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APPROXIMATIONS TO REAL ALGEBRAIC NUMBERS BY ALGEBRAIC NUMBERS OF SMALLER DEGREE

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If β is an algebraic number of degree $n+1$ then the number of solutions α, with α algebraic of degree at most n, to the inequalities

$$(1) \qquad |\beta - \alpha| < \varphi(H(\alpha)) , \quad 1 \leqq H(\alpha) \leqq B$$

is studied using methods developed by Schmidt and Adams for counting solutions to inequalities involving linear forms. In (1) $H(\alpha)$ is a height function which differes slightly from the usual height and φ is a function which decreases to zero.

If $\varphi(y)y^{n+1} \to \infty$ as $y \to \infty$ then the number of solutions is given as an integral plus an error term. If $\varphi(y)y^{n+1}$ is constant then the number of solutions is either bounded or asymptotic to $C \log B$ for some constant C.

1. Introduction and statement of theorems. In this paper the methods of Schmidt [5] and Adams [1, 2], for counting solutions to inequalities involving linear forms with algebraic coefficients will be adapted to prove two theorems on the approximation of algebraic numbers by algebraic numbers of smaller degree.

In what follows β will denote a fixed real algebraic number of degree $n+1$, $\varphi(y)$ a decreasing function which is positive for $y > 0$ and which tends to zero, and $P(T)$ a polynomial of degree $k \leqq n$ with integer coefficients. If

$$P(T) = q_k T^k + \cdots + q_1 T + q_0$$

then the height of P is defined by $H(P) = \max(|q_1|, \cdots, |q_k|)$. This is not the usual definition of height, but it is more convenient for technical reasons. In the situation we are considering we will have q_0 uniquely determined by $q_1, \cdots, q_k$ and $|q_0| = O(H(P))$. If α is an algebraic number whose minimal polynomial over $\boldsymbol{Z}$ is $P(T)$ then the height of α is defined by $H(\alpha) = H(P)$. The number of solutions to

$$(1) \qquad |\beta - \alpha| < \varphi(H(\alpha)) , \quad 1 \leqq H(\alpha) \leqq B$$

with α algebraic of degree at most n will be denoted by $\Gamma(\varphi, B)$. There are two cases in which we will find $\Gamma(\varphi, \beta)$.

THEOREM 1. *Assume that, as $y \to \infty$, $\chi(y) = \varphi(y)y^{n+1} \to \infty$ and $\varphi(y)y^{n+\delta} \to 0$ for some $\delta > 0$. If $n \geqq 2$ we have*

$$\Gamma(\varphi, B) = C\int_{\max|y_i|\leqq B} \varphi(\max|y_i|) \times |ny_n\beta^{n-1} + (n-1)y_{n-1}\beta^{n-2} + \cdots + y_1|\, dy_1 \cdots dy_n + O\left(\int_1^B y^n\varphi(y)\chi(y)^{-1/(n+1)}dy\right).$$

In the case $\chi(y)$ is constant we get:

THEOREM 2. *There exists a constant C_0 such that if $C > C_0$ then there is a $C' > 0$ with $\Gamma(Cy^{-(n+1)}, B) \sim C' \log B$ and if $C < C_0$ then $\Gamma(Cy^{-(n+1)}, B) = O(1)$.*

2. A related problem. What we will actually count is $\Delta(\varphi, B)$, which is the number of primitive polynomials of degree at most n such that

$$(2) \qquad 0 < P(\beta) < |P'(\beta)|\varphi(H(P)), \quad 1 \leqq H(P) \leqq B$$

wxere $P'(T)$ is the derivative of $P(T)$. In order to get the theorems of the previous section we now prove:

THEOREM 3. *If $\chi(y)$ is bounded below and $\varphi(y)y^{n+\delta} \to 0$ for some $\delta > 0$ then*

$$\Delta((1 - Cy^{-(1+\delta)})\varphi, B) \leqq \Gamma(\varphi, B) \leqq \Delta((1 + Cy^{-(1+\delta)})\varphi, B)$$

where the constant C depends on β and δ.

