ will be called the period map.

THEOREM 2.1. *The period map $X_0 \to \mathfrak{H}^-/\bar{\Gamma}$ is algebraic and extends to a regular map $X \to M_{\bar{\Gamma}}$ where $M_{\bar{\Gamma}}$ is the modular curve $\mathfrak{H}^*/\bar{\Gamma}$ and where $\mathfrak{H}^* = \mathfrak{H} \cup \{\boldsymbol{Q}\}$.*

Proof. Let $x \in X - X_0$. Choose a disc about x in X with local parameter t and select branches of ω_1, ω_2 single-valued in a fixed sector of the disc. Now because Λ has regular singular points there is an integer N such that $t^N\omega_1/\omega_2$ remains bounded as $t \to 0$ in the sector. It follows easily from this pole-like behavior that the map is algebraic and it must extend as both X and $M_{\bar{\Gamma}}$ are complete smooth curves.

Note that if the corresponding elliptic surface E/X has a singular fibre at $x \in X - X_0$ of type I_b or I_b^* $b \geqq 1$ then the local monodromy of Λ will be parabolic and x will map to a cusp, and for types II, II*, III, III*, IV, IV* x will map to an elliptic point. The only other possibility is local monodromy $\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ which yields some noncusp. (See Kodaira [5] for a description of these fibre types.)

From the commutative diagram above we can obtain immediate results:

PROPOSITION 1.3. $[\mathrm{PSL}_2(\boldsymbol{Z})\colon \bar{\Gamma}] \mid$ *valence* $\mathscr{J}$, *that is, the index of the projective monodromy group divides the valence of* $\mathscr{J}$.

Proof. Obvious.

REMARK. Using the formulas for the numerical invariants of a basic surface (Kodaira [6]), one can show, for example, that if $E/\boldsymbol{P}^1_{\boldsymbol{C}}$ is a K_3 surface then valence $\mathscr{J} \leqq 24$. It follows that the monodromy has index $\leqq 48$. This in turn limits the nature of the $K(X)$-rational torsion on E^{gen} as an elliptic curve over $K(X)$.

THEOREM 1.4. *Suppose* $X_0 \xrightarrow{\mathscr{J}} \mathfrak{H}^-/\mathrm{PSL}_2(\boldsymbol{Z})$ (*or* $X \xrightarrow{\mathscr{J}} \boldsymbol{P}^1_{\boldsymbol{C}}$) *is abelian Galois with no* 2 *or* 3 *torsion. Then* $\bar{\Gamma} = \mathrm{PSL}_2(\boldsymbol{Z})$.

Proof. Under the hypotheses of the theorem, a standard fact in the theory of modular functions gives $\bar{\Gamma}$ normal in $\mathrm{PSL}_2(\boldsymbol{Z})$. We have

$$\begin{tikzcd}
PSL_2(Z)/\bar{\Gamma} \arrow[rr, "\cong"] \arrow[d] & & \text{Abelian group with no 2 or 3 torsion} \\
PSL_2(Z)/\bar{\Gamma}\cdot\Delta \arrow[urr, "*"'] & &
\end{tikzcd}$$

where Δ is the commutator subgroup of $\mathrm{PSL}_2(\boldsymbol{Z})$. But $\mathrm{PSL}_2(\boldsymbol{Z})/\Delta$ surjects on $\mathrm{PSL}_2(\boldsymbol{Z})/\bar{\Gamma}\cdot\Delta$ and $H_1(\mathrm{PSL}_2(\boldsymbol{Z}), \boldsymbol{Z}) = \mathrm{PSL}_2(\boldsymbol{Z})/\Delta$ is $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$. Thus * is the zero map and $\bar{\Gamma} = \mathrm{PSL}_2(\boldsymbol{Z})$.

Now suppose we have a basic surface E/X. Assume the projective monodromy group of E/X is all of $\mathrm{PSL}_2(\boldsymbol{Z})$.

THEOREM 1.5. *Let* $\hat{X} \to X$ *be any abelian Galois extension of* X *with no* 2 *or* 3 *torsion. Then the projective monodromy of* $\hat{E} = E \times_X \hat{X}$ (*minimal smooth model*/$\hat{X}$) *is also* $\mathrm{PSL}_2(\boldsymbol{Z})$.

Proof. After removing a suitable set of points from both $\hat{X}$ and X we have

$$\begin{tikzcd}
\hat{E}_0 \arrow[r] \arrow[d] & E_0 \arrow[d] \\
\hat{X}_0 \arrow[r] & X_0
\end{tikzcd}$$

with $\hat{X}_0 \to X_0$ étale Galois (thereby a covering may). Then

$$\pi_1(\hat{X}_0) \rightarrowtail \pi_1(X_0) \twoheadrightarrow \mathrm{PSL}_2(\boldsymbol{Z}) ,$$

with $\pi_1(\hat{X}_0)$ normal in $\pi_1(X_0)$ and the image of $\pi_1(\hat{X}_0) = \bar{\Gamma}$, the monodromy of $\hat{E}_0$, also normal in $\mathrm{PSL}_2(\boldsymbol{Z})$. So

$$\pi_1(X_0)/\pi_1(\hat{X}_0) \twoheadrightarrow \mathrm{PSL}_2(\boldsymbol{Z})/\bar{\Gamma} .$$

On the other hand, $\pi_1(X_0)/\pi_1(\hat{X}_0) \cong \mathrm{Gal}\,(\hat{X}_0/X_0)$. Thus $\mathrm{PSL}_2(\boldsymbol{Z})/\bar{\Gamma}$ is abelian with no 2 or 3 torsion and as above we conclude $\bar{\Gamma} = \mathrm{PSL}_2(\boldsymbol{Z})$.

