PACIFIC JOURNAL OF MATHEMATICS
Vol. 93, No. 2, 1981

EXTENDING WHITNEY MAPS

L. E. WARD, JR.

The following theorems are proved. (1) If X is a continuum then any Whitney map for $C(X)$, the space of subcontinua of X, can be extended to a Whitney map for 2^X, the space of nonempty closed subsets of X. (2) If Y is a continuum and X is a subcontinuum of Y then any Whitney map for $C(X)$ (resp., 2^X) can be extended to a Whitney map for $C(Y)$ (resp., 2^Y). The proofs entail recasting these problems in the more inclusive setting of partially ordered spaces and then employing results of Nachbin.

1. Introduction. In this paper a *continuum* is a compact connected metric space. If X is a continuum then 2^X (respectively, $C(X)$) is the hyperspace of nonempty closed subsets (respectively, subcontinua) of X, endowed with the Hausdorff metric. If $\Lambda \subset 2^X$ and if Λ contains all of the singleton subsets of X, then a *Whitney map* for Λ is a continuous function $\omega\colon \Lambda \to [0, +\infty)$ such that $\omega(\{x\}) = 0$ for each $x \in X$ and $\omega(A) < \omega(B)$ whenever A and B are members of Λ and A is properly contained in B.

Among the many interesting and heretofore unsolved problems in the theory of hyperspaces are the following. Nadler ([4], 14.71.5) has asked if every Whitney map for $C(X)$ can be extended to a Whitney map for 2^X. A related question (due to Bruce Hughes and communicated to me by Professor Carl Eberhart) asks whether a Whitney map for $C(X)$ can always be extended to a Whitney map for $C(Y)$ if X is a subcontinuum of Y. We shall answer these questions in the affirmative. The keystone of our approach is to recast the problem in the more general setting of partially ordered spaces, whereupon Nachbin's order-theoretic analog of Tietze's theorem [3] provides an essential ingredient of the proof.

At this point it is worth recalling that in [6] we promoted the notion—certainly not new—that some problems concerning hyperspaces become more tractable if the hyperspace is regarded as a special type of partially ordered space. There is a substantial literature dealing with the latter which can then be utilized. The present paper constitutes further evidence in support of this view.

2. Definitions and known results. A *partially ordered space* is a topological space P endowed with a partial order $\leq$ whose graph is a closed subset of $P \times P$. It is known (see, for example, [2], p. 167) that if X is a regular Hausdorff space then 2^X is a partially ordered space with respect to inclusion. If P is a partially ordered space and $x \in P$, we write $L(x) = \{p \in P: p \leqq x\}$ and $M(x) = \{p \in P: x \leqq p\}$, and if $A \subset P$ then

$$L(A) = \cup \{L(x): x \in A\} ,$$
$$M(A) = \cup \{M(x): x \in A\} .$$

An element m of a partially ordered space P is *minimal* (*maximal*) if, whenever $x \in P$ and $x \leqq m (m \leqq x)$, it follows that $x = m$. The set of minimal elements of P is denoted $\operatorname{Min} P$; the set of maximal elements is denoted $\operatorname{Max} P$. It is known [5] that if P is compact and $x \in P$ then $L(x)$ meets $\operatorname{Min} P$ and $M(x)$ meets $\operatorname{Max} P$.

A *Whitney map* for a partially ordered space P is a continuous function $\omega: P \to [0, 1]$ which satisfies

(i) if $x \in \operatorname{Min} P$ then $\omega(x) = 0$,
(ii) if $x \in \operatorname{Max} P$ then $\omega(x) = 1$,
(iii) if $x < y$ in P then $\omega(x) < \omega(y)$.

It is obvious that if $P = 2^X$ for some continuum X and if ω satisfies (i), (ii) and (iii), then ω is a Whitney map in the hyperspace sense. Moreover, if X is a continuum then a Whitney map for 2^X is, up to a constant factor, a Whitney map in the sense of partially ordered spaces.

It is well-known (for example, see the discussion in [4], pp. 24-27) that 2^X admits a Whitney map whenever X is a continuum. In a recent note [6] the author generalized this result to an appropriate class of partially ordered spaces, as follows.