Before beginning the proof we set up some notation. If P is any polynomial with $P(\alpha) = 0$ then we let $\alpha = \alpha_1$ and denote the other roots of P by $\alpha_2, \cdots, \alpha_k$. We use $f \ll g$ to indicate $f < Cg$ where C may depend on β and δ. C will denote a constant which will not always be the same each time it appears.

We will use the following facts:

$$(3) \qquad |\alpha - \alpha_i| - |\beta - \alpha| \leqq |\beta - \alpha_i| \leqq |\alpha - \alpha_i| + |\beta - \alpha| .$$

If $P(\alpha) = 0$ then (1) is equivalent to

$$(4) \qquad |P(\beta)| \leqq |q_k(\beta - \alpha_2) \cdots (\beta - \alpha_k)|\varphi(H(\alpha)) .$$

(5) Since φ is bounded above there are only a finite number of α satisfying (1) with $H(\alpha)$ less than a given bound.

We now prove some lemmas.

LEMMA 1. *If $P(\alpha) = 0$ then $|\alpha| \gg H(P)^{-1}$.*

Proof. If $|\alpha| \geqq 1$ this is clear. If not, since $q_k\alpha^k + \cdots + q_0 = 0$ we get

$$|q_0\alpha^{-1}| = |q_k\alpha^{k-1} + \cdots + q_1| \ll H(P)$$

which proves the lemma.

LEMMA 2. *If $P(\alpha) = 0$ and $|\beta - \alpha|$ is bounded then*

$$|\alpha_i| \ll H(P)\ ,\quad 1 \leqq i \leqq k\ .$$

Proof. If $|\alpha_i| \leqq 1$ the result is clear. If not, since $P(\alpha) = 0$ and $|\beta - \alpha|$ is bounded, then $|q_0| \ll H(P)$ and so $|q_k\alpha_i| = |q_{k-1} + \cdots + q_0\alpha^{-(k+1)}| \ll H(P)$ as desired.

LEMMA 3. *If $|\beta - \alpha|$ is bounded and α, α_i, α_j are roots of P with $\alpha_i \neq \alpha_j$ then*

$$|\alpha_i - \alpha_j| \gg H(P)^{-(k-1)}\ .$$

Proof. Since the proof is similar to Lemma 1 of [4] we only sketch it. The polynomial

$$Q(T) = q_k^{2k-2}\prod_{i<j}\{T - (\alpha_i - \alpha_j)^2\}$$

has integer coefficients and roots $(\alpha_i - \alpha_j)^2$, so by Lemma 1 we have $|\alpha_i - \alpha_j|^2 \gg H(Q)^{-1}$. By examining the coefficients of Q we see $H(Q) \gg H(P)^{2k-2}$ which gives the desired result.

LEMMA 4. (1) *implies*

$$0 < P(\beta) < |P'(\beta)|\varphi(h(P))(1 + CH(P)^{-(1+\delta)})$$

where P is the minimal polynomial for α over $\boldsymbol{Z}$, which is chosen so that $P(\beta) > 0$.

Proof. From (1), (3) and Lemma 3 we get

$$\left|\frac{\beta - \alpha}{\beta - \alpha_i}\right| \ll \frac{\varphi(H(\alpha))}{H(\alpha)^{-(k-1)} - \varphi(H(\alpha))} = \left(\frac{H(\alpha)^{1+\delta}}{H(\alpha)^{n+\delta}\varphi(H(\alpha))} - 1\right)^{-1}$$
$$\ll \varphi(H(\alpha))^{-(1+\delta)}$$

since $H(\alpha)^{n+\delta}\varphi(H(\alpha))$ is bounded above. Thus

$$\left|\frac{x_k(\beta - \alpha)\cdots(\beta - \alpha_k)}{P'(\beta)}\right| = \left|1 + \sum_{i\neq 1}\frac{\beta - \alpha}{\beta - \alpha_i}\right| \leqq \frac{1}{1 - CH(\alpha)^{-(1+\delta)}}$$
$$\leqq 1 + CH(\alpha)^{-(1+\delta)}\ .$$

Since (1) is equivalent to (4) this proves the lemma.