THEOREM 1.6. *Let E/X be an elliptic surface over X with functional invariant $\mathscr{J}$. Suppose $X \xrightarrow{\mathscr{J}} \boldsymbol{P}^1_{\boldsymbol{C}}$ exhibits X as a solvable Galois extension which admits a tower having no 2 or 3 torsion. Then the projective monodromy of E/X is all of* $\mathrm{PSL}_2(\boldsymbol{Z})$.

Proof. Clear.

REMARK. Clearly $\bar{\Gamma} = \mathrm{PSL}_2(\boldsymbol{Z})$ if and only if the monodromy $\Gamma = \mathrm{SL}_2(\boldsymbol{Z})$. Thus $\Gamma = \mathrm{SL}_2(\boldsymbol{Z})$ in all the above results.

COROLLARY 1.7. *If the monodromy is all of* $\mathrm{SL}_2(\boldsymbol{Z})$, *as it is in the above cases, the generic fibre $E^{\mathrm{gen}}/K(X)$ as an elliptic curve over $K(X)$ has no nonzero $K(X)$-rational division points.*

Proof. See Stiller [12].

Let E/X be a basic surface. The homological invariant is a locally constant $\boldsymbol{Z} \oplus \boldsymbol{Z}$ sheaf on some $X_0 \to X$ Zariski open. Let $x_0 \in X_0$ be a base point. We can interpret the homological invariant as an action of $G = \pi_1(X_0, x_0)$ on $H^1(E_{x_0}, \boldsymbol{Z}) = A(\cong \boldsymbol{Z} \oplus \boldsymbol{Z})$ where E_{x_0} is the fibre over x_0. Thus there is a map

$$G \xrightarrow{\rho_Z} \mathrm{Aut}_Z\,(H^1(E_{x_0}, \boldsymbol{Z})) ,$$

and A is naturally a $\boldsymbol{Z}G$-module.

THEOREM 1.8. *Let $A \otimes_Z \boldsymbol{C} \cong H^1(E_{x_0}, \boldsymbol{C}) = V$. Now $G \xrightarrow{\rho_C} \mathrm{Aut}_C\,(V)$, so V is a $\boldsymbol{C}G$-module. We claim V is a simple $\boldsymbol{C}G$-module,* i.e., *the representation is irreducible.*

Proof. Say $H \subset V$ is a one dimensional invariant subspace. Thus V has a basis where the representation takes the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} .$$

Now V also has an underlying $\boldsymbol{Z}$-structure so that the representation may be taken into $\mathrm{SL}_2(\boldsymbol{Z})$. Hence there is an $M \in \mathrm{GL}_2(\boldsymbol{C})$ conjugating one form to the other. An easy computation shows that in $\mathrm{SL}_2(\boldsymbol{Z})$ we could get only parabolic elements fixing i^∞ (assuming we normalize so $\pm\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$, $n > 0$ is in the monodromy group $\Gamma \subset \mathrm{SL}_2(\boldsymbol{Z})$. This contradicts $\Gamma \subset \mathrm{SL}_2(\boldsymbol{Z})$ of finite index.

We now consider when A is simple as a $\boldsymbol{Z}G$-module. It is easy to see that A has no G-invariant subgroups of rank 1. However suppose we have $H \subsetneqq A$ a G-invariant subgroup of rank 2. Obviously upon tensoring with $\boldsymbol{Q}$ we have $H \otimes_Z \boldsymbol{Q} \cong A \otimes_Z \boldsymbol{Q}$. If we view A as $\boldsymbol{Z}\omega_1 + \boldsymbol{Z}\omega_2$, ω_1, ω_2 a lattice for E_{x_0}, then H corresponds to an invariant sublattice and clearly gives rise to another K-basis for our K-equation Λ which is not $\boldsymbol{Z}$-equivalent to ω_1, ω_2 i.e., the new basis ω_1', ω_2' is not a scalar or $\mathrm{SL}_2(\boldsymbol{Z})$ combination of ω_1, ω_2. This new basis for Λ determines another elliptic surface E'/X whose generic fibre will be isogeneous to that of E/X. Conversely every elliptic surface E'/X which is generically isogeneous to E/X over $K(X)$ arise out of another K-basis for Λ. (See Stiller [12] and the remarks in §0.)

2. Invariants. Let E and E' be two elliptic surfaces over a common base curve X with function field denoted $K(X)$. We shall assume as before that both E and E' have nonconstant functional invariant, admit a section over X, and that they are free of exceptional curves of the first kind in the fibres. The section corresponds to a $K(X)$-rational point on the generic fibre E^{gen}, E'^{gen}. In this way E^{gen}, E'^{gen} can be viewed as elliptic curves/$K(X)$.

