PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 2, 1981

bo-RESOLUTIONS

MARK MAHOWALD

This paper considers the Adams-Novikov type spectral sequence with bo as the spectrum. The action of the generator of $\pi_8 bo$ in the spectral sequence is completely determined. The result is a complete determination of the v_1-periodic homotopy of the stable sphere.

1. Introduction. Let bo be the Ω-spectrum representing connected real K-theory. This spectrum is a ring spectrum with a unit and $H^*(bo) = A/A(Sq^1, Sq^2)$. (Unless otherwise noted, A is the mod 2 Steenrod algebra, all coefficient groups are $\boldsymbol{Z}_2$, and all spaces are localized at 2.)

Associated to a spectrum with unit, like bo, we have a tower of spectra

$$\begin{array}{ccccccccc}
S^0 & \longleftarrow & S_1 & \longleftarrow & S_2 & \longleftarrow \cdots \longleftarrow & S_s & \longleftarrow & S_{s+1} \longleftarrow \cdots \\
\downarrow & & \downarrow{\scriptstyle id \wedge \iota} & & \downarrow & & \downarrow & & \\
bo & & S_1 \wedge bo & & S_2 \wedge bo & & S_s \wedge bo & &
\end{array}$$

where $S_s \wedge bo \xleftarrow{id \wedge \iota} S_s \leftarrow S_{s+1}$ is a fibration and $l\colon S^0 \to bo$ is the unit. If we use the homotopy functor π^*, we get an exact couple with $E_1^{s,t} = \pi_{t-s}(S_s \wedge bo)$. Under reasonable hypothesis on the spectrum, $E_\infty^{*,*}$ is an associated graded group of $\pi_*(S^0)$. This is true for bo since $\pi_j(S_s) = 0$ for $j < 3s$ and so for $t - s < 3s$, $E_r^{s,t} = E_\infty^{s,t}$ for large enough r. This spectral sequence will be written $\{E_r(S^0, bo, \pi)\}$.

Clearly $\pi_* bo$ acts on E_1 but d_1 is not a $\pi_* bo$ module map. Nevertheless, if two classes in E_1 are related by the action of a class in $\pi_* bo$ and they both survive to E_∞, then we will say that these two classes are still related in this manner. In particular the class which generates $\pi_8 bo$ is a basic periodicity class which we will call v_1^4. (The name is suggested by BP-theory and is discussed in [3].) A class such that all multiples of it with v_1^{4k} are non zero is called a v_1-periodic class. Classes in E_1 which survive to E_∞ but for which all $\pi_* bo$ compositions except the identity do not survive will be said to generate a $\boldsymbol{Z}_2$ vector space. Our main theorem is:

THEOREM 1.1.

(a)
$$\begin{aligned}
E_\infty^{0,t}(S^0, bo, \pi) &= \boldsymbol{Z} \quad t = 0 \\
&= \boldsymbol{Z}_2 \quad t \equiv 1, 2 \bmod 8 \\
&= 0 \quad \textit{all other } t\,.
\end{aligned}$$

The classes for $t \equiv 1$ *and* $2 \bmod 8$ *are* v_1*-periodic.*

(b)
$$\begin{aligned} E_\infty^{1,t}(S^0, bo, \pi) &= \boldsymbol{Z}_2\rho(k) && t = 4k \\ &= \boldsymbol{Z}_2 && t = 1, 2 \bmod 8 \\ &= 0 && \textit{otherwise} \end{aligned}$$

where $\rho(k)$ *is defined by* $4k \equiv 2^{\rho(k)-1} \bmod 2^{\rho(k)}$ *and all the classes are* v_1*-periodic.*

(c) $E_\infty^{s,t}(S^0, bo, \pi) = 0$ *for* $6s > t + 12$ *and is a* $\boldsymbol{Z}_2$ *vector space (as a* $\pi_*(bo)$ *module) for all* $s > 1$ *and all* t.

This result was essentially announced in [6]. Since that time Milgram [12] and Carlsson [1], [2] have also investigated bo resolutions. The setting in the unstable range has also been investigated and results which are based on Theorem 1.1 were described in [7]. A second paper will be devoted to discussing that material.

The edge result stated in Theorem 1.1 part (c) guarantees that the only classes which have a v_1 type periodicity acting on them are described in parts (a) and (b). Other classes may admit finite multiples of v_1 but must form finite families. This is discussed more fully in §6 and also in [3].

The paper is organized as follows. Section 2 states the main ancillary results, which are Theorems 2.4, 2.5 and 2.7. Most of the section is devoted to definitions and the introduction of terminology. In addition consequences of 2.5 and 2.7 are obtained. The proof of 2.5 and 2.7 occupies §3. Section 4 contains the proof of 2.4. Section 5 analyzes the bo resolution and contains the proof of the main theorem. Section 6 discusses v_1-periodicity but a more detailed discussion is contained in [3].

2. The homotopy of $bo \wedge bo$. Let $g\colon S^3 \to B^3O$ represent a generator. Let $\Omega^2 g\colon \Omega^2 S^3 \to BO$ be the double loop map. Clearly the space $\Omega^2 S^3$ is homotopically equivalent to $S^1 \times W$ where W is the fiber of $\Omega^2 S^3 \to S^1$. Let $F_n(\Omega^2 S^3)$ be the May filtration. (Compare [11] and [12].) This induces a filtration on W, $F_n(W)$. Let $f_n\colon F_{2n}(W) \to BO$ be the composite $F_{2n}(W) \subset F_{2n}(\Omega^2 S^3) \subset \Omega^2 S^3 \xrightarrow{\Omega^2 g} BO$. Let $B(n) = T(f_n)$, the Thom spectrum defined by f_n. The following are some properties of $B(n)$ which follow immediately from the definition. (See [4] and [5] for calculations of this type.)

2.1. $B(n) \wedge B(m) \to B(n + m)$. The induced map in homology is onto if $n = 2^i$ and $m < 2^i$.

This follows from the fact that $F_{2n} \times F_{2m} \to F_{2(n+m)}$ using the multiplication in Ω^2S^3. The homology of Ω^2S^3 is a polynomial ring $Z_2[x_1, x_3, \cdots]$ with $\dim x_{(2^i-1)} = 2^i - 1$. Then $H_*(W) = Z_2[x_1^2, x_3, x_7, \cdots]$. If we assign a degree to x_i by $\deg x_i = (i+1)/2$ then $H_*(F_{2n})$ is the vector space generated by all monomials in x_i of degree $\leqq 2n$. These facts give 2.1 when translated to Thom spectra.

2.2. $B(n) \subset B(n+1)$ and $\bigcup_{n=0}^{\infty} B(n) = K(Z, 0)$(localized at 2). This is discussed in detail in [4].

2.3. If $M_1(k) = A/A(Sq^1, \chi Sq^n \mid n > 2k)$ then $H^*(B(n)) = M_1(n)$. (We will sometimes denote $M_1(k)$ as $M(2k) \bigotimes_{A_0} Z_2$ where A_0 is the subalgebra of A generalized by Sq^1.)

This follows immediately from 2.2 since $H^*(K(Z, 0)) \cong A \bigotimes_{A_0} \circ Z_2$.

Using the spectra $B(n)$ we define $\Omega_+ = \bigvee_{i=0}^{\infty} \Sigma^{4i} B(i)$. The key result about $bo \wedge bo$ is the following.

THEOREM 2.4. *There is a homotopy equivalence between $bo \wedge bo$ and $\Omega_+ \wedge bo$. (All spaces are localized at* 2.)

This theorem was first proved in 1969 [6]. Milgram [12] found another proof which proceeds from a different point of view. The proof given here, in spirit, is like the original one but the intervening years have seen a development particularly in the Thom spectrum approach. The following proof has profitted from this.