From the preceding lemmas we see that each solution to (1) gives rise to a solution to (2) with $\varphi(H(\alpha))$ replaced by $(1 + CH(P)^{-(1+\delta)})\varphi(H(P))$, if $H(P)$ is large. To show

$$\Gamma(\varphi, B) \leqq \Delta((1 + Cy^{-(1+\delta)})\varphi, B) + O(1)$$

we will show that, if $H(\alpha)$ is large, all these solutions are distinct, i.e., two conjugates cannot both satisfy (1). Assume this is not true, then $P(\alpha_1) = P(\alpha_2) = 0$ where α_1 and α_2 satisfy (1). Thus

$$|P(\beta)| \leqq |x_k|\varphi(H(P))^2 \prod_{i>2} |\beta - \alpha_i|$$

which by Lemma 2 implies

$$|P(\beta)| \ll \varphi(H(P))^2 H(P)^{n-1} .$$

Since β is algebraic of degree $n + 1$ we also have

$$|P(\beta)| \gg H(P)^{-n}$$

because the norm of $P(\beta)$ is bounded below. Comparing these two bounds gives

$$1 \ll \varphi(H(P))H(P)^{n-1/2}$$

which contradicts $\varphi(y)y^{n+\delta} \to 0$ and so gives the desired inequality.

For the other inequality of Theorem 3 we start with a polynomial P of degree $k \leqq n$ and let $\alpha = \alpha_1$ be a root of P nearest to β. Then (2) is equivalent to

$$|\beta - \alpha| \leqq \left|1 + \sum_{i \neq 1} \frac{\beta - \alpha}{\beta - \alpha_i}\right| \varphi(H(P))$$

which implies $|\beta - \alpha|$ is bounded. Thus (3) and $\varphi(y)y^{n-\delta} \to 0$ imply as above

$$\left|1 + \sum_{i \neq 1} \frac{\beta - \alpha}{\beta - \alpha_i}\right| \leqq 1 + CH(P)^{-(1+\delta)} .$$

Also using $|\beta - \alpha|$ bounded we can show that $H(P)$ can be assumed to be large.

LEMMA 5. *If α is as above then no other root of P can satisfy $|\beta - \alpha_i| \ll \varphi(H(P))$ for $H(P)$ large, in particular $\alpha \neq \alpha_i$ for $i \neq 1$.*

Proof. As above we can use $|P(\beta)|$ bounded below and Lemma 2 to get $1 \ll \varphi(H(P))H(P)^{n-1/2}$ which is a contradiction of $H(P)$ is large enough.

We now have that each solution to (2) gives at least one solution to

$$|\beta - \alpha| \leqq (1 + CH(P))^{-(1+\delta)}\varphi(H(P)) .$$

We will now show that we get only one solution and that $H(P)$ can be replaced by $H(\alpha)$. This follows easily from:

LEMMA 6. *If $H(P)$ is large then* (2) *implies that the degree of α is n and so P is the minimal polynomial for α. Since $P(\beta) > 0$ there is only one P giving rise to α.*

Proof. Assume the degree of α is at most $n-1$, so by a theorem of Schmidt [6, Theorem 2], $|\beta - \alpha| \gg H(\alpha)^{-(n+\delta)}$. If $Q(T)$ is the minimal polynomial for α then $P(\alpha) = 0$ implies $Q \mid P$ and so

$$H(Q) = H(\alpha) \ll H(P) \quad [3, \text{ page } 14] .$$

Thus, with (2) and Schmidt's Theorem, imply

$$H(\alpha)^{-(n+\delta)} \ll \varphi(H(P)) \ll \varphi(CH(\alpha))$$

which contradicts $\varphi(y)y^{n+\delta} \to 0$. Finally we note that, since P is primitive and $P(\beta) > 0$, P is unique.