DEFINITION 2.1. E and E' are said to be generically isogeneous over X if the genetic fibres of E and E' are isogeneous over $K(X)$.

One should note that this definition is equivalent to the existence of a rational map

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

which over a Zariski open subset of X is a regular fibre isogeny. Such a map need not extend to all of E as examples show (see §3). Assume E and E' are generically isogeneous and let ϕ be the above rational map. Choose X_0 Zariski-open in X so that $E_0 = \pi^{-1}(X_0)$ and $E_0' = \pi'^{-1}(X_0)$ contain no degenerate fibre and so that

$$\begin{tikzcd} \psi \mid E_0 \colon E_0 \arrow[rr] \arrow[dr] & & E_0' \arrow[dl] \\ & X_0 & \end{tikzcd}$$

is a regular fibre by fibre isogeny. Let D_0 and D_0' be the Gauss-Manin connexion (Katz and Oda [4]):

$$D_0\colon H^1_{DR}(E_0/X_0) \longrightarrow \Omega^1_{X_0/C} \otimes_{\mathcal{O}_{X_0}/C} H^1_{DR}(E_0/X_0)$$

etc. for D_0', where H^1_{DR} is the first hyperderived functor of direct image applied to $\Omega^{\cdot}_{E_0/X_0}$, the relative algebraic DeRham complex. Note

$$H^1_{DR}(E_0/X_0) \cong R^1\pi_*(\boldsymbol{C})$$
$$H^1_{DR}(E_0'/X_0) \cong R^1\pi'_*(\boldsymbol{C}) .$$

Thus we have two flat vector bundles of rank two on X_0.

THEOREM 2.2. *E and E'; are generically isogeneous if and only if the resulting flat vector bundles are isomorphic.*

Proof. See Stiller [12].

This flat bundle can be represented by a second order algebraic differential equation X

$$\frac{d^2f}{dx^2} + P\frac{df}{dx} + Qf = 0$$

where $P, Q \in K(X)$ and $x \in K(X)$ nonconstant. The resulting differential equation will be a K-equation Λ. However, it will possess two K-bases ω_1, ω_2 and ω_1', ω_2' such that E will be the surface associated to ω_1, ω_2 and E' to ω_1', ω_2'. Thus the functional invariant $\mathscr{J}$ of E will be $J(\omega_1/\omega_2)$ and $\mathscr{J}'$ of E will be $J(\omega_1'/\omega_2')$, J the elliptic modular function. Moreover the homological invariants of E, E' will correspond to the monodromy representations given by ω_1, ω_2 and ω_1', ω_2' respectively. Since both ω_1, ω_2 and ω_1', ω_2' are bases of solutions of the same differential equation the representations are complex equivalent. However they will not be equivalent over $\mathrm{SL}_2(\boldsymbol{Z})$. It can be shown (Stiller [12]) that there exists $M \in \mathrm{GL}_2{}^+(\boldsymbol{Q})$ such that

$$M(\omega_1/\omega_2) = (\omega_1'/\omega_2') .$$

Of course if M is in $\mathrm{SL}_2(\boldsymbol{Z})$ or scalar then we will have $E \cong E'$.

THEOREM 2.3. *If E and E' are generally isogeneous over X then the period maps commute, that is there exists a regular map Ψ such that the diagram*

$$
\begin{tikzcd}
 & & \mathfrak{H}^+/\bar{\Gamma}' \arrow[dd] \\
X \arrow[urr, "\text{period}"] \arrow[r, "\text{period}"] \arrow[drr, dashed, "\mathscr{J}'"] \arrow[ddr, "\mathscr{J}"'] & \mathfrak{H}^+/\bar{\Gamma} \arrow[ur, "\Psi"'] \arrow[dd] & \\
 & & \mathfrak{H}^+/PSL_2(\boldsymbol{Z}) \\
 & \mathfrak{H}^+/PSL_2(\boldsymbol{Z}) \arrow[ur, equal] &
\end{tikzcd}
$$

commutes. Here $\mathfrak{H}^+$ is the upper-half-plane union the appropriate cusps and $\bar{\Gamma}$, $\bar{\Gamma}'$ are the projective monodromy groups in $\mathrm{PSL}_2(\boldsymbol{Z})$ *of E and E' respectively.*

Proof. Let $\rho\colon \pi_1(X_0) \to \mathrm{SL}_2(\boldsymbol{Z})$ and $\rho'\colon \pi_1(X_0) \to \mathrm{SL}_2(\boldsymbol{Z})$ be the monodromy representations (homological invariants) of E and E' respectively. For $\gamma \in \pi_1(X_0)$ we have

$$M\rho(\gamma)M^{-1} = \rho'(\gamma)$$

for some $M \in \mathrm{GL}_2^+(\boldsymbol{Q})$. We have

$$
\begin{tikzcd}
 & \mathfrak{H}^- \\
X_0 \arrow[ur, "\omega_1'/\omega_2'"] \arrow[dr, "\omega_1/\omega_2"'] & \\
 & \mathfrak{H}^-
\end{tikzcd}
$$

where $\mathfrak{H}^-$ is the upper-half-plane minus a finite number of $\mathrm{SL}_2(\boldsymbol{Z})$ orbits, map $\mathfrak{H}^-$ to $\mathfrak{H}^-$ by $\tau \to M\tau$. It is easy to check that this descends to a well-defined map

$$\mathfrak{H}^-/\bar{\Gamma} \longrightarrow \mathfrak{H}^-/\bar{\Gamma}'$$

since $M\bar{\Gamma}M^{-1} = \bar{\Gamma}'$. It then follows that the diagram commutes.