THEOREM 2.1. *If P is a compact metric partially ordered space such that* $\operatorname{Min} P$ *and* $\operatorname{Max} P$ *are disjoint closed sets, then P admits a Whitney map.*

At this point it is helpful to take cognizance of several results of Nachbin [3] for partially ordered spaces. The statements given here for Nachbin's results differ slightly from those in [3], but they follow easily. In particular, Nachbin's order-theoretic version of Tietze's Theorem (2.4) is stated here only for compact partially ordered spaces, whereas the original result was established in the more general setting of "normally ordered" spaces.

THEOREM 2.2. *If K is a compact subset of a partially ordered space, then $L(K)$ and $M(K)$ are closed sets.*

THEOREM 2.3. *If x and y are elements of a compact partially ordered space and if $M(x) \cap L(y) = \varnothing$, then there are disjoint open sets U and V such that $x \in U = M(U)$ and $y \in V = L(V)$.*

If A and B are partially ordered sets then a function $f: A \to B$ is said to be *order-preserving* if, whenever $x \leqq y$ in A, it follows that $f(x) \leqq f(y)$.

THEOREM 2.4. *If Q is a closed subset of the compact partially ordered space P and if $f: Q \to [0, 1]$ is a continuous order-preserving function, then there exists a continuous order-preserving function $g: P \to [0, 1]$ such that $g | Q = f$.*

3. Extending Whitney maps for partially ordered spaces. Our main result is the following theorem.

THEOREM 3.1. *Let P be a compact metric partially ordered space such that* $\operatorname{Min} P$ *and* $\operatorname{Max} P$ *are disjoint closed sets and let Q be a closed subset of P such that* $\operatorname{Min} Q \subset \operatorname{Min} P$ *and* $\operatorname{Max} Q \subset \operatorname{Max} P$. *Then a Whitney map for Q can be extended to a Whitney map for P.*

The proof of (3.1) depends on a delicate application of (2.4). To facilitate this we first obtain a lemma.

LEMMA 3.2. *Suppose P is a compact partially ordered space such that* $\operatorname{Min} P$ *and* $\operatorname{Max} P$ *are disjoint closed sets, Q is a closed subset containing* $(\operatorname{Min} P) \cup (\operatorname{Max} P)$, *and suppose A and B are disjoint nonempty closed subsets such that $A = M(A)$ and $B = L(B)$. If $f: Q \to [0, 1]$ is a continuous order-preserving function such that $f | (\operatorname{Min} P) \equiv 0$ and $f | (\operatorname{Max} P) \equiv 1$, then f admits a continuous order-preserving extension $\bar{f}: P \to [0, 1]$ such that $\bar{f}(a) \geqq \inf f | (A \cap Q)$ for each $a \in A$ and $\bar{f}(b) \leqq \sup f | (B \cap Q)$ for each $b \in B$.*

Proof. By (2.4) the function $f | (A \cap Q)$ admits a continuous order-preserving extension

$$f_1: A \longrightarrow [\inf f | (A \cap Q), 1] ,$$

and the function $f | (B \cap Q)$ admits a continuous order-preserving extension

$$f_0: B \longrightarrow [0, \sup f | (B \cap Q)] .$$

The mapping $f \cup f_0 \cup f_1$ is a continuous order-preserving function defined on the closed set $Q \cup A \cup B$, and another application of (2.4) yields the desired function $\bar{f}: P \to [0, 1]$.

We turn now to proof of (3.1). Let ω_Q be a Whitney map for Q. We may extend ω_Q at once to $(\operatorname{Min} P) \cup (\operatorname{Max} P)$ by letting $\omega_Q | (\operatorname{Min} P) \equiv 0$ and $\omega_Q | (\operatorname{Max} P) \equiv 1$, so there is no loss of generality if we assume Q contains $(\operatorname{Min} P) \cup (\operatorname{Max} P)$, and hence (3.2) may be applied. We employ a variation on an argument due to Carruth [1]. Suppose $\mathscr{U}$ is a countable base for the topology of P, and let $\mathscr{B}$ denote the family of all pairs (U, V) of members of $\mathscr{U}$ such that $M(\bar{U}) \cap L(\bar{V}) = \emptyset$. Then $\mathscr{B}$ is also countable and we may enumerate its elements:

$$\mathscr{B} = \{(U_n, V_n) \colon n = 1, 2, \cdots\} .$$

By (2.2) the sets $M(\bar{U}_n)$ and $L(\bar{V}_n)$ are closed, so by (3.2), for each positive integer n there is a continuous order-preserving extension $f_n \colon P \rightarrow [0, 1]$ of ω_Q such that

$$f_n(a) \geqq \inf \omega_Q \mid (M(\bar{U}_n) \cap Q) \quad \text{if} \quad a \in M(\bar{U}_n) ,$$
$$f_n(b) \leqq \sup \omega_Q \mid (L(\bar{V}_n) \cap Q) \quad \text{if} \quad b \in L(\bar{V}_n) .$$