To get the desired inequality we let

$$\varphi'(y) = (1 + Cy^{-(1+\delta)})\varphi(y) .$$

What we have shown above is

$$\Delta(\varphi, B) \leqq \Gamma((1 + Cy^{-(1+\delta)})\varphi, B) + O(1) .$$

This implies

$$\Delta((1 - Cy^{-(1+\delta)})\varphi', B) \leqq \Gamma(\varphi', B) + O(1)$$

which finishes the proof of Theorem 3.

3. The proof of Theorem 1. Instead of Theorem 1 we prove the following, from which Theorem 1 easily follows.

THEOREM 1′. *Let $1, \beta_1, \cdots, \beta_n$ be a basis of a real number field K of degree $n+1$. Assume $\varphi(y)$ is decreasing and $\chi(y) = \varphi(y)y^{n+1}$ tends to infinity. Let $L(y_1, \cdots, y_n)$ be a linear form and let $N(B)$ be the number of solutions to*

$$(6) \qquad \begin{aligned} 0 &< q_1\beta_1 + q_2\beta_2 + \cdots + q_n\beta_n - p \\ &< \varphi(\max q_i)|L(q_1, \cdots, q_n)| , \quad 1 \leqq q_i \leqq B \end{aligned}$$

where $p, q_1, \cdots, q_n \in \boldsymbol{Z}$ are relatively prime. If $n \geqq 2$

$$\begin{aligned} N(B) = C\int_{1 \leqq y_i \leqq B} &\varphi(\max y_i)|L(y_1, \cdots, y_n)| dy_1 \cdots dy_n \\ &+ O\Big(\int_1^B y^n\varphi(y)\chi(2y)^{1/(n+1)}\, dy\Big) . \end{aligned}$$

The proof is similar to that of Schmidt [5] and so some of the details will be omitted. For any $\xi \in K$ we let $\xi = \xi^{(0)}$ and denote its conjugates by $\xi^{(i)}$, $1 \leqq i \leqq n$. Let $r + 2s = n$ and let

$$\xi^{*(i)} = \begin{cases} \xi^{(i)} & 0 \leqq i \leqq r \\ \operatorname{Re}(\xi^{(i)}) & r+1 \leqq i \leqq r+s \\ \operatorname{Im}(\xi^{(i)}) & r+s+1 \leqq i \leqq n \end{cases}$$

where $\xi^{(i)}$, $0 \leqq i \leqq r$, are the real conjugates of ξ and if $r+1 \leqq i \leqq r+s$ then $\xi^{(i)} = \bar{\xi}^{(i+s)}$. If M is a module in K then $\Lambda(M) = \{(\xi^{*(0)}, \cdots, \xi^{*(n)})\}$ is a lattice. Let $(c_0, \cdots, c_n)\Lambda$ be the set of all $(\mu_0, \cdots, \mu_n)$ where

$$\mu_i = \begin{cases} c_i\lambda_i & 0 \leqq i \leqq r \\ c_i\lambda_i - c_{i+s}\lambda_{i+s} & r+1 \leqq i \leqq r+s \\ c_i\lambda_{i-s} - c_{i+s}\lambda_i & r+s+1 \leqq i \leqq n \end{cases}$$

where $(\lambda_0, \cdots, \lambda_n) \in \Lambda$. If $\theta \in K$, let $\theta\Lambda = (\theta^{*(0)}, \cdots, \theta^{*(n)})\Lambda$ so $\theta\Lambda(M) = \Lambda(\theta M)$.

