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OSCILLATION CRITERIA FOR GENERAL LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Lovelady has recently proved the following oscillation theorem.

THEOREM. **Let $n \geqq 4$ be even and q: $[a, \infty) \to (0, \infty)$ be continuous. If $\int^{\infty} t^{n-2} q(t)dt < \infty$ and the second order equation**

$$\frac{d^2 z}{dt^2} + \left(\frac{1}{(n-3)!}\int_t^{\infty}(s-t)^{n-3}q(s)ds\right)z = 0$$

is oscillatory, then the nth order equation

$$x^{(n)} + q(t)x = 0$$

is oscillatory.

In this paper the above theorem will be extended to a class of differential equations of the form

$$\frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{x}{p_0(t)} + q(t)x = 0 .$$

Let $n \geqq 4$ be an even number, let p_i, $0 \leqq i \leqq n$, and q be positive continuous functions on $[a, \infty)$, and consider the linear differential equation

$$L_n x + q(t)x = 0 , \tag{1}$$

where L_n denotes the general disconjugate operator

$$L_n = \frac{1}{p_n(t)}\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{p_1(t)}\frac{d}{dt}\frac{\cdot}{p_0(t)} . \tag{2}$$

We introduce the notation:

$$\begin{aligned} &D^0(x; p_0)(t) = \frac{x(t)}{p_0(t)} , \\ &D^j(x; p_0, \cdots, p_j)(t) = \frac{1}{p_j(t)}\frac{d}{dt}D^{j-1}(x; p_0, \cdots, p_{j-1})(t) , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \leqq j \leqq n . \end{aligned} \tag{3}$$

The differential operator L_n defined by (2) can then be rewritten as

$$L_n = D^n(\cdot; p_0, \cdots, p_n) .$$

The domain $\mathscr{D}(L_n)$ of L_n is defined to be the set of all functions x: $[a, \infty) \to R$ such that $D^j(x; p_0, \cdots, p_j)(t)$, $0 \leqq j \leqq n$, exist and are continuous on $[a, \infty)$. By a solution of equation (1) we mean a function $x \in \mathscr{D}(L_n)$ which satisfies (1) on $[a, \infty)$. A nontrivial solution of (1) is called oscillatory if the set of its zeros is unbounded, and it is called nonoscillatory otherwise. Equation (1) itself is said to be oscillatory if all of its nontrivial solutions are oscillatory.

The study of the oscillatory behavior of higher-order ordinary differential equations goes back to Kneser [12] and has received a great deal of attention up to the present. For typical results on the subject we refer to the papers [1, 2, 4-6, 8, 10, 11, 13, 14, 16, 18].

In what follows we are primarily interested in the situation in which equation (1) is oscillatory. We have been motivated by the observation that there are very few effective criteria for equation (1) with general L_n to be oscillatory, though equation (1) and its nonlinear analogue have been the object of intensive investigations in recent years. The desired oscillation criterion is established in § 2. It generalizes an interesting oscillation theorem of Lovelady [15] for the particular equation $x^{(n)} + q(t)x = 0$.

1. Preliminaries. We begin by formulating preparatory results which are needed in proving the main theorem in the next section.

Let $i_k \in \{1, \cdots, n-1\}$, $1 \leqq k \leqq n-1$, and $t, s \in [a, \infty)$. Generalizing upon notation introduced by Willett [19], we define

$$\begin{aligned} &I_0 = 1 , \\ &I_k(t, s; p_{i_k}, \cdots, p_{i_1}) = \int_s^t p_{i_k}(u) I_{k-1}(u, s; p_{i_{k-1}}, \cdots, p_{i_1}) du . \end{aligned} \tag{4}$$

It is easy to verify that for $1 \leqq k \leqq n-1$

$$I_k(t, s; p_{i_k}, \cdots, p_{i_1}) = (-1)^k I_k(s, t; p_{i_1}, \cdots, p_{i_k}) , \tag{5}$$

$$I_k(t, s; p_{i_k}, \cdots, p_{i_1}) = \int_s^t p_{i_1}(u) I_{k-1}(t, u; p_{i_k}, \cdots, p_{i_2}) du . \tag{6}$$

For convenience of notation we put

$$J_i(t, s) = p_0(t) I_i(t, s; p_1, \cdots, p_i) , \qquad J_i(t) = J_i(t, a) , \tag{7}$$

$$K_i(t, s) = p_n(t) I_i(t, s; p_{n-1}, \cdots, p_{n-i}) , \qquad K_i(t) = K_i(t, a) . \tag{8}$$

