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HOMOTOPY DIMENSION OF SOME ORBIT SPACES

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The homotopy dimension of a compact absolute neighborhood retract space X is defined to be the least dimension among all the finite CW-complexes which have the same homotopy type of X. We show that actions of finite groups or actions of tori (with finite orbit types) on a finite-dimensional compact absolute neighborhood retract X do not raise homotopy dimension if the homotopy dimension of X is not two.

1. Introduction and preliminaries. Through this note, all actions are of finite types.

In [7], Oliver gave an affirmative answer to Conner's conjecture: "*The orbit space of an action of a compact Lie group on a finite-dimensional* AR *is an* AR". From West [10], it follows that every compact absolute neighborhood retract X (CANR X) has the homotopy type of a finite complex. So, we can define the homotopy dimension (h.d.) of a CANR X by

$$\text{h.d.}\,(X) = \min\{\dim K \mid K \text{ is a finite complex and } K \cong X\}\,.$$

On the other hand, Conner [5] has shown that the orbit space of an action of a compact Lie group on a finite-dimensional CANR is a CANR. It is natural to wonder whether the actions of a compact Lie group on a CANR can raise the homotopy dimension. We will show that the homotopy dimension does not increase when h.d. $(X) \neq 2$ and when the action comes from either a finite group or a toral group.

Combining a well-known result of Wall (Thm. F, [8]) and the result of West [10] (mentioned above), we can easily obtain the following lemma that will be needed in the sequel.

LEMMA 0. *A* CANR *has the homotopy type of a k-dimensional finite complex if and only if $H_q(\tilde{X}; Z) = 0$ for all $q > k$ and $H^{k+1}(X; \beta) = 0$ for every coefficient bundle β of $Z\pi_1(X)$-modules over X if $k \neq 2$. Moreover, if $H_q(\tilde{X}; Z) = 0$ for $q > 2$ and $H^3(X; \beta) = 0$; then* h.d. $(X) \leqq 3$.

2. Orbits of action of finite groups. Let G be a cyclic group of order p with a genertor g. The notation in [1] will be used as follows $1 - g$ and $1 + g + \cdots + g^{p-1}$ will be denoted respectively by τ and σ. If one of these is denoted by ρ, the other will be denoted by $\bar{\rho}$. If β is a sheaf of Z_p-modules over X/G, let $\underline{A}$ denote the sheaf

$$\{H^0(\pi^{-1}y; \pi^*\beta|\pi^{-1}y) | y \in X/G\} \quad \text{over} \quad X/G ,$$

where $\pi^*\beta$ is the pull back of β associated with the orbit map $\pi: X \to X/G$. If U is an open subset of X/G, let $\underline{A}_U$ denote the sheaf

$$[\cup \{H^0(\pi^{-1}y; \pi^*\beta|\pi^{-1}y) | y \in U\}] \cup [\{0_y | y \in X/G\}]$$

and let $\underline{A}_F$ (F closed in X/G) denote $\underline{A}/\underline{A}_{(X/G)-F}$ (refer to page 41 of [1]).

It will be convenient to establish the following preliminary lemmas before we begin the proof of the main result.

LEMMA 1. *Let $G = Z_p$, p prime, act on a* CANR X *with fixed point set F. Assume that $m = \dim X < \infty$ and that β_p is a bundle of coefficients of $Z_p\pi_1(X/G)$- modules over X/G. If* h.d. $(X) \leqq k$, *then $H^q(X/G; \beta_p) = 0$ for all $q \geqq k + 1$.*

Proof. Think of ρ and $\bar{\rho}$ as endomorphisms of the sheaf $\underline{A}$ and denote their images respectively by $\rho\underline{A}$ and $\bar{\rho}\underline{A}$. Since Z_p is a field, it follows that the following sequence of sheaves over X/G

$$0 \longrightarrow \bar{\rho}\underline{A} \longrightarrow \underline{A} \xrightarrow{\rho \oplus \eta} \rho\underline{A} \oplus \underline{A}_F \longrightarrow 0$$

is exact, where $\bar{\rho}\underline{A} \to \underline{A}$ is the inclusion and where $\eta: \underline{A} \to \underline{A}_F$ is the quotient homomorphism (Lemma 4.1 of [1]). This sequence induces an exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow H^n(X/G; \bar{\rho}\underline{A}) &\longrightarrow H^n(X/G; \underline{A}) \\ &\longrightarrow H^n(H/G; \rho\underline{A}) \oplus H^n(X/G; \underline{A}_F) \longrightarrow \cdots . \end{aligned}$$