The first step is to get an algebraic version of 2.4. We need to show that as modules over the Steenrod algebra $H^*(bo \wedge bo)$ and $H^*(\Omega_+ \wedge bo)$ are isomorphic. Recall that $H^*(bo) = A \bigotimes_{A_1} Z_2$ where $A_1 \subset A$, is the sub Hopf algebra generated by Sq^1 and Sq^2. Since the A-module structure of $A \bigotimes_{A_1} Z_2 \otimes M$ is determined completely by the A_1-module structure of M, we need to determine the A_1-module structure of $A \bigotimes_{A_1} Z_2$. Theorem 2.5 will do this but it needs some introduction.

Using the Cartan basis representation we have a Z_2-basis for A given by Sq^I where I is admissible. ($I = (i_1, i_2, \cdots, i_l)$ and $i_j \geqq 2i_{j+1}$.) Then χSq^I is also a basis where χ is the anti-automorphism of A. Since $Sq^1Sq^{2n} = Sq^{2n+1}$ and $Sq^1Sq^{4n} + Sq^1Sq^{4n}Sq^1 = Sq^{4n+2}$, we see that in order for a class χSq^I to be nonzero in $A \bigotimes_{A_1} Z_2$, $i_1 \equiv 0 \bmod 4$. Also since $Sq^2Sq^{4n-1} + Sq^1Sq^{4n} = Sq^{4n}Sq^1$ we see that $i_2 \equiv 0(2)$. A Z_2-bases for $A \bigotimes_{A_1} Z_2$ is given by considering χSq^I with $i_1 \equiv 0(4)$ and $i_2 \equiv 0(2)$. The proof of 2.5 will give this. A Z_2-basis for $M_1(k)$ can be given by χSq^I, I admissible, $i_1 \equiv 0(2)$, and $i_1 \leqq 2k$. Thus each class in $M_1(k)$ can be identified with a class in A by this basic choice.

Then $M_1(k)\chi Sq^{4k}$ will mean the action of A on χSq^{4k} restricted to the subspace of A represented by $M_1(k)$.

These comments nearly complete the proof of the following. What remains is to show that there are no further relations and that the left A_1 action is correct.

THEOREM 2.5. *Let* $g^*: \bigoplus_{r\geq 0} \sum^{4k} M_1(k) \to A \bigotimes_{A_1} Z_2$ *be defined by* $\Sigma^{4k} M_1(k) \to M_1(k)\chi Sq^{4k} \subset A \bigotimes_{A_1} Z_2$. *Then* g^* *is an isomorphism of left* A_1*-modules.*

The proof will be given in §3.

The next step is to calculate the homotopy of $bo \wedge bo$. We will use the Adams spectral sequence and, in view of 2.5, all we need to do is to calculate $\mathrm{Ext}_{A_1}(M_1(k), Z_2)$.

The key step in this calculation will be the following. First some notation. Let $\{bo^i\}$ be a minimal Adams resolution for bo. By this we mean that the spaces bo^i fit into a sequence

$$bo \longleftarrow bo^1 \longleftarrow \cdots \xleftarrow{p_i} bo^i$$

and the fiber of $bo^i \leftarrow bo^{i+1}$ is a generalized Eilenberg MacLane space $\prod_t K(\mathrm{Ext}_{A_1}^{i,t}(Z_2, Z_2), t-i-1)$ and p_i^* is zero. Let $\alpha(k)$ be the number of 1's in the dyaic expansion of k.

PROPOSITION 2.6. *If* $k \equiv 0(2)$ *then* $A \bigotimes_{A_1} Z_2 \otimes M_1(k)$ *is stably* A *isomorphic to* $H^*(bo^{2k-\alpha(k)})$. *If* $k = 2l+1$, *then* $A \bigotimes_{A_1} Z_2 \otimes M_1(k)$ *is stably* A *isomorphic to* $H^*(b\,\mathrm{spin}^{4l-\alpha})$. ($b\,\mathrm{spin}^i$ *is defined analogously to* bo^i.)

This will be proved in §3. Note that $b\,\mathrm{spin} = M_1(1) \wedge bo$, see [10]. This proposition yields immediately.

THEOREM 2.7. *For* $s > 0$ *and if* $k = 2l$, *then* $\mathrm{Ext}_{A_1}^{s,t}(M_1(2l), Z_2) = \mathrm{Ext}_{A_1}^{s+4l-\alpha(l),t+4l-\alpha(l)}(Z_2, Z_2)$. *For* $s > 0$ *and if* $k = 2l+1$, *then*

$$\mathrm{Ext}_{A_1}^{s,t}(M_1(2l+1), Z_2) = \mathrm{Ext}_{A_1}^{s+4l-\alpha(l),t+4l-\alpha(l)}(M_1(1), Z_2)\,.$$

Theorems 2.5 and 2.7 can be combined to give:

THEOREM 2.8. *If* $s > 0$ *then*

$$\mathrm{Ext}_A^{s,t}(H^*(bo \wedge bo), Z_2)$$
$$= \bigoplus_{l=0}^{\infty} [\mathrm{Ext}_{A_1}^{2+4l-\alpha(l),t-4l-\alpha(l)}(Z_2, Z_2) \oplus \mathrm{Ext}_{A_1}^{s+4l-\alpha(l),t-4l-\alpha(l)-4}(M_1(1), Z_2)\,.$$

Since both $\mathrm{Ext}_{A_1}^{s,t}(Z_2, Z_2)$ and $\mathrm{Ext}_{A_1}^{s,t}(M_1(1), Z_2)$ are zero for $t - s \equiv 3(4)$, there can be no differentials in the Adams spectral sequence for $bo \wedge bo$. This gives:

THEOREM 2.9. *In the Adams spectral sequence for* $bo \wedge bo$, $E_2 = E_\infty$.

This completes the homotopy calculations for $bo \wedge bo$, (modulo the calculations done in §3 for 2.5 and 2.7). The rest of the proof of 2.4 will be done in §4.

3. The proof of several results from §2. (The first part of this section is heavily influenced by Peterson's lectures [13].)

We begin by studying the left A_1-module structure of $A \otimes_{A_1} Z_2$. It is easier to do these calculations in the dual. We have $i: A \to A \otimes_{A_1} Z_2$. In each gradation each side is a finite dimensional vector space over Z_2. In the dual we have $Z_2[\xi_1, \cdots] = A^* \supset (A \otimes_{A_1} Z_2)^*$.

The subset is more easily identified in the image of A^* under χ, the anti-isomorphism. We represent this image as $\chi[(A \otimes_{A_1} Z_2)^*]$.

PROPOSITION 3.1. *As left A-modules*

$$\chi[(A \otimes_{A_1} Z_2)^*] = Z_2[\xi_1^4, \xi_2^2, \xi_3, \cdots] .$$

Proof. Since $A \otimes_{A_1} Z_2 = A/A(Sq^1, Sq^2)$ we have $A \oplus A \xrightarrow{R(Sq^2) \oplus R(Sq^1)} A \to A/A(Sq^1, Sq^2) \to 0$ which gives $A^* \oplus A^* \xleftarrow{L(Sq^1) \oplus L(Sq^1)} A^* \leftarrow \{A/A(Sq^1, Sq^2)\}^* \leftarrow 0$ and finally $A^* \oplus A^* \xleftarrow{R(Sq^2) \oplus R(Sq^1)} A^* \leftarrow \chi[(A \otimes_{A_1} Z_2)^*] \leftarrow 0$. But $\xi_k Sq = \xi_k + \xi_{k-1}$. Hence the proposition follows.