Note that Ψ is an isomorphism of Riemann surfaces $M_{\bar{\Gamma}}$ and $M_{\bar{\Gamma}'}$. Thus:

COROLLARY 2.4. *The global projective monodromy groups must have the same index if E and E' are generically isogeneous* i.e., $[\mathrm{PSL}_2(\boldsymbol{Z})\colon \bar{\Gamma}] = [\mathrm{PSL}_2(\boldsymbol{Z})\colon \bar{\Gamma}']$.

COROLLARY 2.5. *If E and E' are generically isogeneous then the functional invariants $\mathscr{J}$ and $\mathscr{J}'$ have the same valence, that is the same number of poles.*

THEOREM 2.6. *If E and E' are generically isogeneous then all the betti numbers b_i $i = 0, \cdots, 4$, the geometric genus p_g and the irregularity q of E and E' are the same.*

Proof. We appeal to the formulas of Kodaira [6] and general relationships among the numerical invariants. Let g be the genus of X then $b_1 = b_3 = 2q = 2g$ and $b_0 = b_4 = 1$. So only p_g and b_2 are of interest; and quality between one of these for E and E' implies equality for the other. Recall Kodaira's formula (Kodaira [6]):

$$12(p_a + 1) = \mu + 6 \sum_{b \geq 0} v(I_b^*) + 2v(\mathrm{II}) + 10v(\mathrm{II}^*) + 3v(\mathrm{III}) + 9v(\mathrm{III}^*) + 4v(\mathrm{IV}) + 8v(\mathrm{IV}^*) ,$$

where p_a is the arithmetic genus of the surface, $v(T)$ is the number of singular fibres of type T (for types see Kodaira's list in Kodaira [5]), and μ is the valence of the functional invariant. As $p_a = p_g - q$ it will be enough to show that the sum on the right hand side is invariant under genetic isogeny. By Corollary 2.5 above μ is invariant. Now each fibre type has an associated matrix in $\mathrm{SL}_2(\boldsymbol{Z})$ which represents the local monodromy up to conjugation in $\mathrm{SL}_2(\boldsymbol{Z})$. For example,

$$\begin{array}{lll} I_b & \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} & b > 0 \\ I_b^* & \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} & b > 0 \\ I_0^* & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . & \end{array}$$

Since E and E' are generically isogeneous, the local monodromy at $x \in X$ of either E or E' is $\mathrm{GL}_2(\boldsymbol{C})$-equivalent to the local monodromy for Λ. Thus trace is preserved and it follows that $\sum_{b \geq 0} v(I_b^*)$ is preserved as is $\sum_{b \geq 1} v(I_b)$. Note that the actual fibre type may *not* be preserved (see examples § 3 where type I_2 becomes I_4 etc.). Of course type I_0^* is preserved. In all of the remaining cases the type will be preserved. For example trace considerations show that a fibre of type II on E must correspond to one of type II or II* on E'. In order for the type to change there would have to be a matrix $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}_2^+(\boldsymbol{Q})$ with

$$N \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} N .$$

But this forces $a = -d$ and $b - a = c$. Thus $\det N = -a^2 - b^2 + ab = -(a^2 - ab + b^2) = -((a - 1/2\, b)^2 + 3/4\, b^2) \leq 0$ a contradiction. The same sort of calculation shows II*, III, III*, IV, IV* are preserved. Thus the entire sum is preserved as we desired to show.

Lastly we wish to investigate the Picard numbers ρ and ρ' of E and E'. Recall that ρ is the rank of the Néron-Severi group which is the group of divisors modulo algebraic equivalence.

THEOREM 2.7. *If E and E' are generically isogeneous then their Picard numbers ρ, ρ' are equal.*

Proof. We make use of a formula appearing in Shioda [11]

$$\rho = r + 2 + \sum_v (m_v - 1)\ .$$

Here ρ is the Picard number, r is rank of the group of $K(X)$-rational points of the generic fibre E^{gen}, E'^{gen} which is an elliptic curve over $K(X)$, and m_v is the number of irreducible components of the fibres where v runs over the singular fibres. Since E and E' are generically isogeneous r is preserved. As we observed in Theorem 2.6 all fibre types are preserved except possibly I_b, I_b^* $b \geqq 1$. Now a fibre of type I_b $b \geqq 1$ has b components and a fibre of type I_b^* $b \geqq 1$ has $b + 4$. Let b_v be the indices of the type I_{b_v} that occur as v runs over the singular fibres and b_v^* the indices of the type $I_{b_v}^*$. Now $\sum_v b_v + \sum_v b_v^*$ is μ the valence of the functional invariant which is preserved. Also the number of type I_b and the number of type I_b^* are fixed by trace considerations.