LEMMA 1. *If $x \in \mathscr{D}(L_n)$, then for $t, s \in [a, \infty)$ and $0 \leqq i < k \leqq n-1$*

$$\begin{aligned} &D^i(x; p_0, \cdots, p_i)(t) - D^i(x; p_0, \cdots, p_i)(s) \\ &\quad = \sum_{j=i+1}^{k} (-1)^{j-i} D^j(x; p_0, \cdots, p_j)(s) I_{j-i}(s, t; p_j, \cdots, p_{i+1}) \\ &\qquad + (-1)^{k-i+1} \int_t^s I_{k-i}(u, t; p_k, \cdots, p_{i+1}) p_{k+1}(u) \\ &\qquad \times D^{k+1}(x; p_0, \cdots, p_{k+1})(u) du . \end{aligned} \tag{9}$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

LEMMA 2. *If there exists an eventually positive function* $y \in \mathscr{D}(L_n)$ *satisfying*

$$L_n y + q(t)y \leqq 0 \tag{10}$$

for all large t, *then equation* (1) *has an eventually positive solution.*

This lemma exhibits an important relationship between the differential equation (1) and the differential inequality (10). For the proof see Čanturija [3].

In what follows we assume that

$$\int_a^\infty p_i(t)dt = \infty \qquad \text{for} \quad 1 \leqq i \leqq n-1\,. \tag{11}$$

The operator L_n satisfying condition (11) is said to be in canonical form. It is known that any operator L_n of the form (2) can always be represented in canonical form in an essentially unique way (see Trench [17]).

LEMMA 3. *Suppose* (11) *holds. If* $x \in \mathscr{D}(L_n)$ *satisfies* $x(t)L_n x(t) < 0$ *on* $[t_0, \infty)$, *then there exist an odd integer* l, $1 \leqq l \leqq n-1$, *and a* $t_1 > t_0$ *such that*

$$x(t)D^j(x; p_0, \cdots, p_j)(t) > 0 \quad on \quad [t_1, \infty) \quad for \quad 0 \leqq j \leqq l\,, \tag{12}$$

$$(-1)^{j-l}x(t)D^j(x; p_0, \cdots, p_j)(t) > 0 \quad on \quad [t_1, \infty) \quad for \quad l+1 \leqq j \leqq n\,. \tag{13}$$

This lemma generalizes a well-known lemma of Kiguradze [9] and can be proved similarly.

2. Main Result. The best oscillation theorem known to date for equation (1) is the following theorem due to Trench [18].

THEOREM A. *Suppose* (11) *holds. If*

$$\int^\infty J_{i-1}(t)K_{n-i-1}(t)q(t)dt = \infty \qquad for \quad i = 1, 3, \cdots, n-1\,, \tag{14}$$

then equation (1) *is oscillatory.*

A question naturally arises as to what will happen when condition (14) is violated. In fact, Theorem A cannot cover an important class of Euler's equations of the form

$$\frac{d^m}{dt^m} t^{\alpha+m} \frac{d^m x}{dt^m} + ct^{\alpha-m} x = 0 \,, \qquad t \geqq 1 \,, \tag{15}$$

where α and $c > 0$ are constants with $\alpha + m \leqq 1$, since in this case the integrals appearing in (14) converge.

An answer to this question is given in the following theorem, which reduces the oscillation of equation (1) to the oscillation of a certain set of second order linear differential equations.

THEOREM B. *Suppose* $n \geqq 4$, (11) *holds, and the integrals in* (14) *converge. Define*

$$q_i(t) = p_{i+1}(t) \int_t^\infty J_{i-1}(u, t) K_{n-i-2}(u, t) q(u) du \,, \qquad i = 1, 3, \cdots, n-3 \,; \tag{16}$$

$$q_{n-1}(t) = p_{n-2}(t) \int_t^\infty J_{n-3}(u, t) K_0(u, t) q(u) du \,. \tag{17}$$

Then equation (1) *is oscillatory if the second order equations*

$$\frac{d}{dt} \frac{1}{p_i(t)} \frac{dz}{dt} + q_i(t) z = 0 \,, \qquad i = 1, 3, \cdots, n-1 \,, \tag{18}$$

are oscillatory.