We first determine $M(H)$, the number of relatively prime solutions to (6) with $Hf^{-1} < \max q_i \leqq H$ where $f > 1$ is fixed. Let $\lambda = q_1\beta_1 + \cdots + q_n\beta_n - p$ and let M be the module consisting of all such λ, then $\Lambda(M) = \{\lambda^{*(0)}, \cdots, \lambda^{*(n)}\}$. Let A_1 be the linear transformation $A_1(p, q_1, \cdots, q_n) = (\lambda, q_1, \cdots, q_n)$ so $\det(A_1) = 1$. Let $A_2 = (a_{ij})$, $0 \leqq i, j \leqq n$, where

$$a_{00} = 1\,, \qquad a_{0j} = 0\,, \qquad a_{i0} = 0 \qquad \text{and}$$
$$a_{ij} = \begin{cases} \beta_j^{(i)} - \beta_j & 1 \leqq i \leqq r\,, \quad j \neq 0 \\ \operatorname{Re}(\beta_j^{(i)} - \beta_j) & r+1 \leqq i \leqq r+s\,, \quad j \neq 0 \\ \operatorname{Im}(\beta_j^{(i)}) & r+s+1 \leqq i \leqq n\,, \quad j \neq 0\,. \end{cases}$$

Let $\bar{A}_2 = (a_{ij})$, $1 \leqq i, j \leqq n$, so $\det(\bar{A}_2) = \det(A_2) = \pm\det(\Lambda(M))$ which is nonzero. Let

$$\begin{aligned} \bar{A}_2(q_1, \cdots, q_n)^t &= (\lambda^{*(1)} - \lambda^{*(0)}, \cdots, \lambda^{*(r+s)} - \lambda^{*(0)}, \lambda^{*(r+s+1)}, \cdots, \lambda^{*(n)}) \\ &= \bar{\lambda}\,. \end{aligned}$$

From (6) we see that we wish to count the number of primitive points of $\Lambda(n)$ with

$$0 \leqq \lambda^{*(0)} \leqq \varphi(\max \bar{A}_2^{-1}\bar{\lambda})|L(\bar{A}_2^{-1}\bar{\lambda})|$$
$$\frac{H}{f} < \max(\bar{A}_2^{-1}\bar{\lambda}) \leqq H\,.$$

We call the region of $\boldsymbol{R}^{n+1}$ defined by these inequalities D. Its volume is given by

$$(7)\quad \frac{V(D)}{\det(\Lambda(M))} = \int_{H/F<\max y_i \leqq H;\, y_i>0} \varphi(\max y_i)|L(y_1, \cdots, y_n)|\, dy_1 \cdots dy_n .$$

The lattice will be transformed as in [5].

LEMMA 7. *Let M be a module in K. Then there are $\theta_1, \cdots, \theta_{r+s}$ in K with $\theta_j^{(i)} > 0$, $1 \leqq i, j \leqq r+s$ with $\det(\log|\theta_j^{(i)}|) \neq 0$ and $\theta_j \Lambda(M) = \Lambda(M)$.*

Proof. See [5].

We let $u = \varphi(H)^{-1/(n+1)}$. Define $\mu_1, \cdots, \mu_{r+s}$ by

$$u = |\theta_1^{(i)}|^{\mu_1} \cdots |\theta_{r+s}^{(i)}|^{\mu_{r+s}}, \quad 1 \leqq i \leqq r+s$$

and m_i by $|m_i - \mu_i| \leqq 1/2$. So $\kappa = \kappa^{(0)} = \theta_1^{m_1} \cdots \theta_{r+s}^{m_{r+s}}$ is a unit of K and there exists a c such that

$$cu \leqq \kappa^{(i)} \leqq cu, \quad 1 \leqq i \leqq n$$
$$c^{-n}u^{-n} \leqq \kappa^{(0)} \leqq c^n u^{-n} .$$