We are interested in the invariance of the sum $\sum_v m_v - 1$ where v runs only over singular fibres of types I_b, I_b^* $b \geqq 1$. This becomes

$$\sum_v m_v - 1 + \sum_{v^*} m_v^* - 1$$

where v runs over types I_b $b \geqq 1$ and v^* runs over types I_b^*. By our remarks above this is

$$\begin{aligned}\sum_v b_v - 1 + \sum_{v^*} ((b_{v^*}^* + 4) - 1) &= \sum_v b_v + \sum_{v^*} b_{v^*}^* - \sum_{b \geqq 1} v(I_b) + 3 \sum_{b \geqq 1} v(I_b^*) \\ &= \mu - \sum_{b \geqq 1} v(I_b) + 3 \sum_{b \geqq 1} v(I_b^*)\ .\end{aligned}$$

Now μ is the valence of the functional invariant and so invariant by Corollary 2.5 and the sums are invariant by trace considerations.

Thus the Picard number is invariant.

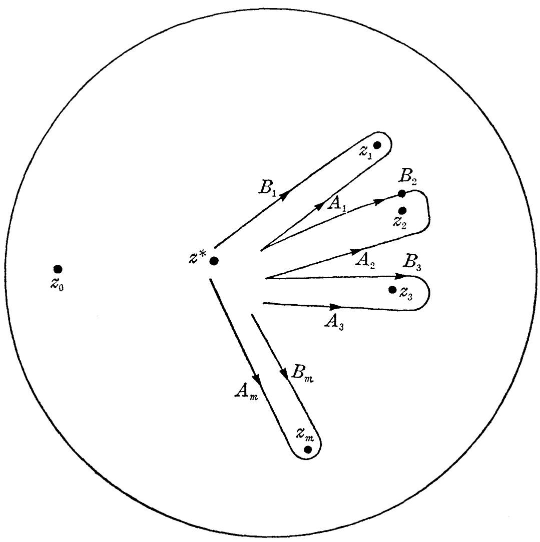
Again, we remark that under generic isogeny fibre types need not be preserved—see example in § 3.

3. Computing monodromy and examples. Let X be a complete smooth curve/C with function field $C(z, w)$. Here w is given as an algebraic function of z by an irreducible polynomial

$$w^n + a_{n-1}(z)w^{n-1} + \cdots + a_0(z) = 0 \ , \quad a_i(z) \in \boldsymbol{C}(z) \ . \tag{3.1}$$

Our purpose will be to give a method for computing the monodromy of a basic surface E/X with functional invariant $\mathcal{J} = z$. We begin with the SK-case, i.e., $\lambda = 1$.

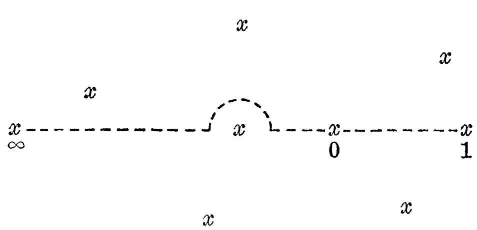
Let $T = \{z_1 = \infty, z_2 = 0, z_3 = 1, z_4, \cdots, z_m\}$ be the branch points of (3.1) on the z-sphere including ∞, 0, 1. Pick a base point z_0 and another point z^*; letting $T^* = T \cup \{z^*\}$. We suppose given slits L_i from z^* to z_i, $i = 1, \cdots, m$. Each slit has "two sides" A_i, B_i oriented to run from z^* to z_i with B_i being the side which maintains the sphere to the left. So $B_1A_1^{-1}B_2A_2^{-1} \cdots B_mA_m^{-1}$ is positively oriented closed curve about z_0 on the sphere.



Let $w_1, \cdots, w_n$ be n distinct function elements at z_0 of w. Assume given permutations $\pi_1, \cdots, \pi_m$ of $\{1, \cdots, n\}$ where analytic continuation of w_k across L_j from B_j to A_j leads to $w_{\pi_j(k)}$. Consider the free group Π on $\pi_1, \cdots, \pi_m$, and fix a function element say w_1 at z_0. Let $X_0 = X - \{\text{all points over } T^*\}$.

Now we look at the z-sphere less T^*.