Define $\omega_P \colon P \rightarrow [0, 1]$ by $\omega_P = \sum 2^{-n} f_n$. Obviously ω_P is continuous and ω_P is an extension of ω_Q. Since each f_n is order-preserving, so is ω_P. Thus it remains to show that if $x < y$ in P then $\omega_P(x) < \omega_P(y)$. Clearly, it is sufficient to verify the existence of a positive integer n such that $f_n(x) < f_n(y)$.

Let $t_x = \sup \omega_Q \mid (L(x) \cap Q)$ and $t_y = \inf \omega_Q \mid (M(y) \cap Q)$. Since ω_Q is a Whitney map it follows that $t_x < t_y$. Let $0 < \varepsilon < (t_y - t_x)/2$. By (2.3) there are disjoint open sets U and V such that $x \in V = L(V)$ and $y \in U = M(U)$, and by a straightforward compactness argument we may assume that $\omega_Q(V \cap Q) \subset [0, t_x + \varepsilon)$ and $\omega_Q(U \cap Q) \subset (t_y - \varepsilon, 1]$. It follows that there is a positive integer n such that $x \in V_n \subset \bar{V}_n \subset V$ and $y \in U_n \subset \bar{U}_n \subset U$, from which we conclude that

$$f_n(x) \leqq t_x + \varepsilon < t_y - \varepsilon \leqq f_n(y) .$$

The proof is complete.

COROLLARY 3.3. *If X is a continuum then any Whitney map for $C(X)$ can be extended to a Whitney map for 2^X.*

COROLLARY 3.4. *If Y is a continuum and X is a subcontinuum of Y, then any Whitney map for $C(X)$* (resp., 2^X) *can be extended to a Whitney map for $C(Y)$* (resp., 2^Y).

Proof. We give the proof for $C(X)$; the proof for 2^X follows similarly. Clearly $C(X)$ is a closed subset of $C(Y)$ and $\operatorname{Min} C(X) \subset \operatorname{Min} C(Y)$. However, $\operatorname{Max} C(X) = \{X\}$ is not a subset of $\operatorname{Max} C(Y)$. This deficiency is readily corrected by defining $Q = C(X) \cup \{Y\}$ so that $\operatorname{Max} Q = \{Y\} = \operatorname{Max} C(Y)$. If ω_X is a Whitney map for $C(X)$ with $\omega_X(X) = 1$, let $\omega_Q \colon Q \rightarrow [0, 1]$ be defined by

$$\omega_Q \mid C(X) = \frac{1}{2}\omega_X ,$$
$$\omega_Q(Y) = 1 .$$

Theorem 3.1 now applies and ω_Q extends to a Whitney map ω_Y for $C(Y)$. Clearly, $2\omega_Y$ is the desired extension of ω_X.

It is worth remarking that the family of mappings $f_n: P \to [0, 1]$ does not, in general, distinguish points of P and hence does not generate an order homeomorphism of P into the Hilbert cube. However, Carruth [1] has shown that such order homeomorphisms exist for all compact metric partially ordered spaces. The following question arises naturally.

Problem 3.5. *Let ω_P be a Whitney map for the compact metric partially ordered space P. Under what conditions does there exist an order-homeomorphism $\varphi: P \to H$, the Hilbert cube, so that $\omega \mid \varphi(P) = \omega_P \varphi^{-1}$, where $\omega: H \to [0, 1]$ is the Whitney map defined by $\omega(x) = \sum 2^{-n} x_n$?*

REFERENCES

1. J. H. Carruth, *A note on partially ordered compacta*, Pacific J. Math., **24** (1968), 229-231.
2. K. Kuratowski, *Topology I*, Academic Press, 1966.
3. L. Nachbin, *Topology and Order*, Van Nostrand, 1965.
4. S. B. Nadler, Jr., *Hyperspaces of Sets*, Dekker, 1978.
5. L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc., **5** (1954), 144-161.
6. ———, *A note on Whitney maps*, Canad. Math. Bull., to appear.

Received September 17, 1979.

UNIVERSITY OF OREGON
EUGENE, OR 97403