Let $H^n(\rho)$ denote $H^n(X/G; \rho\underline{A})$. Observe $H^n(X/G; \underline{A}_F) = H^n(F; \beta_p|F)$; then, from the above cohomology sequence and the fact that $H^n(X/G; \underline{A}) \cong H^n(X; \pi^*\beta_p)$ (see page 35, [1]), there are the following exact sequences:

$$\begin{array}{llll} H^q(X; \pi^*\beta_p) & \longrightarrow H^q(\sigma) & \oplus H^q(F; \beta_p|F) & \longrightarrow H^{q+1}(\tau) , \\ H^{q+1}(X; \pi^*\beta_p) & \longrightarrow H^{q+1}(\tau) & \oplus H^{q+1}(F; \beta_p|F) & \longrightarrow H^{q+2}(\sigma) , \\ \vdots & \vdots & \vdots & \vdots \\ H^m(X; \pi^*\beta_p) & \longrightarrow H^m(\rho) & \oplus H^m(F; \beta_p|F) & \longrightarrow H^{m+1}(\bar{\rho}) . \end{array}$$

Since h.d. $(X) \leqq k$, it follows from Lemma 0 that $H^n(X, \pi^*\beta_p) = 0$, for all $n \geqq q \geqq k + 1$. On the other hand, $H^{m+1}(\bar{\rho}) = 0$ since $\dim X = m < \infty$. Thus, we can show inductively that

(1) $H^q(X/G, F; \beta_p) = H^q(\sigma) = 0$, and

(2) $H^q(F; \beta_p|F) = 0$.

Hence, from the exact sequence of the pair $(X/G, F)$,

$$\cdots \longrightarrow H^q(X/G, F; \beta_p) \longrightarrow H^q(X/G; \beta_p) \longrightarrow H^q(F; \beta_p | F) \longrightarrow \cdots ,$$

it follows that $H^q(X/G; \beta_p) = 0$; and the proof of lemma is complete.

LEMMA 2. *Let $G = Z_p$, p prime, act on a* CANR X *with fixed point set $F \neq \emptyset$. Assume that* $\dim X = m < \infty$ *and that β is a bundle of coefficients of $Z\pi_1$-modules over X/G. Then $H^q(X/G; \beta) = 0$ for all $q \geqq k + 1$, if* h.d. $(X) \leqq k$.

Proof. Consider the following diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^q(X/G; \beta) & \xrightarrow{\times p} & H^q(X/G; \beta) & \longrightarrow & H^q(X/G; \beta_p) & \longrightarrow \cdots \\ & {\scriptstyle \pi^*}\searrow & & \nearrow{\scriptstyle \mu^*} & & & \\ & & H^q(X; \pi^*\beta) = 0 & & & & \end{array}$$

where μ^* is the transfer map [1] and where the horizontal exact sequence is from the exact sequence of bundles of coefficients over X/G:

$$0 \longrightarrow \beta \xrightarrow{\times p} \beta \longrightarrow \beta_p \longrightarrow 0 .$$

So, it follows easily that $H^q(X/G; \beta) = 0$ if $q \geqq k + 1$, since $H^q(X; \pi^*\beta) = 0$ by Lemma 0 and $H^q(X/G; \beta_p) = 0$ by Lemma 1. The proof is now complete.