Proof. Suppose $x(t)$ is a nonoscillatory solution of (1). We may suppose $x(t)$ is eventually positive. Let $t_0 \geqq a$ be such that $x(t) > 0$ for $t \geqq t_0$. Lemma 3 implies that there exists an odd integer l, $1 \leqq l \leqq n-1$, such that (12) and (13) hold for $t \geqq t_1$, provided $t_1 > t_0$ is sufficiently large.

Suppose $1 \leqq l \leqq n-3$. Then, from Lemma 1 applied to $x(t)$ with $i = l+1$, $k = n-1$ and $s \geqq t \geqq t_1$ it follows that

$$\begin{aligned} &D^{l+1}(x; p_0, \cdots, p_{l+1})(t) - D^{l+1}(x; p_0, \cdots, p_{l+1})(s) \\ &\quad = \sum_{j=l+2}^{n-1} (-1)^{j-l-1} D^j(x; p_0, \cdots, p_j)(s) I_{j-l-1}(s, t; p_j, \cdots, p_{l+2}) \\ &\qquad + (-1)^{n-l-1} \int_t^s I_{n-l-2}(u, t; p_{n-1}, \cdots, p_{l+2}) p_n(u) D^n(x; p_0, \cdots, p_n)(u) du \,. \end{aligned}$$

Using (12) and (13) in the above and letting $s \to \infty$, we have

$$\begin{aligned} &-D^{l+1}(x; p_0, \cdots, p_{l+1})(t) \\ &\qquad \geqq \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \cdots, p_{l+2}) q(u) x(u) du \end{aligned} \tag{19}$$

for $t \geqq t_1$. If $l \geqq 3$, then using Lemma 1 again (with $i = 0$, $k = l-2$, $s = t_1$ and $t \geqq t_1$) and (5), we get

$$
\begin{aligned}
&D^0(x; p_0)(t) - D^0(x; p_0)(t_1) \\
&\quad = \sum_{j=1}^{l-2} (-1)^j D^j(x; p_0, \cdots, p_j)(t_1) I_j(t_1, t; p_j, \cdots, p_1) \\
&\qquad + (-1)^{l-1} \int_t^{t_1} I_{l-2}(u, t; p_{l-2}, \cdots, p_1) p_{l-1}(u) D^{l-1}(x; p_0, \cdots, p_{l-1})(u) du \\
&\quad = \sum_{j=1}^{l-2} D^j(x; p_0, \cdots, p_j)(t_1) I_j(t, t_1; p_1, \cdots, p_j) \\
&\qquad + \int_t^{t_1} I_{l-2}(u, t; p_{l-2}, \cdots, p_1) p_{l-1}(u) D^{l-1}(x; p_0, \cdots, p_{l-1})(u) du \, .
\end{aligned}
$$

Thus in view of (12) we obtain

$$
D^0(x; p_0)(t) \geqq \int_{t_1}^t I_{l-2}(t, u; p_1, \cdots, p_{l-2}) p_{l-1}(u) D^{l-1}(x; p_0, \cdots, p_{l-1})(u) du \tag{20}
$$

for $t \geqq t_1$. Combining (19) with (20) yields

$$
\begin{aligned}
&-D^{l+1}(x; p_0, \cdots, p_{l+1})(t) \\
&\quad \geqq \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \cdots, p_{l+2}) q(u) p_0(u) \\
&\qquad \times \int_{t_1}^u I_{l-2}(u, v; p_1, \cdots, p_{l-2}) p_{l-1}(v) D^{l-1}(x; p_0, \cdots, p_{l-1})(v) dv du \\
&\quad \geqq \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \cdots, p_{l+2}) q(u) p_0(u) \\
&\qquad \times \int_t^u I_{l-2}(u, v; p_1, \cdots, p_{l-2}) p_{l-1}(v) D^{l-1}(x; p_0, \cdots, p_{l-1})(v) dv du
\end{aligned}
$$

for $t \geqq t_1$. Since $D^{l-1}(x; p_0, \cdots, p_{l-1})$ is increasing, we conclude from the above that