Define c_i by $\kappa^{*(0)} = u^{-n}c_0$ and $\kappa^{*(i)} = uc_i$, $1 \leqq i \leqq n$. So $c_0 \cdots c_r(c_{r+1}^2 + c_{r+s+1}^2) \cdots (c_{r+s}^2 + c_n^2) = 1$ and $|c_i| \leqq c^n$. Let $\Lambda' = (c_0, \cdots, c_n)\Lambda(M)$ then $\det \Lambda' = \det \Lambda(M)$ and $\kappa\Lambda(M) = \Lambda(M)$ is the set of all $(u^{-n}\nu_0, u\nu_1, \cdots, u\nu_n)$ where $(\nu_0, \cdots, \nu_n) \in \Lambda'$. We therefore wish to count the number of primitive lattice points of Λ' with

$$(8)\quad \begin{cases} 0 < \nu_0 u^{-n} < (\max \bar{A}_2^{-1}\bar{\lambda})L(\bar{A}_2^{-1}\bar{\lambda}) \\ Hf^{-1} < \max \bar{A}^{-1}\bar{\lambda} \leqq H \end{cases}$$

where $\bar{\lambda} = (u\nu_1 - u^{-n}\nu_0, \cdots, u\nu_n - u^{-n}\nu_0)$. Let $\bar{A}_2^{-1}\bar{\lambda} = u\bar{\gamma} - u^{-n}\nu_0\bar{\gamma}_0$ where $\bar{\gamma}, \bar{\gamma}_0 \in \boldsymbol{R}^n$. Since $u^{-n}\nu_0$ is bounded, if we count the number of solutions to

$$(9)\quad \begin{cases} 0 < \nu_0 u^{-n} < \varphi(u\bar{\gamma})|L(u\bar{\gamma})| \\ Hf^{-1} < \max(u\bar{\gamma}) \leqq H \end{cases}$$

instead of (8) we will get an error term which is a constant times the volume of the boundary of the region given by (9), which can be absorbed into the error term we will get. The actual counting is done in the next lemma.

LEMMA 8. *Let Λ be a lattice, D a bounded set in $\boldsymbol{R}^{n+1}$, ε the minimum diameter ef all fundamental parallelepipeds of Λ and $D(\varepsilon)$ the set of all points with distance at most ε from the boundary of D. If $n \geqq 2$ the number of primitive lattice points of D is given by*

$$\Big(\sum_{k=1}^{\infty}\frac{\mu(k)}{k^{n+1}}\Big)\frac{\mathrm{Vol}\,(D)}{\det \Lambda} + O(\mathrm{Vol}\,(D(\varepsilon))) .$$

Proof. The proof follows easily from Lemma 4 in [5].

We now let $\pi(y) = y\varphi(y)^{1/n+1}$ and from (9) we can check that $\mathrm{Vol}\,(D(\varepsilon)) = O(\pi(H)^n)$ and

$$\pi(H)^n = O\Big(\int_{H/f^2}^{H/f} y^n\varphi(y)\chi(f^2y)^{-1/(n+1)}\,dy\Big) .$$

If $H_0 > f^2$ is fixed and if $H_0 f < B$ we get

$$N(B) = M(B) + M(B/f) + M(B/f^2) + \cdots + M(H_0) + O(1) .$$

From Lemma 8 and (7) we then get

$$N(B) = C\int_{1\leqq y_i\leqq B}\varphi(\max y_i)|L(y_1, \cdots, y_n)|\,dy_1, \cdots, dy_n + O\Big(\int_1^{B/f} y^n\varphi(y)\chi(f^2y)^{-1/(n+1)}\,dy\Big)$$

with $C = \sum_{k=1}^{\infty}\mu(k)/k^{n+1}$ and this proves Theorem 1$'$.

To prove Theorem 1 we must count solutions for each possible choice of sign for $(p, q_1, \cdots, q_n)$ which can be done by letting $\beta_i = \pm\beta^i$. We must also check that the error terms from Theorems 1$'$ and 3 give the desired error.

4. The proof of Theorem 2.

Instead of Theorem 2 we prove the following from which Theorem 2 follows.