LEMMA 3. *Let a finite group G act on X with fixed point set $F \neq \emptyset$. If X has the homotopy type of a simplicial complex K^k, then $H_q(\widetilde{X/G}; Z) = 0$ for all $q > k$.*

Proof. Let $\pi^*(\widetilde{X/G})$ be the pullback of the universal covering space $p: \widetilde{X/G} \rightarrow X/G$ associated with the orbit map $\pi: X \rightarrow X/G$. Then, the induced map $\bar{P}: \pi^*(\widetilde{X/G}) \rightarrow X$ is a covering map and the lifting map π^* of π is the orbit map of the induced action of G on $\pi^*(\widetilde{X/G})$. Now, since $X \cong K^k$, it follows that $H_q(\pi^*(\widetilde{X/G}), Z) = 0$ for $q \geqq k + 1$. Then, the Smith theorem in the integral homology theory shows that $H_q(\widetilde{X/G}, Z) = 0$ for $q \geqq k + 1$. (Similar to the proof of Lemma 2 above by use of the transfer map μ_* on page 119 of [3].) Hence, the proof is complete.

THEOREM 1. *Suppose that a finite group G acts on a finite dimensional* CANR X. *If* h.d. $(X) \leqq k$ *and $k \neq 2$, then* h.d. $(X/G) \leqq k$. *If $k = 2$,* h.d. $(X/G \leqq 3)$.

Proof. *Step* 1. *$G = Z_p$, p prime.*

Case 1. $F = \varnothing$. See Lemma 2 of [6].

Case 2. $F \neq \varnothing$. It follows from Lemma 2 and Lemma 3 above that

(1) $H^q(X/G; \beta) = 0$, $q \geqq k + 1$ and for any bundle coefficient β over X/G,

(2) $H_q(X/G; Z) = 0$, $q \geqq k + 1$.

So, it follows from Lemma 0 that h.d. $(X) \leqq k$.

Step 2. *G is cyclic of order p^n, p prime.* We prove inductively on $|G|$, the order of G. Let H be a subgroup of G of order p^{n-1} then, h.d. $(X/H) \leqq k$ by induction hypothesis and the proof is complete by Step 1.

Step 3. *G is a finite p-group.* First, by an inductive proof as in Step 2 we may assume that G is abelian, since G is solvable. Therefore, we can write $G = Z_p^{n_1} \oplus \cdots \oplus Z_p^{n_k}$. Then, again an inductive proof as above will complete the proof for this case.

Step 4. *General case.* The proof will be similar to that of Theorem III. 5.2 in [1].

Suppose that $|G| = p_1^{n_1} \cdots p_s^{n_s}$ and that K_j is a p_j-Sylow subgroups of G, and denote $\pi_{2,j}$ the canonical map $X/K_j \to X/G$ for $j = 1, 2, \cdots, s$ as in [1]. Define π': $H^*(X/G; \beta) \to \sum_{j=1}^{s} H^*(X/K_j; \pi_2^*{}_j\beta)$ by

$$\pi' = \pi_{2,1}^* + \cdots + \pi_{2,s}^* .$$

Observe that $H^q(X/K_j; \pi^*\beta) = 0$ for $q \geqq k + 1$ and $j = 1, 2, \cdots, s$ by Step 3 above. Hence, if we can show that π' is injective, then $H^q(X/G; \beta) = 0$ for $q \geqq k + 1$. Therefore, the theorem will follow by Lemma 0 and Lemma 3 above.

Now, let μ_j': $H^*(X/K_j; \pi_{2,j}^*\beta) \to H^*(K/G; \beta)$ be the tranfer map [1] such that $\mu_j'\pi_{2,j}^*$ is the multiplication by $|G|/|K_j|$. If $r \in \operatorname{Ker} \pi'$, then we have $|G|/|K_j|)\cdot r = \mu_j'\pi_{2,j}^*(r) = 0$ for each $j = 1, 2, \cdots, s$, since $\pi_{2,j}^* = 0$. Therefore, for each $j = 1, 2, \cdots, s$

$$(p_1^{n_1} \cdots p_{j-1}^{n_{j-1}} p_{j+1}^{n_{j+1}} \cdots p_s^{n_s}) \cdot r = 0 .$$

Since the family $p_1^{n_1} \cdots p_{j+1}^{n_{j-1}} p_{j+1}^{n_{j+1}} \cdots p_s^{n_s}$, $j = 1, 2, \cdots, s$, is relatively prime, it follows that $r = 0$, and the proof is now complete.