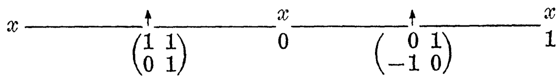
(3.3)



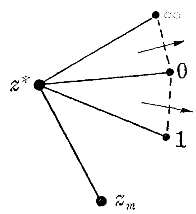
We can slit the z-sphere from 0 to ∞ and 0 to 1 so that the differential equation

$$\frac{d^2f}{dz^2} + \frac{1}{z}\frac{df}{dz} + \frac{((31/144)z - 1/36)}{z^2(z-1)^2}f = 0$$

has single-valued solutions on the remaining part of the z-sphere. Further we can select a branch so that the monodromy across the slits is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as indicated:



The above choices for L_i, A_i, B_i, z_0, z^* and the slits above can all be made so that the picture becomes:



Let $\Pi_{w_1} \subset \Pi$ be all words γ such that $1\gamma = 1$, i.e., the isotropy of 1 under the action of Π on $\{1, \cdots, n\}$. (Note $\pi_1 \cdot \cdots \cdot \pi_m$ acts like the identity.) Now let $x_1 \cdots x_r$ be the points of X over T^* i.e., $X - X_0 = \{x_1, \cdots, x_r\}$. Choose words $\gamma_1, \cdots, \gamma_r$ and $C_1, \cdots, C_g$, $D_1, \cdots, D_g$ in the π_j's which lie in Π_{w_1} and represent a basis for $\pi_1(X_0, w_1/z_0)$. Thus γ_i represents a simple loop about x_i; C_i, D_i's are various cycles and as permutations

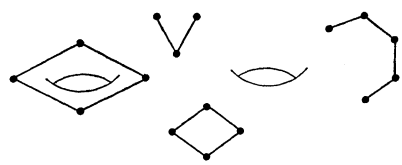
$$\prod_{i=1}^{g} C_i D_i C_i^{-1} D_i^{-1} \prod_{i=1}^{r} \gamma_i = \text{identity} .$$

THEOREM 3.1. *We can select a* $K =$ *basis of solutions* ω_1, ω_2 *of our SK-equation on* X *at* w_1/z_0 *so that* $\mathscr{J} = J(\omega_1/\omega_2)$ *and if* γ_i, C_i, D_i *is* $\pi_{f(1)}^{\varepsilon_1} \cdots \pi_{f(q)}^{\varepsilon_q}$, $\varepsilon_i = \pm 1$, $f_{(i)} \in \{1, \cdots, m\}$, *as a word, then the monodromy matrix for* ω_1, ω_2 *is*:

$$M_{f(1)}^{\varepsilon_1} \cdots M_{f(q)}^{\varepsilon_q}$$

where $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $M_4 = \cdots = M_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Obvious. Essentially we have pulled our single-valued branch of the solutions back to a single-valued branch on X minus some slits:



We are also able to keep track of how these solutions of our SK-equation change as we cross a slit.

Now adding λ to get the general K-equation amounts to changing sign around a path if λ changes sign.

We now apply this and/or similar techniques to calculate certain global monodromy representations. Let us work on the z-sphere, taking $\mathscr{J} = 1/(1 - z^{12k})$. We take the SK-case so that the only singular fibres are of type I_1 at the $12k$-roots of 1. Note $\mathscr{J} = 0$ at $z = \infty$ and $\mathscr{J} = 1$ at $z = 0$ but the fibres are good. On $\boldsymbol{P}_C^1 - \{\infty\}$ the family is

$$y^2 = 4x^3 - \frac{27}{z^{12k}}x - \frac{27}{z^{12k}}\ .$$

Note this appears to be bad at $z = 0$ but taking

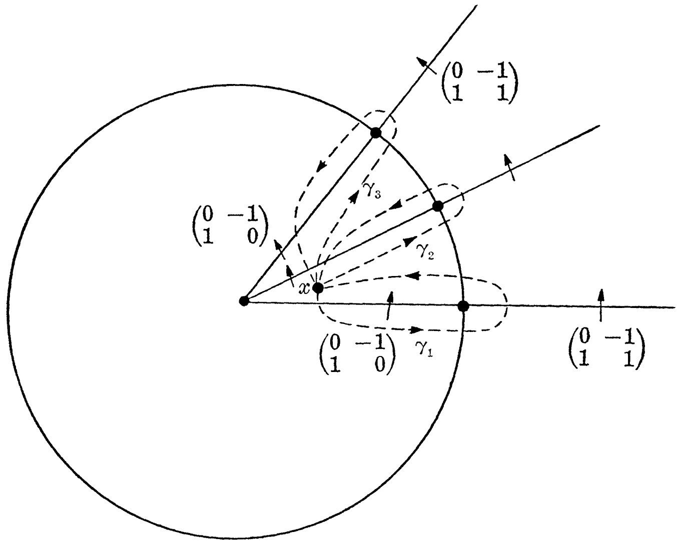
$$g = \frac{z^{3k}}{3} \quad \text{then} \quad g^4\frac{27}{z^{12k}} = 3 \qquad \text{and} \qquad g^6\frac{27}{z^{12}} = z^{6k}\ .$$

So

$$y^2 = 4x^3 - 3x - z^{6k}$$

also describes the family (Sasai [10]). As usual we take a single-valued branch on the $\mathscr{J}$-sphere and lift:

As $\mathscr{J} = 1/(1 - z^{12k})$, the path from 0 to ∞ in the $\mathscr{J}$-plane lifts to the radial lines from ∞ to a $12k$-root of unity, and the path from 1 to ∞ lifts to the radial lines from 0 to a $12k$-root of unity. The picture is shown on the following page.



Pick a base point x as marked above. Let $\gamma_1 \cdots \gamma_{12k}$ be the loops pictured. Then the representation is given by

$$\gamma_1 \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\gamma_2 \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$\vdots$$
$$\gamma_{2n-1} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\gamma_{2n} \longrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

(The reader will observe that this agrees with Sasai who obtains the result in a different manner (Sasai [10]).)