$$
\begin{aligned}
&-D^{l+1}(x; p_0, \cdots, p_{l+1})(t) \\
&\quad \geqq D^{l-1}(x; p_0, \cdots, p_{l-1})(t) \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \cdots, p_{l+2}) \\
&\qquad \times q(u) p_0(u) \int_t^u I_{l-2}(u, v; p_1, \cdots, p_{l-2}) p_{l-1}(v) dv du \\
&\quad = D^{l-1}(x; p_0, \cdots, p_{l-1})(t) \int_t^\infty p_n(u) I_{n-l-2}(u, t; p_{n-1}, \cdots, p_{l+2}) \\
&\qquad \times q(u) p_0(u) I_{l-1}(u, t; p_1, \cdots, p_{l-1}) du \, ,
\end{aligned} \tag{21}
$$

where we have used formula (6). Let $y(t)$ be given by

$$
y(t) = D^{l-1}(x; p_0, \cdots, p_{l-1})(t) \, .
$$

Note that $y(t) > 0$ and in view of (21)

$$
-D^{l+1}(x; p_0, \cdots, p_{l+1})(t) \geqq y(t) \int_t^\infty K_{n-l-2}(u, t) J_{l-1}(u, t) q(u) du \tag{22}
$$

for $3 \leqq l < n-1$ and $t \geqq t_1$. That (22) is true for $l=1$ follows immediately from (19). Since $dy(t)/dt = p_l(t)D^l(x; p_0, \cdots, p_l)(t)$, we have

$$\frac{d}{dt}\frac{1}{p_l(t)}\frac{dy(t)}{dt} = p_{l+1}(t)D^{l+1}(x; p_0, \cdots, p_{l+1})(t) ,$$

which together with (22) implies

$$\frac{d}{dt}\frac{1}{p_l(t)}\frac{dy(t)}{dt} + q_l(t)y(t) \leqq 0 ,$$

where $q_l(t)$ is defined by (16). Now from Lemma 2 it follows that the equation

$$\frac{d}{dt}\frac{1}{p_l(t)}\frac{dz}{dt} + q_l(t)z = 0$$

has a nonoscillatory solution. But this is impossible by hypothesis.

Finally, suppose $l = n-1$. Integrating (1), we have

$$D^{n-1}(x; p_0, \cdots, p_{n-1})(t) \geqq \int_t^\infty p_n(u)q(u)x(u)du , \qquad t \geqq t_1 . \tag{23}$$

On the other hand, application of Lemma 1 to the case where $i = 0$, $k = n-3$, $s = t_1$ and $t \geqq t_1$ shows that

$$\begin{aligned} &D^0(x; p_0)(t) - D^0(x; p_0)(t_1) \\ &\quad = \sum_{j=1}^{n-3}(-1)^j D^j(x; p_0, \cdots, p_j)(t_1)I_j(t_1, t; p_j, \cdots, p_1) \\ &\qquad + (-1)^{n-2}\int_t^{t_1} I_{n-3}(u, t; p_{n-3}, \cdots, p_1)p_{n-2}(u)D^{n-2}(x; p_0, \cdots, p_{n-2})(u)du \\ &\quad = \sum_{j=1}^{n-3} D^j(x; p_0, \cdots, p_j)(t_1)I_j(t, t_1; p_1, \cdots, p_j) \\ &\qquad + \int_{t_1}^t I_{n-3}(t, u; p_1, \cdots, p_{n-3})p_{n-2}(u)D^{n-2}(x; p_0, \cdots, p_{n-2})(u)du . \end{aligned}$$

This implies that

$$\begin{aligned} &D^0(x; p_0)(t) \\ &\qquad \geqq \int_{t_1}^t I_{n-3}(t, u; p_1, \cdots, p_{n-3})p_{n-2}(u)D^{n-2}(x; p_0, \cdots, p_{n-2})(u)du \end{aligned} \tag{24}$$

for $t \geqq t_1$. From (23) and (24) we obtain

$$\begin{aligned} &D^{n-1}(x; p_0, \cdots, p_{n-1})(t) \\ &\qquad \geqq \int_t^\infty p_n(u)q(u)p_0(u)\int_{t_1}^u I_{n-3}(u, v; p_1, \cdots, p_{n-3})p_{n-2}(v) \\ &\qquad\quad \times D^{n-2}(x; p_0, \cdots, p_{n-2})(v)dvdu \end{aligned}$$

$$\geqq \int_t^\infty p_n(u)q(u)p_0(u)\int_t^u I_{n-3}(u, v; p_1, \cdots, p_{n-3})p_{n-2}(v)$$
$$\times D^{n-2}(x; p_0, \cdots, p_{n-2})(v)dvdu$$
$$= \int_t^\infty \left(\int_v^\infty p_n(u)p_0(u)I_{n-3}(u, v; p_1, \cdots, p_{n-3})q(u)du\right) p_{n-2}(v)$$
$$\times D^{n-2}(x; p_0, \cdots, p_{n-2})(v)dv .$$