3. Orbits of actions of total groups.

LEMMA 4. *Suppose that the circle group S^1 acts on a finite-*

dimensional CANR X. *If* h.d. $(X) \leqq k$, *then* $H^q(X/S^1; \beta) = 0$ *for all* $q \geqq k+1$ *and for all bundles of coefficients* β *over* X/S^1.

Proof. Assume that $H_1, \cdots, H_s$ are finite isotropy subgroups of the action. Let G be a finite cyclic subgroup of S^1 such that $H_1, \cdots, H_s$ are subgroups of G. Then h.d. $(X/G) \leqq k$ by the theorem above. So, we may assume that the action is semi-free, i.e., it has only two orbit types $\{e\}$ and S^1. Let β be a bundle of coefficients of $Z\pi_1$-modules over X/S^1, where $\pi_1 = \pi_1(X/S^1)$. From Lemma 0, it follows that $H^q(X; \pi^*\beta) = 0$ for all $q \geqq k+1$, where $\pi: X \to X/S^1$ is the orbit map.

Case 1. $F = \varnothing$. Since the action is free, $\{H^0(\pi^{-1}y; \pi^*\beta): y \in X/S^1\} = \beta$ and $\{H^1(\pi^{-1}y; \pi^*\beta): y \in X/S^1\} = \beta$. An observations on Leray spectral sequence (as in Case 2) proves the lemma for this case.

Case 2. $F \neq \varnothing$. Since $\pi^{-1}(y) = \{e\}$ or S^1, we have

(1) $E_2^{q,0} = H^q(X/S^1; H^0(\pi^{-1}y; \pi^*\beta|\pi^{-1}y)) = H^q(X/S^1; \beta)$,

(2) $E_2^{q,1} = H^q(X/S^1; H^1(\pi^{-1}y; \pi^*\beta|\pi^{-1}y)) = H^q(X/S^1, F; \beta)$, and

(3) $E_2^{q,s} = 0$ if $s \geqq 2$.

We now proceed by induction on q. Since $\dim X < \infty$, we may assume that $H^q(X/S^1; \beta) = 0$ for $q \geqq k+2$, then we will show that $H^{k+1}(X/S^1; \beta) = 0$.

Step 1. *To show that* $H^q(X/S^1, F; \beta) = 0$ *for* $q \geqq k+1$. By the induction hypothesis, we observe that for each $q \geqq k+2$, the E_2-term, $E_2^{q,0}$, of the Leray spectral sequence for the map π (page 140, [2]) is trivial, since $E_2^{q,0} = H^q(X/S^1; \beta)$ by (1). Observing the Leray spectral sequence $\{E_2^{q,s}\}$ of π, we can show that for all $r \geqq 2$

(a) $E_r^{k+1,1} = E_2^{k+1,1}$,

and

(b) $E_r^{k+2,0} = 0$;

therefore,

(a^1) $E_\infty^{k+1,1} = H^{k+1}(X/S^1, F; \beta)$ by (2),

and

(b^1) $E_\infty^{k+2,0} = 0$.

Now, from the convergence of $\{E_2^{q,s}\}$ to $H^*(X; \pi^*\beta)$ and from the fact that $H^{k+2}(X; \pi^*\beta) = 0$ by Lemma 0, we can show that $H^{k+1}(X/S^1, F; \beta) = 0$.