Assign to each ξ_i degree 2^{i-1} and each monomial $\xi^I = \xi_1^{i_1}\xi_2^{i_2} \cdots$, the degree $\Sigma i_j 2^{j-1}$.

Let N_{4n} be the $\boldsymbol{Z}_2$ vector space generated by monomials of degree $4n$. Then $\boldsymbol{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \cdots] \cong \bigoplus_n N_{4n}$.

PROPOSITION 3.2. *As left A_1-modules* $\boldsymbol{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \cdots] \cong \bigoplus_n N_{4n}$.

Proof. The left A action is given by $Sq\xi_k = \xi_k + \xi_{k-1}^2$. In the absence of $\xi_1, \xi_1^2, \xi_1^3, \xi_2$ and products, degree $(Sq^1\xi^I)$ = degree ξ^I and degree $(Sq^2\xi^I)$ = degree ξ^I (of course 0 has every degree).

PROPOSITION 3.3. $\chi N_{4n}^* = M_1(n)$.

Proof. Using the multiplication A^* and the multiplicative nature of the degree we have maps

$$\chi N_{4n}^* \longrightarrow \chi N_{4j}^* \otimes \chi N_{4k}^* \quad j + k = n$$

which are monomorphism if $n \neq 2^i$ and $4j = 2^i$ and $4k < 2^i$. If $n = 2^i$ then the class corresponding to ξ_{i+2} generates the kernel. Now using the obvious isomorphism $\chi N_4^* = M_1(1)$ and the kind of argument of [5] §4 we get the result.

Combining 3.2 and 3.3 we get the proof of 2.5. Using this explicit calculation we also can get the following. Let $Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$. Then Q_j acts as a differential in M for any A (or A_1) module M.

PROPOSITION 3.4. [13] *$H_*(\chi(A \otimes Z_2)^*, Q_0) = Z_2[\xi_1^4]$ and*

$$H_*(\chi(A \underset{A_1}{\otimes} Z_2)^*, Q_1) \cong E(\xi_2^2, \xi_3^2, \cdots) .$$

Proof. Both of these are easy calculations from 3.2.

COROLLARY 3.5. *$H^*(M_1(2^i), Q_0) = Z_2$ generated by the class in dimension 0 and $H^*(M_1(2^i), Q_1) = Z_2$ generated by the class in dimension $2^{i+1} - 2$.*

We now begin the proof of 2.6. The strategy will be to construct spaces $R(n)$ with the properties:

(3.6.1) There is an A-module map $f_n^*: H^*(B(n)) \to H^*(R(n))$ which is a stable A_1-isomorphism.

(3.6.2) There are maps $g_n: R(n) \wedge bo \to bo^{2n-\alpha(n)}$ $n \equiv 0(2)$ and

$$g_n: R(n) \wedge bo \longrightarrow b\,\mathrm{spin}^{2n-1-\alpha(n)} \quad n \equiv 1(2)$$

and g_n^* are stable A-isomorphisms.

It would be nice to construct a map f_n which induces the cohomology map but we have not done so.

Let Y_i be the ith stage in an Adams resolution for S^0 constructed by iterating the sequence $Y_1 \to S^0 \to K(Z_2, 0)$. Then $Y_i = Y_j \wedge Y_{i-j}$ for any i and j.

LEMMA 3.6.3. *$\mathrm{Ext}_A^{s,t}(H^*(X \wedge Y_\sigma), Z_2) \cong \mathrm{Ext}_A^{s+\sigma,t+\sigma}(H^*(X), Z_2)$ for $s > 0$ and any spectrum X.*

The proof is immediate.

We will also use the following obvious lemma.

LEMMA 3.7. *If there is a map $f: X \to Y$ such that*

$$f_{\#}: \mathrm{Ext}_A^{s,t}(H^*(X), Z_2) \longrightarrow \mathrm{Ext}_A^{s,t}(H^*(Y), Z_2)$$

is an isomorphism for $s > 0$ *then* $H^*(Y)$ *is stably A-isomorphic to* $H^*(X)$.

Let $R(2^{i-1})$ be the cofiber of the standard map $RP^{2^{i+1}-2} \to S^0$. (λ is the stable adjoint of the composite $RP \to SO \to \Omega^\infty S^\infty$.) Since $S^0 \underset{\iota}{\to} bo^i \to bo$, where ι is a generator, is 2^i times a generator, the composite $P^{2^{i+1}-2} \to S^0 \to bo^{2-1}$ is inessential if $i > 0$. Hence there is a map

$$\tilde{g}_{2i-1}\colon R(2^{i-1}) \longrightarrow bo^{2^i-1} = bo^{2^i-\alpha 2^i}\ .$$

For $n = \sum_{i=0}^{k} \varepsilon_i 2^i$ and $\varepsilon_i = 0$ or 1 let $R(n) = \wedge R(\varepsilon_i 2^i)$ where $R(0) = S^0$. If $n \equiv 0(2)$ there is a map $\tilde{g}_n\colon R(n) \to \bigwedge_{i=1}^{k} bo^{\varepsilon_i(2^{i+1}-1)}$. By 3.6.3 $H^*(\wedge bo^{\varepsilon_i(2^{i+1}-1)})$ is stably A-isomorphic to $H^*(bo^{2n-\alpha(n)}{}^{bo\wedge\cdots\wedge bo})$. The map g_n results by multiplying both sides by bo and multiplying out.

If $n \equiv 1 \bmod 2$ then $\varepsilon_1 = 1$ and $R(n) = R(n-1) \wedge R(1)$. This gives $R(n-1) \wedge R(1) \to bo^{2(n-1)-\alpha(n-1)} \wedge R(1)$. But since $R(1) \wedge bo = b\,\mathrm{spin}$, $H^*(bo^{2(n-1)-\alpha(n-1)} \wedge R(1)) \cong H^*(b\,\mathrm{spin}^{2(n-1)-\alpha(n-1)})$. This completes the construction of the maps g_n.

LEMMA 3.8. *The maps* g_n *induce stable A-isomorphisms.*

Proof. By the way the maps are constructed it will be sufficient to show that the lemma is true if $n = 2^i$. Since $R(1) \wedge bo = b\,\mathrm{spin}$ ($R(1) = B(1)$) there is nothing to show in this case. In general we will show that

$$\mathrm{Ext}_{A_1}^{s,t}(H^*(R^{2^{i-1}}, Z_2)) \cong \mathrm{Ext}_{A_1}^{s+2^{i-1},t+2^{i-1}}(Z_2, Z_2) = \mathrm{Ext}_A^{s,t}(H^*(bo^{2^i-1}), Z_2)\ .$$

Then we will show that g_{2^i} induces this isomorphism. This will complete the proof. Let the image of $A_1 \otimes (\bigoplus_{\infty=0}^{4n} H^*(R(2^i))$ in $H^*(R(2^i))$ be F_n. Then $F_n/F_{n-1} \cong A_1 \otimes_{A_0} Z_2$. Thus there is a filtration $H^*(R(2^i))$ so that the associated graded is just $[\bigoplus_{y=0}^{2^i-1} \Sigma^{4j} A_1 \otimes_{A_0} Z_2] \oplus \Sigma^{2^{i+1}-4} M_1(1)$. There is a spectral sequence going from $\mathrm{Ext}_{A_1}(\ , Z_2)$ of this associative graded module to $\mathrm{Ext}_{A_1}(H^*(R(2^i), Z_2))$.
But

$$\begin{aligned}\mathrm{Ext}_{A_1}&\Big(\Big(\bigoplus_{j=0}^{2^i-1} \Sigma^{4j} A_1 \otimes_{A_0} Z_2\Big) \oplus \Sigma^{2^{i+1}-4} M_1(1), Z_2\Big)\\ &= \bigoplus_{j=0}^{2^{i-1}-1} \mathrm{Ext}_{A_0}^{s,t+4j}(Z_2, Z_2) \oplus \mathrm{Ext}_{A_1}^{s,t+2^{i-1}-4}(M_1(1), Z_2)\ .\end{aligned}$$

So there cannot be any differentials. These groups are easily seen to be the same as $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(bo^{2^{i-1}}), Z_2)$.