It follows that for $t \geqq t_1$

$$D^{n-1}(x; p_0, \cdots, p_{n-1})(t)$$
$$\geqq \int_t^\infty \left(\int_v^\infty J_{n-3}(u, v)K_0(u, v)q(u)du\right) p_{n-2}(v)D^{n-2}(x; p_0, \cdots, p_{n-2})(v)dv .$$

Integrating the above inequality from t_1 to t, we see that the positive function $w(t) = D^{n-2}(x; p_0, \cdots, p_{n-2})(t)$ satisfies

$$w(t) \geqq w(t_1) + \int_{t_1}^t p_{n-1}(u)\int_u^\infty q_{n-1}(v)w(v)dvdu \tag{25}$$

for $t \geqq t_1$, where $q_{n-1}(t)$ is given by (17). Denote the right hand side of (25) by $y(t)$. By differentiation

$$\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{dy(t)}{dt} + q_{n-1}(t)w(t) = 0 , \qquad t \geqq t_1 ,$$

and so

$$\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{dy(t)}{dt} + q_{n-1}(t)y(t) \leqq 0 , \qquad t \geqq t_1 .$$

Again by Lemma 2 we see that the equation

$$\frac{d}{dt}\frac{1}{p_{n-1}(t)}\frac{dz}{dt} + q_{n-1}(t)z = 0$$

has a nonoscillatory solution, contradicting the hypothesis. This completes the proof in the case $l = n - 1$.

REMARK. According to a classical oscillation criterion of Hille [7] equations (18) are oscillatory if

$$\liminf_{t\to\infty} \int_a^t p_i(s)ds \cdot \int_t^\infty q_i(s)ds > \frac{1}{4} , \qquad i = 1, 3, \cdots, n-1 . \tag{26}$$

It is not difficult to see that, when specialized to the particular equation

$$\frac{d^m}{dt^m}\frac{1}{p_m(t)}\frac{d^m x}{dt^m} + q(t)x = 0 , \tag{27}$$

Theorem B yields the following result which contains the theorem of Lovelady stated at the beginning of this paper.

COROLLARY. *Suppose that* $\int^{\infty} p_m(t)dt = \infty$. *Suppose moreover that:*

(i) *if* $m = 2$, *then the equation*

$$\frac{d^2z}{dt^2} + \left(p_m(t) \int_t^{\infty} (u - t)q(u)du\right) z = 0 \tag{28}$$

is oscillatory;

(ii) *if* $m > 2$ *is even, then the equations*

$$\frac{d^2z}{dt^2} + \left(\frac{1}{(m-1)!\,(m-3)!} \times \int_t^{\infty} \left(\int_t^u (u - v)^{m-1}(v - t)^{m-3} p_m(v)dv\right) q(u)du\right) z = 0\,, \tag{29}$$

$$\frac{d^2z}{dt^2} + \left(\frac{p_m(t)}{(m-1)!\,(m-2)!} \int_t^{\infty} (u - t)^{2m-3} q(u)du\right) z = 0 \tag{30}$$

are oscillatory; and

(iii) *if* $m > 2$ *is odd, then the equations* (29) *and*

$$\frac{d}{dt}\frac{1}{p_m(t)}\frac{dz}{dt} + \left(\frac{1}{(m-1)!\,(m-2)!} \int_t^{\infty} (u - t)^{2m-3} q(u)du\right) z = 0 \tag{31}$$

are oscillatory. Then equation (27) *is oscillatory.*

EXAMPLE. Consider the Euler equation

$$\frac{d^m}{dt^m} t^{\alpha+m} \frac{d^m x}{dt^m} + ct^{\alpha-m}x = 0\,, \qquad t \geqq 1\,, \tag{15}$$

where α and $c > 0$ are real constants, and $\alpha \leqq -m + 1$.