We now consider the case of $A_{(\mathscr{J},\lambda)}$ where $\mathscr{J}$ is unramified over 0, 1, ∞, i.e., $X \xrightarrow{\mathscr{J}} \boldsymbol{P}^1_{\boldsymbol{C}}$ is unramified over 0, 1, ∞. This is in some sense the general case. The reader should have no trouble seeing:

THEOREM 3.2. *Let X be any curve and E/X any elliptic surface with functional invariant $\mathscr{J}$ unramified over* 0, 1, ∞. *Then the global monodromy group $\Gamma \subset \mathrm{SL}_2(\boldsymbol{Z})$ is in fact $\mathrm{SL}_2(\boldsymbol{Z})$.*

The reader should refer to Corollary 1.7 which shows that such an E/X then has no $K(X)$-rational division point on the generic fibre.

Further, as is obvious, one can choose cycles on X so that the monodromy is $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ around all cycles.

We present an example which will give us a case where the isogeny phenomenon occurs. (See Ince [3] for details on differential equations with regular singular points, local monodromy, exponents, etc.) Let

$$\mathcal{J} = \frac{4}{27}\frac{(1-z+z^2)^3}{z^2(1-z)^2}$$

on the z-sphere. $\mathcal{J}$ has a double pole at 0, 1, ∞, a triple zero at $e^{2\pi i/6}$ and $e^{-2\pi i/6}$, and a double one at -1, 2, 1/2. An easy calculation yields that the SK-equation for this $\mathcal{J}$ has exponents $\pm 1/2$ at $e^{2\pi i/6}$, $e^{-2\pi i/6}$ and 1/2, 3/2 at -1, 2, 1/2. At 0, 1, ∞ the exponents are 0, 0. Let λ^2 have divisor:

$$1(\infty) + 1(e^{2\pi i/6}) + 1(e^{-2\pi i/6}) - 1(-1) - 1(2) - 1(1/2)\ .$$

The differential equation $\Lambda_{(\mathcal{J},\lambda)}$ with λ, $\mathcal{J}$ as above is holomorphic at $e^{2\pi i/6}$, $e^{-2\pi i/6}$, -1, 2, 1/2 with exponents 0, 0 at 0 and 1 and exponents 1/2, 1/2 at ∞. The equation is therefore the hypergeometric equation:

$$\frac{d^2f}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1}\right)\frac{df}{dz} + \left(\frac{1/4}{z(z-1)}\right)f = 0 \tag{1.1}$$

which is that for ${}_2F_1(1/2, 1/2; 1; z)$. The group is easily seen to be $\Gamma(2)$ in either case (with or without λ). Recall $\Gamma(2) = \left\{M \in \mathrm{SL}_2(\boldsymbol{Z}) \,\middle|\, M \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \bmod 2\right\}$. Let $\Gamma_0(4) = \left\{M \in \mathrm{SL}_2(\boldsymbol{Z}) \,\middle|\, M \equiv \begin{pmatrix}*&*\\0&*\end{pmatrix} \bmod 4\right\}$. Both $\Gamma(2)$ and $\Gamma_0(4)$ are of index 6 in $\mathrm{SL}_2(\boldsymbol{Z})$ and $\begin{pmatrix}1/2&0\\0&1\end{pmatrix}\Gamma(2)\begin{pmatrix}2&0\\0&1\end{pmatrix} = \Gamma_0(4)$. As above take

$$\mathcal{J} = \frac{4}{27}\frac{(1-z+z^2)^3}{z^2(1-z)^2}$$

but instead take

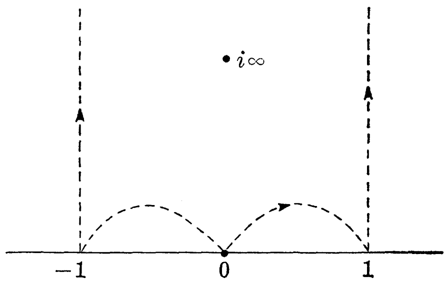
$$\lambda^2 = \frac{z - e^{2\pi i/6}}{z - e^{-2\pi i/6}}\ .$$

The K-equation $\Lambda = \Lambda_{(\mathcal{J},\lambda)}$ has global group $\Gamma(2)$. The fibres are:

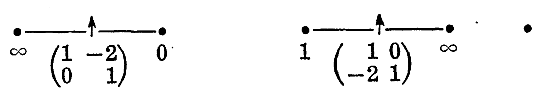
at	∞	type	I_2	at	-1	type	I_0^*
at	0	type	I_2	at	2	type	I_0^*
at	1	type	I_2	at	1/2	type	I_0^*

also the valence of $\mathcal{J}$ is 6. Applying the well-known formulas

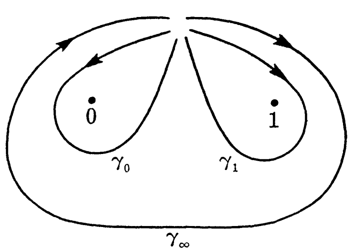
(Kodaira [12]) yields $p_a = 1$, $p_g = 1$, $q = 0$, that is, the surface E/X (functional invariant $\mathscr{J}$) is $K3$. We need to compute the global representation. We first compute the monodromy of Equation (1.1). We consider the usual fundamental domain for $\Gamma(2)$;