We now need to show that g_n induces this isomorphism. Let

$\Sigma^{-1}P^{2^{i+1}}_{2^{i+1}-1} \xrightarrow{q} RP^{2^{i-1}-2}$ be the attaching map of the next two cells i.e., the mapping cone of q is $RP^{2^{i+1}}$. The map q gives a map q^1: $P^{2^{i+1}}_{2^{i+1}-1} \to R(2^i)$. Since the bundle over $P^{2^{i+1}}$ has order 2^i the composite $g_{2^i}q^1$ must be essential; it must be essential in $R(2^i) \wedge bo$ also. There it must represent the composite $P^{2^{i+1}}_{g^{i+1}-1} \to S^{2^{i+1}} \xrightarrow[\text{gen}]{} R(2^i) \wedge bo$. Thus g_{2^i} must induce an isomorphism for $t - s = 2^{i+1}$ but this and the natural bo action forces an isomorphism everywhere.

Next we need to construct maps from $H^*(B(n)) \to H^*(R(n))$. By the definition of $R(n)$ and property 2.1 of $B(n)$ it will be sufficient to construct $F^*_{2^i}$: $H^*(B(2^i)) \to H^*(R(2^i))$. Since $R(1) = B(1)$ we need only consider $i > 0$. Following [5] we have $M(2^{i+1}) \to H^*(\Sigma P^{2^{i+1}-2}_{-1})$ which is an epimorphism of A-modules.

The image of $A(Sq^1, \chi Sq^k \mid k > 2^{i+1}) \to A \to H^*(\Sigma P^{2^{i+1}-2}_{-1})$ is just the class in dimension 1. Hence there is a map between $M_1(2^i)$ and $H^*(R(2^i))$. This is the map $f^*_{2^i}$.

LEMMA 3.9. *The map* $f^*_{2^i}$: $M_1(2^i) \to H^*(R(2^i))$ *is a stable* A_1-*isomorphism.*

Proof. We need check what the map does in Q_i homology. Clearly $f^*_{2^i}$ induces an isomorphism in the Q_0 homology. Since $H_*(M_1(2^i), Q_1) = Z_2$ generated by the class in dimension $2^{i+1} - 2$ and $f^*_{2^i}$ maps this class nontrivally we must verify that its image in $H^*(R(2^i))$ is not a boundary. Indeed it is easy to see that $H_*(H^*(R(2^i), Q_1)) = Z_2$ generated by the class in dimension $2^{i+1} - 2$, (since $Sq^3\alpha^{4n-1} = \alpha^{4n+2}$ and $Sq^2Sq^1\alpha^{4n+1} = \alpha^{4n+4}$). By the theorem of Wall [13] this implies that $f^*_{2^i}$ is a stable A_1-isomorphism.

This now completes the proof of 2.6 since if $n \equiv 0(2)$

$$\begin{aligned}\operatorname{Ext}^{s,t}_A(H^*(B(n) \wedge bo), Z_2) &\overset{(f^*_n)_\#}{\cong} \operatorname{Ext}^{s,t}_A(H^*(R(n) \wedge bo), Z_2)\\ &\overset{g_{n\#}}{\cong} \operatorname{Ext}^{s,t}_A(H^*(bo^{2n-\alpha(n)}), Z_2)\ .\end{aligned}$$

If $n \equiv 1(2)$ we have the analogous formula.

4. The proof of 2.4.

We begin with the following corollary of (3.6.2.)

LEMMA 4.1. *If* $f: B(k) \wedge B(l) \to B(k + l)$ *is the multiplication map of* 2.1 *then* f^* *is a stable* A_1-*isomorphism if* $k = 2^i$ *and* $l < 2^i$.

Proof. Indeed $H^*(B(k) \wedge bo)$ and $H^*(bo^{2k-\alpha(k)})$ are stably A-isomorphic. Thus $bo^{2k-\alpha(k)} \wedge bo^{2l-\alpha(l)} = bo^{2(k+l)-\alpha(k+l)}$ if $k = 2^i$ and $l < 2^i$.

We need to determine what happens if $k = l = 2^i$.

DEFINITION 4.2. A sequence $N' \to N \to N''$ of A_1-modules is stably A_1 exact if there is a free A_1-module which can be added to each term to get an exact sequence.

PROPOSITION 4.3. *A stably A_1 exact sequence of A_1-modules induces a long exact sequence in* Ext *groups for $s > 0$.*

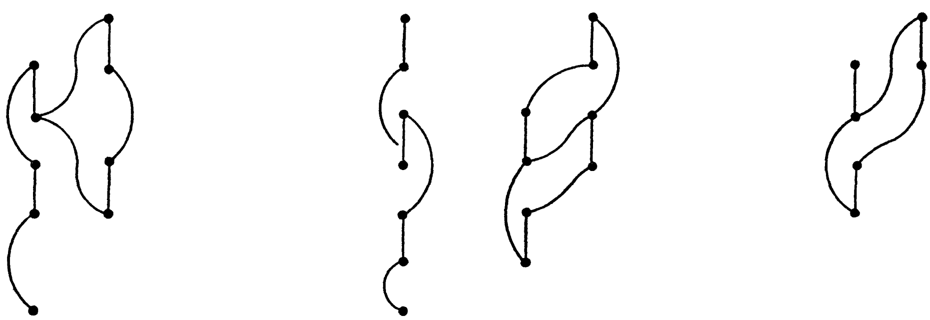
This is immediate.

Motivated by Lemma 3.7 we say f is a stable A-isomorphism through dimension $\bar{t}$ if f induces an isomorphism in $\mathrm{Ext}_A^{s,t}$ for $s > 0$ and $t - s < \bar{t}$.

These ideas can be illustrated by the following example. Consider

$$B(1) \wedge B(1) \longrightarrow B(2) \longrightarrow \Sigma^4 M_{2\iota} \wedge B(1) .$$

The induced maps in cohomology give an A_1-stably exact sequence. Pictorially we have



$B(1) \wedge B(1)$ $\qquad$ $B(2) \vee A_1$ $\qquad$ $\Sigma^4 M_{2\iota} \wedge B(1)$

The curved lines indicate Sq^2 and the straight lines indicate Sq^1.

This example generalizes to give

THEOREM 4.4. *The cohomology sequence induced by*

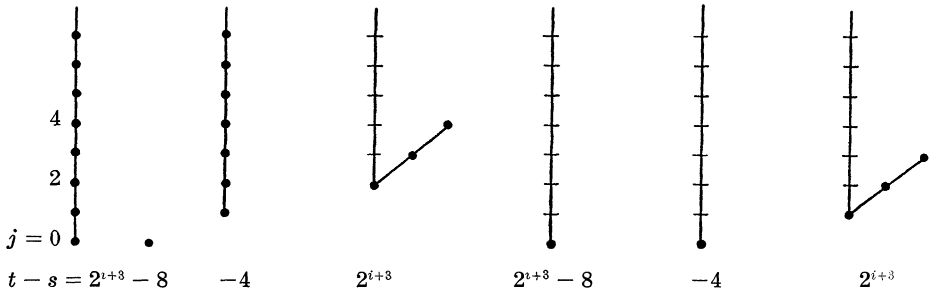
$$B(2^i) \wedge B(2^i) \xrightarrow{\ f\ } B(2^{i+1}) \longrightarrow \Sigma^{2^{i+3}-4} M_{2\iota} \vee B(1)$$

is a stably A_1 exact sequence.