THEOREM 2$'$. *Let* $1, \beta_1, \cdots, \beta_n$ *be as in* § 3. *The number of solutions to*

$$(10)\qquad 0 < q_1\beta_1 + \cdots + q_n\beta_n - p < \Big(\sum_{i=1}^{n} c_iq_i\Big)(\max q_i)^{-(n+1)} ,\quad 1 \leqq q_i \leqq B$$

where $c_i \in \boldsymbol{R}$, *is either* $O(1)$ *or asymptotic to* $C\log B$, *for some* $C > 0$.

The proof of the theorem is similar to those of Adams in [1, 2] and so it will only be sketched. Let $\lambda = q_1\beta_1 + q_2\beta_2 + \cdots + q_n\beta_n - p$, K, M be as before. Let θ be the associated order, i.e., $\theta = \{\alpha \in K: \alpha M \subset M\}$. By Dirichlet's Unit Theorem there exists units $\zeta_1, \cdots, \zeta_{r+s} > 1$ such that the unit group is $U = \{\pm\zeta_1^{\nu_1}\zeta_2^{\nu_2}\cdots\zeta_{r+1}^{\nu_{r+1}}\} = \{\pm\zeta_0^{\nu}\}$ where $\nu = (\nu_1, \cdots, \nu_{r+s})$. We say λ_1 is equivalent to λ_2 if $\lambda_1 = \zeta\lambda_2$ for some $\zeta \in U$. This is an equivalence relation. We note that the $\lambda \in M$ correspond to the possible solutions of (10). As in [2] it is easy to see that we need only count the number of solutions in a fixed equivalence class and we let $\lambda_\nu = \zeta_0^{\nu}\lambda_0$ with $\lambda_0 > 0$, since by (10) we must have $\lambda_\nu > 0$. Write

$$\lambda_\nu = q_{1\nu}\beta_1 + \cdots + q_{n\nu}\beta_n - p_\nu$$

and let $\delta_{i\nu} = q_{i\nu}\lambda_\nu^{1/n}$. If $\max q_{i\nu}$ is large we can assume $q_{i\nu} \neq 0$, for all i, since otherwise we would contradict Schmidt's theorem [6]. We have (10) is equivalent to

$$0 < \delta_{i\nu}^n < \min_j \left(\sum_{k=1}^n c_k \delta_{k\nu} \delta_{j\nu}^{-1} \right), \quad 1 \leqq q_{i\nu} \leqq B . \tag{11}$$

Let R be the region defined by (11), then one can check that R is bounded. Let A_3 be the matrix $(\beta_j^{(i)} - \beta_j)$, $1 \leqq i, j \leqq n$. A_3 is nonsingular since $(\det A_3)^2$ is the discriminant of $1, \beta_1, \cdots, \beta_n$. Define $A_4 = (a_{ij})$ by

$$a_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ \lambda_0^{-1/n} \lambda_0^{(i)-1} & \text{if} \quad i = j . \end{cases}$$

Then

$$\begin{aligned} P_\nu &= A_4 A_3(\delta_{1\nu}, \cdots, \delta_{n\nu}) \\ &= (\zeta^{\nu/n}\zeta^{(1)\nu} - \lambda_\nu \lambda_0^{(1)^{-1}}, \cdots, \zeta^{\nu/n}\zeta^{(n)\nu} - \lambda_\nu \lambda_0^{(n)^{-1}}) \end{aligned}$$

is near

$$Q_\nu = (\zeta^{\nu/n}\zeta^{(1)\nu}, \cdots, \zeta^{\nu/n}\zeta^{(n)\nu}) .$$

Since the last s coordinates of Q_ν are the complex conjugates of the preceding s coordinates we can omit them to get a nonsingular transformation

$$\boldsymbol{R}^n \longrightarrow \boldsymbol{R}^r \times \boldsymbol{C}^{2s} \longrightarrow \boldsymbol{R}^r \times \boldsymbol{C}^s \longrightarrow \boldsymbol{R}^n .$$

We can now apply the uniform distribution theory to count the number of Q_ν in the region $R' = A_4 A_3 R$. This is done exactly as in Lemma 4 of [2] and so we omit it. The result is:

LEMMA 9. *The number of Q_ν in R' with $1 \leqq \nu_1 \leqq N$ is asymptotic to CN or $O(1)$ and the same is true for $1 \leqq -\nu_1 \leqq N$.*

To show it is alright to count the Q_ν instead of the P_ν we prove two lemmas.