The Legendre function λ maps this region to the z-sphere with $\lambda(i\infty) \to 0$, $\lambda(1) \to \infty$, $\lambda(0) \to 1$ sending the imaginary axis $i\infty$ to 0 to the slit 0 to 1 on the real axis, the arc 0 to 1 to the slit 1 to ∞ on the real axis, and finally the line $\operatorname{Re}\tau = 1$ from 1 to $i\infty$ to the imaginary axis ∞ to 0 (Robert [9]). Thus if we slit the z-sphere along the negative real axis and from 1 to ∞, we will be able to find a branch of the quotient of solutions with values in this fundamental domain. Continuation across the slit is obviously



The trace is 2 (not -2) in both cases as the exponents at 0 and 1 are 0, 0. Choosing basis:



yields the representation

$$\gamma_0 \longrightarrow \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$$

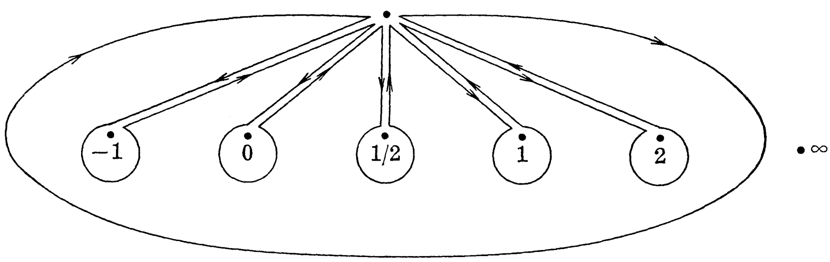
$$\gamma_1 \longrightarrow \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}$$

$$\gamma_\infty \longrightarrow \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -2\\ 2 & -3\end{pmatrix}$$

and $\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&-2\\2&-3\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. This is the monodromy of Equation (1.1). Note at ∞ the exponents are 1/2, 1/2 which corresponds to trace -2 for $\begin{pmatrix}1&-2\\2&-3\end{pmatrix}$. Finally putting in λ (not to be confused with Legendre's λ above) gives

$$\gamma_0 \longrightarrow \begin{pmatrix}1&2\\0&1\end{pmatrix}$$
$$\gamma_1 \longrightarrow \begin{pmatrix}1&0\\-2&1\end{pmatrix}$$
$$\gamma_\infty \longrightarrow \begin{pmatrix}-1&2\\-2&3\end{pmatrix}$$
$$\gamma_{-1},\ \gamma_2,\ \gamma_{1/2} \longrightarrow \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

which is the desired representation where the basis is:



Now if ω_1, ω_2 is the K-basis of $\Lambda=\Lambda_{(\mathscr{J},\lambda)}$ giving this representation ($\mathscr{J}=J(\omega_1/\omega_2)$), then because $\begin{pmatrix}1/2&0\\0&1\end{pmatrix}\Gamma(2)\begin{pmatrix}2&0\\0&1\end{pmatrix}\subset \mathrm{SL}_2(\boldsymbol{Z})$ we have $\begin{pmatrix}\tilde\omega_1\\\tilde\omega_2\end{pmatrix}=\begin{pmatrix}1/2&0\\0&1\end{pmatrix}\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix}$ also a K-basis. This new basis gives another basic elliptic surface $\tilde E/X$ with representation:

$$\gamma_0 \longrightarrow \begin{pmatrix}1&1\\0&1\end{pmatrix} \quad \text{fibre type } I_1$$
$$\gamma_1 \longrightarrow \begin{pmatrix}1&0\\-4&1\end{pmatrix} \quad \text{fibre type } I_4$$
$$\gamma_\infty \longrightarrow \begin{pmatrix}-1&1\\-4&3\end{pmatrix} \quad \text{fibre type } I_1$$
$$\gamma_{-1},\ \gamma_2,\ \gamma_{1/2} \longrightarrow \begin{pmatrix}-1&0\\0&-1\end{pmatrix} \quad \text{fibre type } I_0^* .$$

Note valence $\tilde{\mathscr{J}}=J(\tilde\omega_1/\tilde\omega_2)$ is 6. Thus $\tilde E/X$ is also $K3$. We have a map $E\to\tilde E$ of degree 2 which is a fibre by fibre isogeny almost everywhere. Further the map does not extend since the number of irreducible components in the singular fibres do not agree. We remark that this also proves that E^{gen} (as well as $\tilde{E}^{\text{gen}}$) as an elliptic curve over $K(X)$ has a division point of order 2 rational over $K(X)$.

The reader should note that when the monodromy is all of $\mathrm{SL}_2(\boldsymbol{Z})$ this isogeny phenomenon does not occur and we can conclude that there are no $K(X)$-rational division points on E^{gen} and the isogeny class of E^{gen} over $K(X)$ contains only E^{gen}.

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