Proof. It is sufficient to verify the result for cohomology modules which are stably equivalent. Thus the map f is equivalent by 3.6 to

$$h\colon bo^{2^{i+2}-2} \longrightarrow bo^{2^{i+2}-1}$$

and h in dimension zero is a degree 1 map. (Note that it is not obvious that such a map exists but the multiplication f does exist and via 3.6 we get h.) The following charts will illuminate the argument.



$\mathrm{Ext}_A^{s,t}(H^*bo^{2^{i+2}-2}, Z_2)$ $\qquad$ $\mathrm{Ext}_A^{s,t}(H^*(bo^{2^{i+2}-1}), Z_2)$

Let a be defined by the cofiber sequence

$$\Sigma M_{2\iota} \wedge M_{2\iota} \xrightarrow{a} M_{2\iota} \longrightarrow M_{2\iota} \wedge B(1) .$$

Let j be the composite $\Sigma^{2^{i+3}-5}M_{2\iota} \to S^{2^{i+3}-4} \to bo^{2^{i+2}-2}$.

Clearly $a \cdot j \simeq 0$ and so we have $j_\# \Sigma^{2^{i+3}-5}M_{2\iota} \wedge B(1) \to bo^{2^{i+2}-2}$. Also it is easy to see that $hj_\#$ is null homotopic.

Thus there is a map $\bar{h}$: $bo^{2^{i+2}-2} \bigcup_{hj\#} CM_{2\iota} \wedge B(1) \wedge bo \to bo^{2^{i+2}-1}$.

An easy calculation shows that $\bar{h}$ induces an isomorphism in the Ext_A groups except if $s = 0$ and $t = 2^{i+3} - 6$. This is equivalent to the theorem.

Now we can prove 2.4. We proceed by induction. Let $\Omega^n_+ = \bigvee_{j \leqq n} \Sigma^{4j}B(j)$. Clearly $\Omega^1 \to bo \wedge bo$ and $\Omega^1_+ \wedge bo \to bo \wedge bo$ is a homotopy equivalence through dimension 7.

Suppose we have a map h_i: $\Omega^{2^i-1}_+ \to bo \wedge bo$ such that $\Omega^{2^i-1}_+ \wedge bo \to bo \wedge bo$ is a stable A-isomorphism through dimension $2^{i+2} - 1$. Then by 4.4 we see that $[\Omega^{2^i-1}_+ \wedge (S^0 \vee \Sigma^{2^{i+2}}B(2^{i-1}) \wedge B(2^{i-1}))] \wedge bo \to bo \wedge bo$ exists and gives a stable A-isomorphism through dimension $2^{i+3} - 5$. Consider the diagram

$$\begin{array}{ccccc} \Sigma^{2^{i+3}-5}M_{2\iota} \wedge B(1) & \longrightarrow & \Sigma^{2^{i+2}}B(2^{i-1}) \wedge B(2^{i-1}) \wedge bo & \longrightarrow & bo \wedge bo \\ & & \Big\downarrow & & \\ & & \Sigma^{2^{i+2}}B(2^i) \; . & & \end{array}$$

We need to show that the composite represented by the top row is null homotopic. The composite is clearly filtration $\geqq 1$. Therefore it factors through $\Omega^{2^i-1} \wedge bo$. But $\Omega^{2^i-1} \wedge bo$ is A equivalent to

$$\bigvee_{j<2^i-1} [\Sigma^{8j}bo^{4j-\alpha(j)} \vee \Sigma^{8j+4}b\,\mathrm{spin}^{4j-\alpha(j)}] .$$

The $2^{i+2}-1$ connected cover is just a wedge of $\Sigma^{2^{i+2}}bo$. The generating homotopy classes have various s filtrations depending on which piece of Ω it came from. (The reader is encouraged to draw an Ext diagram using 2.8 for $i = 2, 3$ or 4 to see this clearly.) There is a map of $bo^l \to bo$ for any l and this map induces an isomorphism in homotopy in dimensions greater than $2l + \varepsilon_l$ where $\varepsilon_l = 0$, $l \equiv 0(4)$, $= -1$, $l \equiv 1(4)$, $= -2$, $l \equiv 2, 3(4)$. Thus there is a map of degree divisible by 2^{i+1} on the bottom cell of $\Sigma^{2^{i+2}}B(2^{i-1}) \wedge B(2^{i-1}) \wedge bo \to \Omega^{2^i-1} \wedge bo$ so that the following commutes.

$$\begin{tikzcd}
\Sigma^{2^{i+3}-5}M_{2^i} \wedge B(1) \arrow[r] \arrow[d] & \Sigma^{2^{i+2}}B(2^{i-1}) \wedge B(2^{i-1}) \wedge bo \arrow[r] & bo \wedge bo \arrow[d, "id"] \\
\Sigma^{2^{i+2}}B(2^{i-1}) \wedge B(2^{i-1}) \wedge bo \arrow[r] & \Omega_+^{2^{i-1}-1} \wedge bo \arrow[r] & bo \wedge bo
\end{tikzcd}$$

The bottom row has degree divisible by 2^{i+1} on the bottom cell.

Adding this map to the product map we have a map $\Sigma^{2^{i+2}}B(2^{i-1}) \wedge B(2^{i-1}) \to bo \wedge bo$ of the right degree (1 on the bottom cell) which can be extended over $\Sigma^{2^{i+2}}B(2^i)$. This gives $\Omega_+^{2^i-1} \wedge (S^0 \vee \Sigma^{2^{i+2}}B(2^i)) \to \Omega^{2^{i+1}-1} \to bo \wedge bo$ with the desired properties. This completes the proof.

5. *bo* resolutions. We now are prepared to analyze the *bo* resolution. In order to facilitate the calculation we introduce another spectrum. This and similar spectra are described in [4]. Let $S^5 \to B^2O$ be a generator and let $f\colon \Omega S^5 \to BO$ be the adjoint. Let X be the Thom spectrum of f. (This is called X_5 in [4].) The key properties of X are the following:

5.1. X is a ring spectrum. (See 1.1 of [4].) The proof uses the fact that the Thom spectrum of a bundle classified by an H-map is a ring spectrum.

5.2. $X \wedge X = \Omega S^5_+ \wedge X$ and this homotopy equivalence is induced by the composite map g given by

$$\Omega S^5 \times S\Omega^5 \xrightarrow{\Delta \times id} \Omega S^5 \times \Omega S^5 \times \Omega S^5 \xrightarrow{id \times \mu} \Omega S^5 \times \Omega S^5$$

where Δ is the diagonal and μ is the multiplication. The homotopy equivalence the other way looks the same except Δ is replace by $\bar{\Delta}(x) \to (x, x^{-1})$. We will call that composite h. Clearly g and h are homotopy equivalences and one is the homotopy inverse of the other. (Again see [4] for a fuller discussion.)

5.3. The bundle f has a *bo* orientation $o\colon X \to bo$ and o^* induces an isomorphism in Q_0 homology.

Proof. Consider $S^4 \xrightarrow{i\tilde{1}} S^4 \times \cdots \times S^4 \to \Omega S^5$. Since in the Thom complex of i_1 of, χSq^4 is also nonzero and χSq^i has the same Cartan formula as Sq^i (with $Sq^i \otimes Sq^k$ replaced by $\chi Sq^i \otimes \chi Sq^k$) we see that $\chi Sq^{4k} U$ is nonzero in $H^*(X)$ for each k. But (2.4.1) shows that χSq^{4k} on the fundamental class of bo generates the Q_0 homology.