Step 2. *To show that* $H^q(X, F; \pi^*\beta) = 0$ *for* $q \geqq k+2$. Consider the Leray spectral sequence (page 140, [2]) of the map of pairs $\pi: (X, F) \to (X/S^1, F)$. First we observe that the sheaf $\xi = \{H^0(\pi^{-1}y, \pi^{-1}(y \cap F); \pi^*\beta|\pi^{-1}y) | y \in X/S^1\}$ and the sheaf $\eta = \{H^1(\pi^{-1}y,$

$\pi^{-1}(y \cap F); \pi^*\beta | \pi^{-1}y) | y \in X/S^1\}$ are the same over X/S^1, since $\pi^{-1}(y) = \{e\}$ or S^1. Moreover, from the definition of the relative cohomology (Prop. II. 12.2, [2]), it follows that $H^*(X/S^1, F; \beta) = H^*(X/S^1; \xi)$. Then, from Step 1 it follows that

$$E_2^{q,s} = \begin{cases} H^q(X/S^1, F; \beta) = 0 & \text{if} \quad q \geqq k+1 \\ 0 & \text{if} \quad s \geqq 2 . \end{cases}$$

Therefore, $E_\infty^{q\ s} = 0$ when $q + s \geqq k + 2$. Consequently, for $q \geqq k + 2$ $H^q(X, F; \beta) = 0$, since $\{E_2^{q,s}\}$ converges to $H^*(X, F; \beta)$.

Step 3. *To show that* $H^q(X/S^1; \beta) = 0$ *for* $q \geqq k + 1$. First, from the exact cohomology sequence of the pair (X, F) and from the fact of $H^q(X, F; \pi^*\beta) = 0$ for $q \geqq k + 2$, it follows that $H^q(F; \pi^*\beta | F) = 0$ for $q \geqq k + 1$. Then, we observe that $H^*(F; \pi^*\beta | F) = H^*(F; \beta | F)$, since F is the fixed point set. So, $H^q(F; \beta | F) = 0$ for $q \geqq k + 1$. Therefore, the exactness of the cohomology sequence of the pair $(X/S^1, F)$ shows that $H^q(X/S^1; \beta) = 0$ for $q \geqq k + 1$, since $H^q(X/S^1, F; \beta) = 0$ by Step 1, and the proof of lemma is now complete.

THEOREM 2. *Suppose that* T^m *acts on a finite-dimensional* CANR X. *Then*

(1) h.d. $(X/T^m) \leqq$ h.d. (X) *if* h.d. $(X) \neq 2$,

and

(2) h.d. $(X/T^m) \leqq 3$ *if* h.d. $(X) = 2$.

Proof. By induction m, without loss of generality we only consider the actions of S^1. By Lemmas 0 and 4, we only have to show that $H_q(\widetilde{X/S^1}; Z) = 0$ for all $q \geqq k + 1$. Again, by Lemma 4 above, $H^q(\widetilde{X/S^1}; Z) = 0$ for all $q \geqq k + 1$; therefore Ext $(H_{q-1}(\widetilde{X/S^1}); Z) = 0$ and Hom $(H_q(\widetilde{X/S^1}; Z); Z) = 0$ for all $q \geqq k + 1$ by the universal-coefficient theorem (Thm. 5.5.3 in [8]). Hence, for each $q \geqq k + 1$ Ext $(H_q(\widetilde{X/S^1}; Z); Z) = 0$ and Hom $(H_q(\widetilde{X/S^1}; Z); Z) = 0$; and it follows from Theorem V. 13.7 in [2] that $H_q(\widetilde{X/S^1}; Z) = 0$. The proof is now complete.

COROLLARY. *Let* G *be a compact Lie group such that* $|G/G_0|$ *is finite, where* G_0 *is the torus identity component of* G. *Let* G *act on a finite-dimensional* CANR X. *Then,*

(1) *if* h.d. $(X) \neq 2$, *then* h.d. $(X/G) \leqq$ h.d. (X),

(2) *if* h.d. $(X) = 2$, *then* h.d. $(X/G) \leqq 3$.

We conclude this paper by some remarks.

REMARKS. (1) It is a well-known problem in infinite-dimensional topology to determine whether the orbit space of an action of compact Lie group on the Hilbert cube $\prod_1^\infty [0, 1]$ is a CAR. This explains (maybe) the condition $\dim X < \infty$ in the above statements.

(2) The limitation, when h.d. $(X) = 2$, is from an unsettled problem.

(3) The author does not see how to extend these results for the case of actions of compact Lie groups on a CANR.

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