It is a matter of easy computation to find that the second order equations (28), (29), (30), and (31) associated with (15) reduce respectively to

$$\frac{d^2z}{dt^2} + \frac{c}{\alpha(\alpha - 1)t^2} z = 0\,,$$

$$\frac{d^2z}{dt^2} + \frac{c}{(m-1)!\,\alpha(\alpha - 1) \cdots (\alpha - m + 1)t^2} z = 0\,,$$

$$\frac{d^2z}{dt^2} + \frac{(2m-3)!\,c}{(m-1)!\,(m-2)!\,(\alpha+m-2)(\alpha+m-3) \cdots (\alpha-m+1)t^2} z = 0\,,$$

and

$$\frac{d}{dt}t^{\alpha+m}\frac{dz}{dt}+\frac{(2m-3)!\,ct^{\alpha+m-2}}{(m-1)!\,(m-2)!\,(\alpha+m-2)(\alpha+m-3)\cdots(\alpha-m+1)}z=0\;.$$

Note that these are Euler equations of the second order. Consequently, we conclude that equation (15) is oscillatory provided c is so large that

(i) when $m = 2$, $c > (1/4)\alpha(\alpha-1)$;

(ii) when $m > 2$ is even,

$$c > \frac{1}{4}\max\Big\{(m-1)!\,\alpha(\alpha-1)\cdots(\alpha-m+1),\\ \frac{(m-1)!\,(m-2)!}{(2m-3)!}(\alpha+m-2)(\alpha+m-3)\cdots(\alpha-m+1)\Big\}\;;$$

(iii) when $m > 2$ is odd,

$$c > \frac{1}{4}\max\Big\{(m-1)!\,\alpha(\alpha-1)\cdots(\alpha-m+1),\\ \frac{(m-1)!\,(m-2)!}{(2m-3)!}(\alpha+m-1)^2(\alpha+m-2)(\alpha+m-3)\cdots(\alpha-m+1)\Big\}\;.$$

Let us now turn to the case where $\alpha > -m+1$. To examine this case we consider the fourth order equation

$$\frac{d^2}{dt^2}t^{\alpha+2}\frac{d^2x}{dt^2}+ct^{\alpha-2}x=0\;,\qquad t\geqq 1\;,\tag{32}$$

where we suppose that $\alpha > -1$. We observe that the differential operator $(d^2/dt^2)t^{\alpha+2}(d^2/dt^2)$ can be represented in canonical form as follows:

$$\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}t^{-\alpha}\frac{d}{dt}t^{\alpha+1}\frac{d}{dt}\qquad(-1<\alpha\leqq 0)\;,\tag{33}$$

$$t^{\alpha}\frac{d}{dt}t^{1-\alpha}\frac{d}{dt}t^{\alpha}\frac{d}{dt}t^{1-\alpha}\frac{d}{dt}t^{\alpha}\qquad(0<\alpha<1)\;,\tag{34}$$

$$t^{\alpha}\frac{d^2}{dt^2}t^{2-\alpha}\frac{d^2}{dt^2}t^{\alpha}\qquad(\alpha\geqq 1)\;.\tag{35}$$

Let $0 < \alpha < 1$, for example. Then in view of (34) equation (32) is equivalent to

$$\frac{d}{dt}t^{1-\alpha}\frac{d}{dt}t^{\alpha}\frac{d}{dt}t^{1-\alpha}\frac{dy}{dt}+ct^{-\alpha-2}y=0\;,\tag{36}$$

and, as easily cheked, the second order equations (18) associated with (36) reduce to the single equation

$$\frac{d}{dt}t^{1-\alpha}\frac{dz}{dt}+\frac{c}{\alpha+1}t^{-\alpha-1}z=0\,, \tag{37}$$

which is an Euler equation of the second order. From the remark following the proof of Theorem B equation (37) is oscillatory if

$$\liminf_{t\to\infty}\int_1^t s^{\alpha-1}ds\cdot\int_t^\infty\frac{c}{\alpha+1}s^{-\alpha-1}ds=\frac{c}{\alpha^2(\alpha+1)}>\frac{1}{4}\,.$$

Thus, in case $0<\alpha<1$, equation (32) is oscillatory if $c>\alpha^2(\alpha+1)/4$. Similarly, it can be shown that (32) is oscillatory if $c>\alpha^2(1-\alpha)/4$ in case $-1<\alpha\leqq 0$ and if $c>\alpha(\alpha+1)/4$ in case $\alpha\geqq 1$. It follows that equation (32) is oscillatory for every α provided c is sufficiently large.

The canonical representation of the operator $(d^m/dt^m)t^{\alpha+m}(d^m/dt^m)$ with general $m>2$ and $\alpha>-m+1$ is not known to us.

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