Consider a resolution of S^0 by X. This gives the following diagram

$$\begin{array}{ccc} \bar{X}^2 & \longrightarrow & \bar{X}^2 \wedge X \\ \downarrow & & \\ \bar{X} & \longrightarrow & \bar{X} \wedge X \\ \downarrow & & \\ S^0 & \longrightarrow & X\,. \end{array} \tag{5.4}$$

Since $o\colon X \to bo$ there is a mapping of resolutions

$$\begin{tikzcd} S^0 & \bar{X} \arrow[l] \arrow[d] & \bar{X}^2 \arrow[l] \arrow[d] & \cdots \arrow[l] & \arrow[l] \\ & S_1 \arrow[ul] & S_2 \arrow[l] & \cdots \arrow[l] & \arrow[l] \end{tikzcd} \tag{5.5}$$

For resolutions of this sort the E_1 and E_2-terms are more easily described in the following way. Let Ibo be defined by the cofiber sequence $S^0 \xrightarrow{\iota_0} bo \xrightarrow{p} Ibo$. (IX is similarly defined.) Consider the sequence of spectra where 1 is the identity map and the trailing S^0's are suppressed. In particular

$$bo = bo \wedge S^0 \xrightarrow[p \wedge \iota_0]{} Ibo \wedge bo$$

is the first map.

$$\begin{aligned} &bo \xrightarrow{p \wedge \iota_0} Ibo \wedge bo \xrightarrow[1 \wedge p\iota_0]{} (Ibo)^2 \wedge bo \longrightarrow \cdots \\ &\xrightarrow[1 \wedge p \wedge \iota_0]{} (Ibo)^\sigma \wedge bo \longrightarrow \cdots\,. \end{aligned} \tag{5.6}$$

PROPOSITION 5.7. *The E_1 chain complex of the bo resolution is the chain complex resulting from the sequence of spectra* 5.6 *after applying the homotopy functor.*

This is a standard result.

The results of the first three sections assert that the homotopy of $(Ibo)^\sigma \wedge bo$ is a sum of modules of the form $\pi_*(bo^j)$ for various values of j together with Z_2-summands of Adams filtration zero. *We will always disregard the Z_2-summands of Adams filtration zero* *in this resolution.* The $\pi_*(bo^j)$ modules are given by 3.6. In particular $(Ibo)^\sigma \wedge bo = \Omega^q \wedge bo$ and $H_*(H^*(\Omega^q), Q_0)$ is in $1-1$ correspondence with the distinct bo^j modules. Since the map $o\colon X \to bo$ has the property that o^* induces a Q_0 homology isomorphism we can analyze d_1 by looking at the X-resolution. The first result is the following.

THEOREM 5.8. *The following diagram commutes*

$$\begin{array}{ccc} (IX)^\sigma \wedge X & \xrightarrow{1 \wedge p \wedge \iota_0} & (IX)^{\sigma+1} \wedge X \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ (\Omega S^5)^\sigma \wedge X & \xrightarrow{\delta_0} & (\Omega S^5)^{\sigma+1} \wedge X \end{array}$$

where $\delta_\sigma = \Sigma(-1)^i \delta^i_\sigma$ and δ^i_σ is induced by the following map of spaces:

$$\delta^i_\sigma\colon \underset{\sigma+1\ factors}{\Omega S^5 \times \cdots \times \Omega S^5} \longrightarrow \underset{\sigma+2\ factors}{\Omega S^5 \times \cdots \times \Omega S^5}$$

$$S^i_\sigma = 1 \times^{i-1} \cdots 1, \Delta, 1 \times \cdots \times 1\ .$$

Proof. The map $1 \wedge p \wedge \mathrm{L}_0$ and the resulting chain complex is equivalent to

$$X^{\sigma+1} \xrightarrow{F} X^{\sigma+2}$$

where $F = \Sigma(-1)^i F^i$ and $F^i = 1 \times \cdots \times 1, \iota_0, 1 \cdots 1$ where the ι_0 is in the ith place. Consider the composite

$$\begin{aligned} &(\Omega S^5)^{\sigma+1} \xrightarrow[1\times h\times 1]{} (\Omega S^5)^{\sigma+1} \xrightarrow[1\times h\times 1]{} (\Omega S^5)^{\sigma+1} \longrightarrow \cdots \xrightarrow[h\times 1]{} (\Omega S^5)^{\sigma+1} \\ &\xrightarrow[1\times 1\times \cdots \times_i^* \times 1 \cdots \times 1]{} (\Omega S^5)^{\sigma+2} \xrightarrow[g\times 1]{} (\Omega S^5)^{\sigma+1} \\ &\xrightarrow[1\times g\times 1]{} \cdots \xrightarrow[1\times g]{} (\Omega S^5)^{\sigma+2}\ . \end{aligned}$$

The Thom complex picture is

$$\begin{aligned} &(\Omega S^5_+)^\sigma \wedge X \longrightarrow (\Omega S^5_+)^{\sigma-1} \wedge X \wedge X \longrightarrow \cdots \longrightarrow X^{\sigma+1} \xrightarrow{f^i} X^{\sigma+2} \\ &\qquad \longrightarrow \Omega S^5_+ \wedge X^{\sigma+1} \longrightarrow \cdots \longrightarrow (\Omega S^5_+)^{\sigma+1} \wedge X\ . \end{aligned}$$

The map h is just $h(x, y) = h(x, x^{-1}y)$. The composite $\bar{h}$ which induces the homotopy equivalence $(\Omega S^5_+)^\sigma \wedge X \to X^{\sigma+1}$ is

$$\bar{h}(x_1, \cdots, x_{\sigma+1}) = (x_1, x^{-1}, x_2, x_2^{-1}x_3, \cdots, x_\sigma^{-1}x_{\sigma+1})\ .$$

The "differential" map just inserts a base point in the ith position. The map $\bar{g}$ just unravels the map giving $(x_1, x_2, x_3, \cdots, x_{i-1}, x_i, x_i, x_{i+1}, \cdots, x_{q+1})$. This is the assertion of the theorem.

COROLLARY 5.9. *The maps δ_σ^i for $i \leqq \sigma$ are $\pi_*(X)$ module maps. $\delta_\sigma^{\sigma+1}$ is not a $\pi_*(X)$ module map.*

We will analyze the differential in the bo resolution by the following device. Each class in $H_*(H_*(\Omega^\sigma), Q_0)$ is identified with a class in $H_*((\Omega S^5)^\sigma)$. We will determine what happens to the A_1-module so determined by tracing it in the X-resolution sequence. What we will prove is that by just considering the part of the differential which is a bo map, we get an acylic chain complex except for $\sigma = 0$ and 1 modulo, at each level, elements of Adams filtration zero. To begin we need the following.

THEOREM 5.10. *The construction of the bo splitting can be made so that the following diagram commutes modulo elements of Adams filtration $\geqq 2$.*

$$\begin{array}{ccc} bo \wedge bo & \longleftarrow & \Omega_+ \wedge bo \\ {\scriptstyle o\wedge o}\uparrow & & \uparrow{\scriptstyle q\wedge o} \\ X \wedge X & \longleftarrow & \Omega S^5 \wedge X \end{array}$$

where $q\colon \Omega S^5_+ \to \bigvee_{j\geq 0} \Sigma^{4j} B(j)$ is the stable map $\Omega S^5_+ \to \bigvee_{j\geq 0} \Sigma^j \xrightarrow{V\iota_0} \Omega_+$ and $\iota_0\colon S^0 \to B(j)$ is the inclusion of the bottom cell.