LEMMA 10. *If $P_\nu \in R'$ then $\max q_{i\nu} \doteq \max |\lambda_\nu^{(i)}|$ where $A \doteq B$ means $A \ll B$ and $B \ll A$.*

Proof. Since λ_ν is small

$$A_4 A_3(q_{1\nu}, \cdots, q_{n\nu}) = (\lambda_\nu^{(1)} - \lambda_\nu, \cdots, \lambda_\nu^{(n)} - \lambda_\nu)$$

implies $\max q_{i\nu} \ll \max |\lambda_\nu^{(i)}|$. Since A_4A_3 is invertible we also get the reverse inequality.

LEMMA 11. *If P_ν or Q_ν are in R' then*

$$\log \max |\lambda_\nu^{(i)}| = C_1\nu + O(1)$$

where $C_1 \neq 0$.

Proof. If we let $X_i = (\log |\zeta_i^{1/n}\zeta_i^{(2)}|, \cdots, \log |\zeta_i^{1/n}\zeta_i^{(r+s)}|)$ then as in Lemma 5 of [2] we get

$$\log |\lambda_\nu^{(i)}| = \nu_1 \frac{\left(\sum_{j=1}^{r+s} (-1)^{j-1} \det (X_1, \cdots, \hat{X}_j, \cdots, X_{r+s})(\log |\zeta_j^{(i)}|)\right)}{\det (X_2, \cdots, X_{r+s})} + O(1) .$$

The denominator is nonzere as in [2] and, for any choice of i, the numerator is $1/n$ times the regulator and so nonzero and independent of i, which proves the lemma.

Thus we have that as $|\nu_1| \to \infty$ we must have $\max |\lambda_\nu^{(i)}| \to \infty$ and so $\lambda_\nu \to 0$ which implies $|P_\nu - Q_\nu| \to 0$.

The proof is now finished as in [2]. The number of P in R' with $1 \leqq |\nu_1| \leqq N$ is shown to be asymptotic to CN. Letting $N = |C_1|^{-1} \log B$ this gives $1 \leqq q_{i\nu} \leqq B$ is equivalent to $O(1) \leqq |C_1\nu_1| \leqq \log B + O(1)$. The number of solutions to (10) is then asymptotically the same if $1 \leqq q_{i\nu} \leqq B$ is replaced by $1 \leqq |\nu_1| \leqq |C_1|^{-1} \log B$ and this is CN or $C' \log B$ as desired.

To prove Theorem 2 we must show that adding the restriction that the coefficients of P be relatively prime does not change the form of the result. To do this we note that there is a C_0 such that $0 < P(\beta) < C_0 |P'(\beta)| H(P)^{-(n+1)}$ has only a finite number of solutions. Thus if d divides all the coefficients of P and $0 < P(\beta) < C|P'(\beta)|H(P)^{-(n+1)}$ then $d^{n+1} < C_0^{-1}C$, and so the set of all such d is finite. Let d_0 be the largest such integer, then it is not hard to show that if the number of soultions to $0 < P(\beta) < Cd^{-(n+1)}|P'(\beta)|H(P)^{-(n+1)}$ is asymptotic to $C_d \log B$ then the number of primitive solutions with $d = 1$ is asymptotic to $(\sum_{d=1}^{d_0} \mu(d)C_d) \log B$. Theorems 2′ and 3 can then be combined to give Theorem 2.

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