Proof. The splitting of $X \wedge X$ can be constructed by using the analogous induction argument to that given for $\Omega_+ \wedge X$. The fact that we needed to consider the possibility of modifying the argument at each power of 2 given a possible obstruction to commutativity. But we were careful to show that the modification was always done by a class of high filtration and only involved the bottom class of $B(j)$. Compare the proof of 2.4 given in §4.

Now we are prepared to prove the main theorem. The first step is the following.

THEOREM 5.11. *Each homotopy class in $\pi_*(I^\sigma bo \wedge bo)$ is either of Adams filtration zero, or in the image of d_σ, or is mapped essentially under d_σ, or $\sigma = 0$ or $\sigma = 1$ and the homotopy can be identified with $\pi_*(B(1) \wedge bo)$.*

Proof. The strategy of the proof will be first to look at the part of the differential which preserves Adams filtration. Then we will look at the part which raises filtration by 1 and see that these two considerations cover all the homotopy except as described in the theorem.

The first part we will accomplish by noticing that the resolution is really a standard resolution of an exterior algebra. To set this up recall that $I^{\sigma}bo \wedge bo \to \bigvee \Sigma^{4j_1+\cdots+4j_q}B(j_1) \wedge \cdots \wedge B(j_q) \wedge bo$. We label the wedge on the the right by $\bigvee_{\mathscr{N}_\sigma} \Sigma^{|N|}B(N) \wedge bo$ where $\mathscr{N}_\sigma = \{N = (n_1, \cdots, n_\sigma)\}$, $B(N) = B(n_1) \wedge B(n_2) \wedge B(n_\sigma)$ and $|N| = \Sigma 4N_i$. Suppose $|N'| = |N''|$ with $N' \in \mathscr{N}_\sigma$ and $N'' \in \mathscr{N}_{\sigma+1}$. The map $\Sigma^{|N'|}B(N') \wedge bo \to V_{\mathscr{N}_\sigma}\Sigma^{|N'|}B(N) \wedge bo \xrightarrow{d} V_{\mathscr{N}_{\sigma+1}}\Sigma^{|N'|}B(N) \wedge bo \to \Sigma^{|N'|}B(N'') \wedge bo$ is zero if N' and N'' are not "adjacent". By this we mean that $n_i' = n_i''$ for $i < s$, $n_s' = n_{s+1}''$ and $n_i' = n_{i+1}''$ for $i > s$. The map is a map of degree $\begin{bmatrix} n_i' \\ n_i'' \end{bmatrix}$ between the bottom cells otherwise. This is the content of 5.8. When the degree is 1(mod 2), i.e., when $\begin{bmatrix} n_i' \\ n_i'' \end{bmatrix} = 1$ mod 2 then the composite is a homotopy equivalence modulo a wedge of $K(Z_2)$'s on each side. Indeed

$$\begin{aligned} B(N') \wedge bo \cong bo^{|N'|/2 - \Sigma\alpha(m_j')} &\longrightarrow \begin{bmatrix} n_i' \\ n_i'' \end{bmatrix} B(N'') \wedge bo \\ &\cong \begin{bmatrix} n_i' \\ n_i'' \end{bmatrix} bo^{|N'|/2 - \Sigma\alpha(n_j') + \alpha(n_i') - \alpha(n_{i+1}'')} . \end{aligned}$$

But

$$\left(\alpha\begin{bmatrix} n_i' \\ n_i'' \end{bmatrix} = \alpha(n_i'') + \alpha(n_{i+1}'') - \alpha(n_i')\right) \bmod 2 .$$

Thus to discuss the filtration preserving situation it will be sufficient to determine how the bottom cell of each $B(N)$ maps. To do this we need look at the resolution

$$\Omega \xrightarrow{\Delta} \Omega \wedge \Omega \xrightarrow{\Delta \wedge 1 - 1 \wedge \Delta} \Omega \wedge \Omega \wedge \Omega \longrightarrow \cdots \longrightarrow \tag{5.12}$$

and apply homology and then take the homology with respect to Q_0.

$$H_*(H_*(\Omega), Q_0) \longrightarrow H_*(H_*(\Omega^2), Q_0) \cdots .$$

This complex is just the same as

$$H_*(\Omega S^5_*) \xrightarrow{\Delta_*} H_*(\Omega S^5_* \wedge \Omega S^5_*) \xrightarrow{\delta_2} H_*((\Omega S^5_*)) \xrightarrow{\delta_3} \cdots . \tag{5.13}$$

Finally this complex is just a resolution of an exterior algebra and its homology is a polynomial algebra on generators, $\mathscr{H}_i$ with dimension 2^i and of filtration 1. These generators correspond to $B(2^i)$. Thus there is a way to write $H_*((\Omega S^5)^\sigma)$ as a sum $\mathscr{N}_\sigma^\sigma + \mathscr{C}_\sigma + \mathscr{Z}_\sigma$ so that $\delta_\sigma(\mathscr{C}_\sigma) \cong \mathscr{Z}_{\sigma+1}$ and $\delta_\sigma \mathscr{N}_\sigma^\sigma = 0$. The classes $\mathscr{N}_\sigma^\sigma$ can be chosen so that $\mathscr{N}_\sigma^\sigma = \{(n_i) | n_i = 2^{j_i} \text{ and } j_i \leqq j_{i+1}\}$.

Now we look at $\mathscr{N}_\sigma^\sigma = \{(n_i) | n_i = 2^{j_i} \text{ and } j_i \leqq j_{i+1}\}$.

LEMMA 5.14. *Let $N \in \mathcal{N}_\sigma^\sigma$ be such that $n_{\sigma-1} < n_\sigma$. Then the composite*

$$\Sigma^{|N|}B(N) \wedge bo \longrightarrow \Omega^q \wedge bo \longrightarrow \Omega^{q+1} \wedge bo \longrightarrow \Sigma^{|N|}B(N') \wedge bo$$

where $N' = (n'_s)$ and $n'_s = n_s$, $s < \sigma$, $n'_\sigma = n_{\sigma/2} = n'_{\sigma+1}$ is a homotopy equivalence mod element of Adams filtration zero.

Proof. As in the preceeding proof we compare with X. In this case we have the binomial coefficient $\binom{n_\sigma}{n_{\sigma/2}}$. Recall that $n_\sigma = 2^{j_\sigma}$ and so the coefficient is $2 \times k$ where k is odd.

Now $B(N) = bo^{(|N|-\sigma)/2}$ and $B(N') = bo^{(|N|-\sigma-1)/2}$. Thus a map of Adams filtration 1, but not higher, on the bottom cell induces an isomorphism of homotopy, modulo classes of filtration zero. This proves the lemma.

Note that in this case the classes in the coker are not just Z_2-summands of $\pi_*(B(N'))$ but include classes which are a part of the essential bo homotopy of $B(N)$. They still are classes of Adams filtration zero.

We have completed the proof of the Theorem 5.11. What remains to be proved in the main theorem is "edge" result in part (c) and to consider the map in homotopy induced by $bo \to Ibo \wedge bo \to \Sigma^4 B(1) \wedge bo$. This second part is described completely in [10]. An easy way to see those results from our perspective is to again consider the diagram.

$$\begin{array}{ccc} bo & \xrightarrow{k^1} & \Sigma^4 B(1) \wedge bo \\ \uparrow & & \uparrow \\ X & \xrightarrow[k]{} & \Sigma^4 X \end{array}$$

The fact that k factors through $X \to \Omega S^5 \wedge X$ allows one to show that $k_*(a^{4j}) = j\sigma(a^{4j-4})$. This implies that in k^1 the homotopy in $\pi_{4j}(bo)$ increases in Adams filtration by i where $j = 2^i \times$ odd. From this calculation parts a, and b follow immediately.

To establish the "edge theorem" we first note.

PROPOSITION 5.15. *If X is a space whose cohomology is free over A_1 then $E_2^{s,t}(X, bo, \pi) = \mathrm{Ext}_A^{s,t}(\tilde{H}^*(X), \boldsymbol{Z}_2)$.*

Proof. Let $\bar{A}_1$ be a space such that $\tilde{H}^*(\bar{A}_1) \cong A_1$. Then $bo \wedge \bar{A}_1 = K(\boldsymbol{Z}_2, 0)$. Hence the bo resolution for $\bar{A}_1$ is just an Adams resolution.

We can filter $H^*(S_1)$ by letting

$$F_k = \bigoplus_{i \leq k} \operatorname{im} A_1 \otimes H^i(S_1) .$$

Then F_k/F_{k-1} is as follows.

$$\begin{aligned}
&F_3/F_2 = \Sigma^3 M_1(1) \\
&F_i/F_{i-1} = 0 \quad i = 4, 5, 6, 8, 9, 10, 12, 13, 14 \\
&F_7/F_6 = \Sigma^7(A_1 \otimes_{A_0} Z_2) \\
&F_{11}/F_{10} = \Sigma^{11}(A_1 \otimes_{A_0} Z_2) \oplus \Sigma^{11} M_1(1) .
\end{aligned}$$

Using this filtration we can get an estimate on $S_4 \wedge M_{2^\iota}$. First observe that S_4 is 11-connected. The first portion of $\widetilde{H}^*(S_4 \wedge M_{2^\iota})$ is $M_1(1) \otimes M_1(1) \otimes M_1(1) \otimes M_1(1) \otimes \widetilde{H}^*(M_{2^\iota})$ and a simple calculation shows that this module is stably A_1-isomorphic to $\widetilde{H}^*(S^{24} \wedge M_{2^\iota})$. Hence for $s \geqq 4$ there is a function of t, $f(t)$, such that $E_2^{s,t}(M_{2^\iota}, bo, \pi) = 0$ for $s > f(t)$ if $E_2^{s-4,t-4}(S_4 \wedge M_{2^\iota}, bo, \pi) = 0$. This last expression is valid if $\mathrm{Ext}_A^{s-4,t-16}(A_1, Z_2) = 0$ and $E_2^{s-4,t-28}(M_{2^\iota}, bo, \pi) = 0$. The first occurs for $6s > t + 12$. (This follows easily from a calculation of $\mathrm{Ext}_{A_2}(A_1, Z_2)$.) Using this estimate inductively we see that if $6s > t + 12$ then $6(s - 4) > (t - 28) + 12$. This implies $E_2^{s\ t}(M_{2^\iota}, bo, \pi) = 0$ if $6s > t + 12$.

A much sharper estimate is available. Indeed the theorem is true for $6s > t + 6$ but the proof requires a much sharper analysis of the resolution. Our goal was only to establish the "1/5 line".

6. v_1-periodicity. There are a variety of ways to define v_1 compositions in homotopy. The followihg is equivalent and is convenient for our purposes. Let Y be the stable complex $\Sigma^{-2} M_{2^\iota} \wedge CP^2$. It is easy to see that there is a map $f: \Sigma^2 Y \to Y$ of filtration 1 such that $H^*(Y \cup_f C\Sigma^2 Y)$ is the subalgebra of A generated by Sq^1 and Sq^2. There are four choices of v_1 and we use any one and call the map f, v_1. Let $\gamma: S^j \to X$ be a class. Then there are potentially four maps

(6.1.1) $\Sigma^{j-3} Y \xrightarrow{p_1} S^j \xrightarrow[\gamma]{} X$ where p_1 is of degree 1.

(6.1.2) $\Sigma^{j-2} \xrightarrow{p_2} \Sigma^j M_{2^\iota} \xrightarrow{\gamma^\#} X$ where p_2 is of degree 1 and $\gamma^\#$ is an extension of γ (if it exists).

(6.1.3) $\Sigma^{j-1} Y \xrightarrow[p_3]{} \Sigma^{j-2} P_2^4 \xrightarrow{\gamma^\#} X$ where $P_2^4 = RP^4/S^1$ and p_3 and $\gamma^\#$ are analogously defined.

(6.1.4) $$\Sigma^j Y \xrightarrow{\gamma^\#} Y .$$

If (6.1.1) exists we call the map $\gamma_i^\#$. The first always exists but may be inessential. The others sometime exist and, when they exist, are sometimes essential. If a map of type i exists and the composite

$$\Sigma^{j-4+i+2l} Y \xrightarrow{v_1^l} \Sigma^{j-4+i} Y \xrightarrow[\gamma_i^\#]{} X$$

is essential for all l for every $\gamma^\#$, then we say that γ is v_1-periodic of type i.

To illustrate this we give the following examples. 6.2(a) If η generates $\pi_1(S^0)$ then η is v_1-periodic of type 2, $\eta_1^\#$ is inessential and $\eta_i^\#$, $i = 3$ and 4 do not exist.

(b) If ν generates $\pi_3(S^0)$ then ν is v_1-periodic of type 3 but not of type 1.

(c) The generator of the image of J in the $8k - 1$ stem is v_1-periodic of type 1. The element of order 2 in the image of J in the $8k - 1$ stem is v_1-periodic of type 2.

THEOREM 6.3. *The only homotopy elements in $\pi_*(S^0)$ which are v_1-periodic of any type are those classes discribed in $E^{0,t}(S^0, bo, \pi)$ $E^{1,t}(S^0, bo, \pi)$.*

Note that other classes may have finite v_1-compositions but this theorem describes all the those which admit v_1-composition of all order.

Proof. Let γ be a homotopy class of bo filtration $s \geqq 2$. This gives $S^j \to S_s$, $s \geqq 2$. Now consider the composite if possible $\Sigma^{j-2+i} Y \xrightarrow[v_1]{} \Sigma^{j-4+i} Y \xrightarrow[\gamma_i^\#]{} S_s$. The filtration of $\gamma_i^\# v_1$ is 1 and hence by 1.1.(c), there is a commutative diagram

$$\begin{tikzcd}
\Sigma^{j-2+i} Y \arrow[r, "\gamma_i^i v_1"] \arrow[dr, "f"'] & S_s \arrow[r] & S_s \wedge bo \\
 & \Omega S_{s-1} \wedge bo \arrow[ur, "d_1"'] &
\end{tikzcd}$$

where d_1 is the connecting map in the spectral sequence. Thus $\gamma_i^\# v_1 - pf\colon \Sigma^{j-2+i} Y \to S_s$ is a map which when projected to S^0 represents $\gamma_i^\# v_1$. But $\gamma_i^\# v_1 - pf$ lifts to S_{s+1}. Clearly the edge theorem of 1.1.(c) or just the fact that S_s is $3s - 1$ connected guarantees that some v_1-iterate will lie in a zero group. This gives the theorem.

REMARK. To connect this result with other results about v_1-periodicity we give the following result.

PROPOSITION 6.4. *There is commutative diagram*

$$\begin{array}{ccc} \Sigma^8 Y & \xrightarrow{v_1^4} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ \Sigma^{10} M_{2'} & \xrightarrow[8\sigma^{\#}]{} & \Sigma^2 M_{2'} \end{array}$$

where the maps p_1 *are degree* 1 *and* $8\sigma^{\#}$ *is the coextension over* $\Sigma^{20}M$ *of the extension of* $\Sigma^{10}M_{2'} \xrightarrow[8\sigma^{\#}]{} S^3$.

The result is an easy calculation using the Adams spectral sequence because there are no classes of filtration 4 earlier than $h_0^3 h_3$.

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Received December 22, 1978. This research was supported in part by the NSF grant MCS76-07051.

NORTHWESTERN UNIVERSITY
EVANSTON, IL 60201